In a series of papers [6], [7], [8], [10], Munn has considered the problem of constructing all irreducible representations of a semigroup by matrices over a field. In [10], he showed how to construct all the irreducible representations of an arbitrary inverse semigroup from those of associated Brandt semigroups. In this paper, we generalize the method of [10] to give a construction for the irreducible representations of an arbitrary semigroup from those of certain associated semigroups.

For many types of semigroups, including regular semigroups, periodic semigroups and 0-simple semigroups with non-zero idempotents, the associated semigroups are completely 0-simple. In this case, by means of Clifford's result [1] on the representations of a completely 0-simple semigroup, we can give an explicit method of construction for all irreducible representations.

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1. *-semigroups. In general, a semigroup need have neither a zero nor an identity. However, given any semigroup \( S \), we may embed \( S \) in a semigroup \( S^0 \) which has a zero and which is constructed from \( S \) in the following way. If \( S \) already has a zero and contains at least two members, then \( S = S^0 \); otherwise \( S^0 \) is the semigroup formed from \( S \) by adjoining a new symbol 0 and defining \( a0 = 0 = 0a \) for each \( a \in S^0 = S \cup \{0\} \). The phrase "\( S = S^0 \)" means that \( S \) is a semigroup which has a zero and at least two members.

In a similar way, we can embed a semigroup \( S \) in a semigroup \( S^1 \) that has an identity.

Because of the simple nature of the embedding of a semigroup \( S \) in the corresponding semigroup \( S^0 \), many theorems about semigroups that have no zero may be deduced from corresponding theorems for semigroups that have a zero. In particular, there will be no loss of generality if, in this paper, we consider only semigroups that have a zero.

A homomorphism \( \theta \) of a semigroup \( S = S^0 \) onto a semigroup \( S \) is said to be \( 0 \)-restricted if \( a\theta = 0\theta \) implies \( a = 0 \); the corresponding congruence on \( S \) is also said to be \( 0 \)-restricted.

**Proposition 1.** Let \( S = S^0 \) be a semigroup. Then
\[
\rho = \{(a, b) \in S \times S: \text{for all } s, t \in S^1, sat = 0 \text{ if and only if } sbt = 0\}
\]
is a \( 0 \)-restricted congruence on \( S \). If \( \tau \) is any \( 0 \)-restricted congruence on \( S \), then \( \tau \subseteq \rho \).

**Proof.** The relation \( \rho \) is clearly an equivalence on \( S \). Let \((a, b) \in \rho, x \in S \). Then, for any \( s, t \in S^1, sat = 0 \) if and only if \( sbt = 0 \). Hence, a fortiori, \( saxt = 0 \) if and only if \( sbxt = 0 \); thus \((ax, bx) \in \rho \). Similarly \((xa, xb) \in \rho \) and so \( \rho \) is a congruence on \( S \).

Let \( a \in S \) with \((a, 0) \in \rho \). Then, for any \( s, t \in S^1, sat = 0 \); in particular, \( a = 0 \). Hence \((a, 0) \in \rho \) implies \( a = 0 \) so that \( \rho \) is a \( 0 \)-restricted congruence on \( S \).
Finally, let \( \tau \) be any 0-restricted congruence on \( S \), and let \( (a, b) \in \tau \). Then, by the regularity of \( \tau \) with respect to multiplication, \( (sat, sbt) \in \tau \) for all \( s, t \in S^1 \). Hence, in particular, for all \( s, t \in S^1 \), \( sat = 0 \) if and only if \( sbt = 0 \). This means that \( (a, b) \in \rho \); hence \( \tau \subseteq \rho \).

The fact that \( \rho \) is the maximum 0-restricted congruence on \( S \) may be deduced from the results of Preston [11] on subsets of a semigroup that are congruence classes. A proof is given here for completeness.

The congruence \( \rho \) is of importance because, in many cases, a semigroup \( S = S^0 \) has an image of some particular type under a 0-restricted homomorphism if and only if \( S/\rho \) is of that type. In particular, we have the following result.

**Proposition 2.** Let \( \mathcal{X} \) be a class of semigroups that is closed under homomorphic images. Then a semigroup \( S = S^0 \) has an image in \( \mathcal{X} \) under a 0-restricted homomorphism if and only if \( S/\rho \in \mathcal{X} \).

**Proof.** If \( S/\rho \in \mathcal{X} \), then \( S \) has a 0-restricted homomorphic image in \( \mathcal{X} \). Conversely, let \( \tau \) be a 0-restricted congruence on \( S \) such that \( S/\tau \in \mathcal{X} \). Then, since \( \tau \subseteq \rho \), it follows, from the induced homomorphism theorem, that \( S/\rho \) is a homomorphic image of \( S/\tau \). Hence, by hypothesis, \( S/\rho \in \mathcal{X} \).

Proposition 2 has several interesting corollaries. For example, let \( S = S^0 \) be a regular semigroup. Then, using Proposition 2, we can show that \( S \) has an image under a 0-restricted homomorphism that is an inverse semigroup if and only if, for any idempotents \( e, f, g, h \) of \( S \), \( gefh = 0 \) implies \( gfeh = 0 \).

Munn [10] has shown that the following condition is important in the theory of matrix representations of a semigroup \( S = S^0 \).

\[ C_1: \quad \text{For any } a, x, b \in S, \text{ if } axb = 0, \text{ then } ax = 0 \text{ or } xb = 0. \]

He has also shown that the next condition plays an important part in the theory, if \( S \) is an inverse semigroup.

\[ M_2: \quad \text{If } M \text{ and } N \text{ are nonzero ideals of } S, \text{ then } M \cap N \neq \{0\}. \]

We shall see that, for arbitrary semigroups, condition

\[ C_2: \quad \text{If } a, b \in S \text{ and } aSb = \{0\}, \text{ then } a = 0 \text{ or } b = 0, \]

is more natural. The connection between \( C_2 \) and \( M_2 \) is given by the following proposition.

**Proposition 3.** Let \( S = S^0 \) be a semigroup. Then \( S \) obeys \( C_2 \) if and only if it obeys \( M_2 \) and

\[ C'_2: \quad \text{If } a \in S \text{ and } aSa = \{0\}, \text{ then } a = 0. \]

**Proof.** Suppose first that \( S \) obeys \( C'_2 \); then, clearly, \( S \) obeys \( C_2 \). Let \( M \) and \( N \) be nonzero ideals of \( S \), and let \( a, b \) be nonzero elements of \( M \) and \( N \) respectively. Then \( aSb \subseteq M \cap N \) and, by \( C_2 \), \( aSb \neq \{0\} \). Hence \( S \) obeys \( M_2 \).

Conversely, suppose that \( S \) obeys \( M_2 \) and \( C_2 \). Given nonzero ideals \( M \) and \( N \), let \( a \in M \cap N \setminus \{0\} \). Then, by \( C_2 \), \( axa \neq 0 \) for some \( x \in S \) so that, since \( axa \in M \cdot N, M \cdot N \neq \{0\} \).
In particular, given any nonzero elements \(a, b \in S\), \(S^1 a S^1 \cdot S^1 b S^1 \neq \{0\}\). But
\[
S^1 a S^1 \cdot S^1 b S^1 = S^1 a S b S^1 \cup S^1 a b S^1,
\]
so that \(a S b \neq \{0\}\) or \(a b \neq 0\). If \(a b \neq 0\), then, similarly, \(a S a b \neq \{0\}\) or \(a . a b \neq 0\). Thus, in any case, \(a S b \neq \{0\}\). Hence \(S\) obeys \(C_2\).

**Corollary.** Let \(S = S^0\) be a regular semigroup. Then \(S\) obeys \(C_2\) if and only if it obeys \(M_2\).

We shall make use of Proposition 2 to give a short proof that \(C_1\) and \(C_2\) are necessary and sufficient for a semigroup \(S = S^0\) to have a 0-restricted congruence \(\tau\) such that \(S/\tau\) is completely 0-simple. Since a completely 0-simple semigroup is regular, it follows from Theorem 1 of [9] and the corollary to Proposition 3 that these conditions are necessary. Another proof that \(C_1\) and \(C_2\) are both necessary and sufficient for the existence of a 0-restricted congruence \(\tau\), with \(S/\tau\) completely 0-simple, has been given by Lallement [4].

A semigroup \(S = S^0\) is said to be weakly regular if and only if, for each nonzero member \(a\) of \(S\), there exists \(x \in S\) such that \(a x = a . x \neq 0\).

Weakly regular semigroups have been called \(E\)-inversive, by Clifford and Preston [2], and \(0\)-inversive, by Lallement and Petrich [5].

The proof of the theorem makes use of the following result which is an immediate corollary to Theorem 3 of [5].

**Proposition 4.** Let \(S = S^0\) be a semigroup that obeys \(C_2\). Then \(S\) is completely 0-simple if and only if it is weakly regular and obeys the following weak cancellation law:

\[
C_3: \text{If } a, b, x, y \in S, \text{ then the relations } ax = bx \neq 0 \text{ and } ya = yb \neq 0 \text{ together imply that } a = b.
\]

**Theorem 1.** Let \(S = S^0\) be a semigroup that obeys \(C_1\). Then there is a 0-restricted congruence \(\sigma\) on \(S\) such that \(S/\sigma\) obeys \(C_3\) and such that, if \(\tau\) is any 0-restricted congruence on \(S\) for which \(S/\tau\) obeys \(C_3\), then \(\sigma \subseteq \tau\).

**Proof.** We show first that \(S/\rho\) obeys \(C_3\). Let \(a, b, x, y \in S\) be such that none of the elements \(a x, b x, y a, y b\) is zero. Suppose, further, that \((a x, b x) \in \rho\) and \((y a, y b) \in \rho\). Then \(sat = 0\), for \(s, t \in S^1\), implies \(sa = 0\) or \(at = 0\). For, if \(s, t \in S\), this is immediate from \(C_1\) while, if, for example, \(t \notin S\), then \(s a = 0\). If \(sa = 0\), then \(s E S\) and \(sa = 0\); thus, since \((a x, b x) \in \rho\), \(s b x = 0\). Hence, by \(C_1\), since \(b x \neq 0\), \(s b = 0\); thus \(s b t = 0\). Similarly, \(at = 0\) implies \(s b t = 0\) and so \((a, b) \in \rho\). Thus \(S/\rho\) obeys \(C_3\).

Let \(T\) be the set of 0-restricted congruences \(\tau\) on \(S\) such that \(S/\tau\) obeys \(C_3\); \(T \neq \emptyset\) since \(\rho \in T\). Let \(\sigma = \cap \{\tau: \tau \in T\}\). Then it is immediate that \(\sigma\) is a 0-restricted congruence on \(S\). It is also straightforward to verify that \(S/\sigma\) obeys \(C_3\). Thus, by its definition, \(\sigma\) is the smallest 0-restricted congruence \(\tau\) on \(S\) such that \(S/\tau\) obeys \(C_3\).

**Corollary.** Let \(S = S^0\) be a semigroup. Then there is a 0-restricted congruence \(\tau\) on \(S\) such that \(S/\tau\) is completely 0-simple if and only if \(S\) obeys \(C_1\) and \(C_2\).
Proof. We have already pointed out that conditions $C_1$ and $C_2$ are necessary. To show that the conditions are sufficient, we need only show that $S/p$ is completely 0-simple.

Let $a \in S \setminus \{0\}$; then, by $C_2$, there exists $x \in S$ such that $axa \neq 0$. If $sat = 0$ then, as in the proof of Theorem 1, either $sa = 0$ or $at = 0$. In either case, $saxat = 0$. Conversely, if $saxat = 0$, then also $saxa = 0$ or $axat = 0$. Since $axa \neq 0$, these imply respectively that $sa = 0$ and $at = 0$; hence, in either case, $sat = 0$. Thus $(a, axa) \in p$ and so $S/p$ is regular.

Further, since $S$ obeys $C_2$ and $p$ is a 0-restricted congruence, it is easy to see that $S/p$ obeys $C_2$. By the proof of Theorem 1, $S/p$ obeys $C_3$. Hence $S/p$ obeys the conditions of Proposition 4 and so is completely 0-simple.

Let $S = S^0$ be a semigroup satisfying $C_1$ and let $\sigma$ be the finest 0-restricted congruence $\tau$ on $S$ such that $S/\tau$ obeys $C_3$ (Theorem 1). Then we shall denote $S/\sigma$ by $S^*$. 

Definition. A semigroup $S = S^0$ is called an $M$-semigroup if it satisfies $C_1$ and $C_2$ and is such that $S^*$ is completely 0-simple.

Proposition 5. Let $S = S^0$ be a weakly regular semigroup that obeys $C_1$ and $C_2$. Then $S$ is an $M$-semigroup.

Proof. It is easy to verify that, if $\tau$ is any 0-restricted congruence on $S$, then $S/\tau$ is weakly regular. In particular, since $S$ obeys $C_1$ and $C_2$, $S^*$ is weakly regular and obeys $C_2$. Since $S^*$ obeys $C_3$, it is thus immediate, from Proposition 4, that $S^*$ is completely 0-simple. Thus $S$ is an $M$-semigroup.

Corollary 1. Let $S = S^0$ be a periodic semigroup that satisfies $C_1$ and $C_2$; then $S$ is an $M$-semigroup. In particular, any finite semigroup $S = S^0$ that satisfies $C_1$ and $C_2$ is an $M$-semigroup.

Proof. Let $S$ be a periodic semigroup that obeys $C_1$ and $C_2$. Let $a \in S \setminus \{0\}$. By $C_2$, there exists $x \in S$ such that $axa \neq 0$. By induction on $n$, it follows from $C_1$ that

$$(ax)^n = ax \cdot ax \cdot \ldots \cdot ax \neq 0$$

for any positive integer $n$. Hence, for some positive integer $n$, $(ax)^n$ is a nonzero idempotent of $S$. Thus, since $(ax)^n = a \cdot (xa)^{n-1}x$, $S$ is weakly regular. Hence the result is immediate from Proposition 5.

Corollary 2. Let $S = S^0$ be a semigroup that satisfies $C_1$ and $C_2$ and that obeys the minimal conditions $M_L$ and $M_R$ on principal left and right ideals respectively. Then $S$ is an $M$-semigroup.

Proof. Green [3, Theorem 4] has shown that $M_L$ and $M_R$ together imply the minimal condition $M_J$ on two-sided principal ideals. Hence $S$ has a 0-minimal principal ideal $M$. Since $S$ obeys $C_2$, $M^2 \neq \{0\}$; thus $M$ is 0-simple. Since $S$ obeys $M_L$ and $M_R$, $M$ must contain a 0-minimal left ideal and a 0-minimal right ideal. Hence by [2, Corollary 2.50], $M$ is completely 0-simple; thus it is regular.
Let $a \in S \setminus \{0\}$ and let $x \in M \setminus \{0\}$. Then, by C2, there exists $y \in S$ such that $ayx \neq 0$. Since $x \in M$, so does $ayx$ and hence, since $M$ is regular, there exists $z \in M$ such that $ayz = ayx \cdot z \cdot ayx$.

Let $u = yxz$; then $au$ is a nonzero idempotent of $S$. Hence $S$ is weakly regular.

Another important class of $\mathcal{M}$-semigroups is the class of all 0-simple semigroups that obey C1 and which contain nonzero idempotents. For, suppose that $S = S^0$ is such a semigroup. Then $S^*$ is also 0-simple and contains a nonzero idempotent. Now, if $e, f$ are nonzero idempotents of $S^*$, and $ef = fe \neq 0$, then

$$e \cdot ef = ef \cdot e \neq 0 \quad \text{and} \quad ef \cdot e = fe \cdot e \neq 0.$$ 

Hence, by C3, $ef = f$; similarly, $fe = e$ so that $e = f$. Thus $S^*$ is a 0-simple semigroup that contains a primitive idempotent. But, by [2, §2.7], this means that $S^*$ is completely 0-simple. Thus $S$ satisfies C2 and is an $\mathcal{M}$-semigroup.

In this paper, we shall determine each irreducible 0-restricted representation $\Gamma$ of an arbitrary semigroup $S = S^0$ modulo a representation of $M^*$, where $M$ is a certain ideal of $S$, dependent on $\Gamma$, which obeys C1 and C2. It follows that, if $M$ is an $\mathcal{M}$-semigroup, then the irreducible 0-restricted representations of $S$ are known modulo those of completely 0-simple semigroups and ultimately, by Clifford's result [1], modulo groups.

Munn [9] showed that, if $S = S^0$ is an inverse semigroup that obeys C1 and M2, then $S^* \cong M^*$ for any nonzero ideal $M$ of $S$. This does not hold in general; it need not even hold for an $\mathcal{M}$-semigroup, as the following simple example shows.

Example. Let $S = S^0$ be a completely 0-simple semigroup with no divisors of zero. Suppose further that $S$ is not a group with zero. Let $S^1$ be the semigroup formed by adjoining an identity to $S$. Then $S^1$ has no divisors of zero and so $S^1$ obeys C1 and C2.

Now $S$ is an ideal of $S^1$ and is completely 0-simple, hence clearly $S^* \cong S$. On the other hand $S^1$ has an identity, so that $S^1\ast$ is a group with zero.

If we consider the special case of weakly regular semigroups satisfying C1 and C2 and in which the idempotents commute, it can be shown that $S^*$ is a Brandt semigroup and that, in this case, there is an exact parallel with the results obtained by Munn [9], [10] for inverse semigroups. In particular, as for inverse semigroups, the finest 0-restricted congruence $\sigma$ on $S$ such that $S/\sigma$ obeys C3 has the following simple form (cf. [9, Theorem 2.7]): for $a, b \in S$,

$$ (a, b) \in \sigma \text{ if and only if } a = 0 = b \text{ or } ax = bx \neq 0 \text{ for some } x \in S. $$

We end this section by giving a characterisation, for an arbitrary semigroup $S = S^0$ that obeys C1, of the 0-restricted congruence $\sigma$ on $S$ whose properties were described in Theorem 1. The method of proof is similar to that used by Clifford [11] to describe the minimum cancellative congruence on a semigroup. As we do not need to make use of the construction, we omit the proof.

Let $S = S^0$ be a semigroup. Then, given any relation $\tau$ on $S$, we can construct new relations, from $\tau$, in the following ways.

$$\tau W = \{(a, b) \in S \times S : \text{for some } s, t \in S^1, (at, bt) \in \tau \text{ and } (sa, sb) \in \tau, \text{ where none of } sa, sb, at, bt \text{ is zero} \} \cup \{(0, 0)\};$$
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\[ \tau C^* = \{(a, b) \in S \times S : \text{for some } s, t \in S^1, u, v \in S, a = s u t, b = s u t \}\; \text{where} \; (u, v) \in \tau \];

\[ \tau \circ \tau = \{(a, b) \in S \times S : \text{for some } c \in S, (a, c) \in \tau, (c, b) \in \tau \}; \]

\[ \tau \theta = \tau W \cup \tau C^* \cup (\tau \circ \tau) \text{ and } \tau \theta^n = (\tau \theta^n \backslash \tau \theta^n - 1) \theta. \]

If \( \mathcal{I} \) is the identity congruence on \( S \), we write \( \mathcal{I} \theta^n = \theta^n \).

**Theorem 2.** Let \( S = S^0 \) be a semigroup that obeys \( C_1 \). Let \( \tau \) be any 0-restricted congruence on \( S \). Then the least congruence \( \omega \) on \( S \), containing \( \tau \), such that \( S/\omega \) obeys \( C_3 \) is \( \tau \theta = \bigcup \tau \theta^n \); \( \tau \theta \) is a 0-restricted congruence on \( S \).

In particular, if \( \sigma \) is the least 0-restricted congruence \( \omega \) on \( S \) such that \( S/\omega \) obeys \( C_3 \), then \( \sigma = \theta = \bigcup \theta^n \). If, further, \( S \) is an \( \mathcal{A} \)-semigroup, then \( S/\sigma \) is the maximum completely 0-simple 0-restricted homomorphic image of \( S \).

2. Representations over a field; introduction. Let \( F \) be a field, and let \( n \) be a positive integer; then we denote by \( (F)_n \) the algebra of all \( n \times n \) matrices over \( F \). The \( n \times n \) identity is denoted by \( I_n \).

A *representation* \( \Gamma \) of a semigroup \( S \), of degree \( n \) over a field \( \Phi \), is a homomorphism of \( S \) into the multiplicative semigroup of \( (F)_n \). If \( \Gamma \) is a representation of a semigroup \( S = S^0 \) of degree \( n \) over a field \( \Phi \) then, by convention, we consider \( \Gamma(0) \) to be the \( n \times n \) zero matrix, which we shall also denote by \( 0 \). There is no loss of generality if we restrict \( \Gamma \) in this way; see [10, pp. 167-168].

If \( S \) is a semigroup, and \( S \neq S^0 \), then we may extend any representation \( \Gamma \) of \( S \) to a representation of \( S^0 \) by defining \( \Gamma(0) \) to be the zero matrix. Consequently, it is sufficient to consider semigroups \( S = S^0 \).

Let \( \Gamma \) be a representation of a semigroup \( S = S^0 \), of degree \( n \) over a field \( \Phi \). Then we define

\[ V(\Gamma) = \{ x \in S : \Gamma(x) = 0 \}; \]

\[ r(\Gamma) = \text{least positive integer } s \text{ such that, for some } x \in S, \Gamma(x) \text{ has rank } s; \]

\[ M = M(\Gamma) = \{ x \in S : \text{rank } \Gamma(x) \leq r(\Gamma) \}, \text{where rank } \Gamma(x) \text{ is the usual matrix rank of } \Gamma(x). \]

\( M(\Gamma) \) and \( V(\Gamma) \) are clearly ideals of \( S \), and there is a one-to-one correspondence between the representations \( \Gamma \) of \( S \) that vanish on an ideal \( V \) (i.e. such that \( V = V(\Gamma) \)) and the 0-restricted representations of the Rees quotient semigroup \( S/V \). (A representation \( \Gamma \) of a semigroup \( S = S^0 \) is said to be 0-restricted if \( \Gamma \) is a 0-restricted homomorphism.) It is thus sufficient to consider 0-restricted representations of semigroups; this we do.

Munn [10, §1], has essentially proved the following result.

**Lemma 1.** Let \( \Gamma \) be a 0-restricted representation of a semigroup \( S = S^0 \). Then

(i) \( M \) is an ideal of \( S \) that obeys \( C_1 \),

(ii) \( \Gamma(M) \) obeys \( C_3 \).
A representation $\Gamma$ of a semigroup $S = S^0$, of degree $n$ over a field $\Phi$, is said to be irreducible if $\Gamma(S)$ is an irreducible matrix set, that is, if there is no fixed, nonsingular, $n \times n$ matrix $C$ such that, for each $x \in S$,

$$C\Gamma(x)C^{-1} \text{ has the block form } \begin{bmatrix} \Gamma_1(x) & 0 \\ A & \Gamma_2(x) \end{bmatrix},$$

where $0$ denotes the zero $r \times (n-r)$ matrix, for some $1 \leq r \leq n$. Otherwise, $\Gamma$ is reducible.

Let $\Gamma$ be a representation of a semigroup $S = S^0$, of degree $n$ over a field $\Phi$, and let $T$ be a subset of $S$. Then we denote by $[\Gamma(T)]$ the subspace of $(\Phi)_n$ generated by $\Gamma(T)$. If $T$ is an ideal of $S$, then $[\Gamma(T)]$ is an ideal of the subalgebra $[\Gamma(S)]$ of $(\Phi)_n$. Further $\Gamma(T)$ is an irreducible matrix set if and only if the same is true of $[\Gamma(T)]$.

We now consider irreducible representations. The next two lemmas are classical; proofs may be found in [2, Chapter 5].

**Lemma 2.** An irreducible subalgebra of $(\Phi)_n$ is a simple algebra over $\Phi$.

**Lemma 3.** (Schur’s Lemma) Let $\mathcal{A}$ be an irreducible subalgebra of $(\Phi)_n$. If $C$ is a constant nonzero matrix that commutes with each member of $\mathcal{A}$, then $C$ is nonsingular.

Using Lemmas 2, 3, Munn [7] proves the following result.

**Lemma 4.** Let $\Gamma$ be a $0$-restricted irreducible representation of $S = S^0$, of degree $n$ over a field $\Phi$. Let $\Gamma(T)$ be an irreducible subset of $\Gamma(S)$. Then there exist finite sets $e_1, \ldots, e_r \in T$, $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_{i=1}^r \alpha_i \Gamma(e_i) = I_n.$$

**Lemma 5.** Let $\Gamma$ be a $0$-restricted irreducible representation of $S = S^0$. Then $S$ obeys $C_2$.  

**Proof.** Let $a, b$ be nonzero elements of $S$; then $S^1 a S^1$, $S^1 b S^1$ are nonzero ideals of $S$. If $a S^1 b = \{0\}$, then $S^1 a S^1 S^1 b S^1 = \{0\}$; hence $[\Gamma(S^1 a S^1)].[\Gamma(S^1 b S^1)] = \{0\}$.

By Lemma 2, $[\Gamma(S^1)]$ is a simple algebra; hence

$$[\Gamma(S^1 a S^1)] = [\Gamma(S)] = [\Gamma(S^1 b S^1)].$$

Thus the hypothesis, $a S^1 b = \{0\}$, implies that $[\Gamma(S)].[\Gamma(S)] = \{0\}$. But, by Lemma 4, $I_x \in [\Gamma(S)]$, so this is impossible. Hence $a S^1 b \neq \{0\}$; that is, $a S b \neq \{0\}$ or $a b \neq 0$. Suppose that $a b \neq 0$; then, as above, $a S b a \neq \{0\}$ and so $a S b \neq \{0\}$. Thus $ab \neq 0$. Hence we have the following proposition, which may be used to give a sufficient condition for the existence of non-null representations of a $0$-simple semigroup; we shall consider this point in the next section.
PROPOSITION 6. Let $S = S^0$ be a 0-simple semigroup. Then $S$ obeys $C_2$. Thus $S$ has a completely 0-simple homomorphic image if and only if it obeys $C_1$.

THEOREM 3. Let $S = S^0$ be a 0-simple semigroup, and let $S$ obey $C_1$. Let $S^*$ denote the maximum non-null homomorphic image of $S$ which obeys $C_3$; $S^*$ is clearly 0-simple. Let $\Gamma$ be a non-null representation of $S$, of degree $n$ over a field $\Phi$. Then $\Gamma$ induces a non-null representation $\Gamma^*$ of $S^*$, of degree $n$ over $\Phi$, according to the rule: for each $x \in S^*$,

$$\Gamma^*(\bar{x}) = \Gamma(x),$$

where $x \rightarrow \bar{x}$ is the natural homomorphism of $S$ onto $S^*$.

Conversely, if $\Gamma^*$ is a non-null representation of $S^*$, of degree $n$ over $\Phi$, then the mapping $\Gamma$ of $S$ onto $\Gamma^*(S^*)$, defined by, for each $x \in S$,

$$\Gamma(x) = \Gamma^*(\bar{x}),$$

is a non-null representation of $S$.

Proof. Since $S = M(\Gamma) = M$, $\Gamma(S) = \Gamma(M)$; hence, by Lemma 1, $\Gamma(S)$ obeys $C_3$. Thus $\Gamma(S)$ is a homomorphic image of $S^*$, and it follows, from the induced homomorphism theorem, that the mapping $\Gamma^*$ of $S^*$ onto $\Gamma(S)$, defined by (1), is a representation of $S^*$, of degree $n$ over $\Phi$.

The converse is immediate, since the composition of homomorphisms is a homomorphism.

COROLLARY 1. Let $S = S^0$ be a 0-simple $\mathcal{H}$-semigroup. Then the non-null representations of $S$ are those of its maximum completely 0-simple homomorphic image $S^*$.

COROLLARY 2. Let $S = S^0$ be a 0-simple semigroup with identity. Then $S$ has a non-null representation if and only if it has no divisors of zero. In this case, the non-null representations of $S$ are those of its maximum group-with-zero homomorphic image $S^*$.

Proof. Suppose that $\Gamma$ is a non-null representation of $S$. Then $S$ obeys $C_1$, and is an $\mathcal{H}$-semigroup. Thus $S^*$ is a completely 0-simple semigroup with identity; that is, $S^*$ is a group-with-zero. Hence $S$ has no divisors of zero. The remainder of the result is now immediate from Corollary 1.

Clifford [1] has given a construction for all non-null representations of a completely 0-simple semigroup. Taken with Corollary 1 and Theorem 3, this provides a construction for all representations of a 0-simple $\mathcal{H}$-semigroup. It should be noted however that not every 0-simple semigroup is an $\mathcal{H}$-semigroup. For example, let $S$ be the multiplicative semigroup of all $2 \times 2$ matrices over the reals, of the form

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix},$$

where $a$ and $b$ are positive real numbers; then $S$ is a simple cancellative semigroup [2, Chapter 5, §5, Example 7(b)]. Thus $S^0$ is a 0-simple semigroup that obeys $C_1$ and $C_3$. But $S^0$ has no nonzero idempotents and so is not completely 0-simple.
Theorem 3 shows that, for any 0-simple semigroup \( S = S^0 \), there is a one-to-one correspondence between the representations of \( S \) and those of \( S^* \). It is an easy matter to prove that this correspondence preserves equivalence, decomposition and reduction of representations. For the definitions of equivalence and decomposition of representations, see, for example, [2, Chapter 5].

4. Irreducible representations of an arbitrary semigroup. The main result of this section gives a method of construction for all 0-restricted irreducible representations of an arbitrary semigroup \( S = S^0 \), from those of certain associated semigroups. By Lemma 5, if such a representation exists, then \( S \) satisfies \( C_2 \) and, by Lemma 6 below, so also does any nonzero ideal of \( S \). Further, if \( S \) has the property that each nonzero ideal of \( S \) that satisfies \( C_1 \) is a \( \mathcal{M} \)-semigroup, then each of these associated semigroups is completely 0-simple. In this case, we have an explicit construction for the irreducible 0-restricted representations of \( S \).

**Theorem 4.** Let \( S = S^0 \) be a semigroup which obeys \( C_2 \). Let \( \Gamma \) be a 0-restricted irreducible representation of \( S \), of degree \( n \) over a field \( \Phi \). Then \( \Gamma \) induces a 0-restricted irreducible representation \( \Gamma^* \) of \( M^* \), where \( M = M(\Gamma) \), and there are finite sets of elements \( e_1, \ldots, e_r \in M \), \( \alpha_1, \ldots, \alpha_r \in \Phi \) such that, for each \( x \in S \),

\[
\Gamma(x) = \sum_{i=1}^{r} \alpha_i \Gamma^*(e_i x),
\]

where \( x \rightarrow x \) is the natural homomorphism \( M \rightarrow M^* \).

Conversely, let \( M \) be a nonzero ideal of \( S \) that obeys \( C_1 \), and let \( \Gamma^* \) be a 0-restricted irreducible representation of \( M^* \), of degree \( n \) over \( \Phi \). Then, for any finite sets \( e_1, \ldots, e_r \in M \), \( \alpha_1, \ldots, \alpha_r \in \Phi \) such that

\[
\sum_{i=1}^{r} \alpha_i \Gamma^*(e_i) = I_n,
\]

the mapping \( \Gamma \) of \( S \) into \( (\Phi)_n \), defined by (2), is a 0-restricted irreducible representation of \( S \), of degree \( n \) over \( \Phi \). The representation is independent of the particular choice of elements \( e_i, \alpha_i \) satisfying (3).

Let \( \Gamma_1 \) and \( \Gamma_2 \) be 0-restricted irreducible representations of \( S = S^0 \), defined, as above, from ideals \( M_1 \) and \( M_2 \) of \( S \). Then \( \Gamma_1 \) and \( \Gamma_2 \) are equivalent if and only if they are equivalent on \( M_1 \cap M_2 \).

**Proof.** Let \( M \) and \( \Gamma \) satisfy the hypothesis of the first part of the theorem. By Lemma 1, \( \Gamma(M) \) obeys \( C_3 \). Hence the mapping \( \Gamma^* \) defined by the rule

\[
\Gamma^*(\bar{x}) = \Gamma(x),
\]

for each \( \bar{x} \in M^* \), where \( x \rightarrow \bar{x} \) is the natural homomorphism of \( M \) onto \( M^* \), is a 0-restricted
representation of $M^*$ over $\Phi$, of the same degree as $\Gamma$. Since $M$ is an ideal of $S$, and $\Gamma(S)$ is an irreducible matrix set, it follows from Lemma 2 that

$$[\Gamma^*(M^*)] = [\Gamma(M)] = [\Gamma(S)].$$

Hence $\Gamma^*$ is an irreducible representation of $M^*$.

From Lemma 4, since $\Gamma^*$ is irreducible, there exist $\bar{e}_1, \ldots, \bar{e}_r \in M^*$ and $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_{i=1}^{r} \alpha_i \Gamma^*(\bar{e}_i) = I_n.$$

Choose $e_i \in M$ such that $e_i \to \bar{e}_i$ for each $1 \leq i \leq r$. Then, for each $x \in S$,

$$\Gamma(x) = I_n \Gamma(x) = \left( \sum_{i=1}^{r} \alpha_i \Gamma^*(\bar{e}_i) \right) \Gamma(x).$$

But $\Gamma^*(\bar{e}_i) = \Gamma(e_i)$ for each $1 \leq i \leq r$; hence, since $M$ is an ideal of $S$,

$$\Gamma(x) = \sum_{i=1}^{r} \alpha_i \Gamma(e_i) \Gamma(x) = \sum_{i=1}^{r} \alpha_i \Gamma(e_i) = \sum_{i=1}^{r} \alpha_i \Gamma^*(\bar{e}_i).$$

This completes the proof of the first part.

The proof of the converse follows exactly as in the case of principal irreducible representations; cf. [7, Theorem 1].

Finally, it is clear that the criterion for equivalence is necessary. Suppose that $\Gamma_1$ and $\Gamma_2$ are equivalent on $M_1 \cap M_2$. By $C_2$, $M_1 \cap M_2$ is a nonzero ideal of $S$ and hence

$$[\Gamma_1(M_1 \cap M_2)]$$

is a nonzero ideal of $\Gamma_1(S)$. But, by Lemma 2, this means that $[\Gamma_1(M_1 \cap M_2)] = [\Gamma_1(S)]$. Thus, by Lemma 4, we can choose $e_1, \ldots, e_r \in M_1 \cap M_2$ and $\alpha_1, \ldots, \alpha_r \in \Phi$ such that

$$\sum_{i=1}^{r} \alpha_i \Gamma_1(e_i) = I_n.$$

Since $\Gamma_1$ and $\Gamma_2$ are equivalent on $M_1 \cap M_2$, there exists a nonsingular matrix $A$ such that, for each $m \in M_1 \cap M_2$,

$$\Gamma_2(m) = A \Gamma_1(m) A^{-1}.$$

Hence $\sum_{i=1}^{r} \alpha_i \Gamma_2(e_i) = I_n$; thus, for each $x \in S$,

$$\Gamma_2(x) = \sum_{i=1}^{r} \alpha_i \Gamma_2(e_i) \Gamma(x) = \sum_{i=1}^{r} \alpha_i \Gamma_1(e_i) \Gamma(x) A^{-1} = A \Gamma_1(x) A^{-1}.$$

That is, $\Gamma_1$ and $\Gamma_2$ are equivalent.
Note 1. It can readily be shown that, if, in the above theorem, \( S \) is a regular semigroup, a periodic semigroup, a semigroup satisfying \( M_L \) and \( M_R \), or a 0-simple semigroup containing a nonzero idempotent, then \( M \) is an \( \mathcal{M} \)-semigroup (note Lemma 6); that is, \( M^* \) is completely 0-simple. In this case the 0-restricted irreducible representations of \( S \) can be determined explicitly by means of Clifford's theory of representations of a completely 0-simple semigroup [1].

Note 2. Let \( S = S^0 \) be a semigroup satisfying \( C_2 \) that has a unique minimal nonzero ideal. Then, by the last part of Theorem 4, the irreducible 0-restricted representations of \( S \) are determined, to within equivalence, by those of the unique minimal nonzero of \( S \). That is, in the terminology of [7], they are the principal irreducible 0-restricted representations of \( S \).

We shall end the paper by giving a sufficient condition for the existence of a 0-restricted representation of a semigroup \( S = S^0 \), that obeys \( C_2 \). Before giving this criterion, we shall prove some results about conditions \( C_1 \) and \( C_2 \).

**Lemma 6.** Let \( S = S^0 \) be a semigroup that obeys \( C_2 \). Let \( L \) be a nonzero ideal of \( S \). Then \( L \) obeys \( C_2 \).

**Proof.** Let \( m, n \) be nonzero members of \( L \). Then, by \( C_2 \), there exists \( x \in S \) such that \( mxm \neq 0 \). Again, by \( C_2 \), there exists \( y \in S \) such that \( mxm \cdot y \cdot n \neq 0 \). Let \( u = xmy \); since \( L \) is an ideal of \( S \), \( u \in L \). Then \( mun \neq 0 \) and so \( L \) obeys \( C_2 \).

**Lemma 7.** Let \( S = S^0 \) be a semigroup that obeys \( C_2 \). Then the set of all ideals of \( S \) that obey \( C_1 \) has a unique maximal member \( L \).

**Proof.** Let \( L = \bigcup \{ L_\alpha : \alpha \in \Lambda \} \) be the union of all ideals of \( S \) that obey \( C_1 \). If \( L \neq \{0\} \), let \( a \in L \setminus \{0\} \), and suppose that \( sa \neq 0 \) and \( at \neq 0 \) for \( s, t \in S \); then \( a \in L_\alpha \), for some \( \alpha \in \Lambda \). Since, by Lemma 6, \( L_\alpha \) obeys \( C_2 \), there exist \( m, n \in L_\alpha \) such that \( msa \neq 0 \), \( an \neq 0 \). Since \( L_\alpha \) is an ideal that obeys \( C_1 \), it follows that \( msan \neq 0 \); hence \( sat \neq 0 \). Thus \( L \) obeys \( C_1 \).

**Theorem 5.** Let \( S = S^0 \) be a semigroup that obeys \( C_1 \) and \( C_2 \), and let \( T \) be a nonzero ideal of \( S \). If \( \sigma \) and \( \tau \) denote, respectively, the maximum 0-restricted congruences on \( S \) and \( T \), then \( S/\sigma \cong T/\tau \).

**Proof.** Since \( S \) obeys \( C_1 \) and \( C_2 \), it follows, from Lemma 6, that the same is true of \( T \). From the definitions of \( \sigma \) and \( \tau \), it is clear that, for \( a, b \in T \), if \( (a, b) \in \sigma \) then \( (a, b) \in \tau \). Conversely, let \( (a, b) \in \tau \) and let \( sat = 0 \), where \( s, t \in S \setminus \{0\} \). Since \( T \) is an ideal of \( S \) and \( S \) obeys \( C_2 \), there exist \( m, n \in T \) such that neither of \( ms, tn \) is zero. Then \( msat = 0 \) and so, since \( (a, b) \in \tau \), \( msbn = 0 \). Since \( S \) obeys \( C_1 \), \( msbn = 0 \) implies \( msbt = 0 \) or \( sbtn = 0 \). But neither of \( ms, tn \) is zero so that each of these equations implies \( sbt = 0 \). Similarly, \( sbt = 0 \) implies \( sat = 0 \); hence \( (a, b) \in \sigma \). Thus \( \tau = \sigma \cap (T \times T) \).

Let \( \theta \) denote the natural homomorphism of \( S \) onto \( S/\sigma \). Then, since \( \tau = \sigma \cap (T \times T) \),

\[
T/\tau \cong T\theta.
\]

But, by the proof of the corollary to Theorem 1, \( S\theta = S/\sigma \) is completely 0-simple. Thus, since \( T \) is a nonzero ideal of \( S \), \( T\theta = S\theta \). Hence we have the result.
Let $S = S^0$ be a semigroup that obeys $C_2$, and let $M$ be an ideal of $S$ that obeys $C_1$. Then a sufficient condition for $S$ to have a 0-restricted representation, over a field $\Phi$, defined from $M$ as in Theorem 4, is that $M/\rho$ should have a 0-restricted irreducible representation over $\Phi$. In fact, by Theorem 5, it is sufficient that $L/\rho$ should have a 0-restricted irreducible representation over $\Phi$. By the proof of the corollary to Theorem 1, $L/\rho$ is completely 0-simple; hence we can use Clifford’s results [1] to give necessary and sufficient conditions for $L/\rho$ to have an irreducible 0-restricted representation.

Clifford proves the following. Let $\mathfrak{M}(G; I, \Lambda; P)$ be a regular Rees matrix semigroup over a group with zero $G^0$. Let $\Gamma$ be a representation of $G$ of degree $n$ over a field $\Phi$. Then $\Gamma$ can be extended to a representation of $\mathfrak{M}(G; I, \Lambda; P)$ if and only if the $\Lambda \times I$ block matrix $\Omega$ over $\Phi$, whose $(\lambda, i)$th block is the $n \times n$ matrix $\Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1}p_{1 i})$, has finite rank over $\Phi$. Further, every representation of $\mathfrak{M}(G; I, \Lambda; P)$ is the extension of some representation of $G$; in particular, the irreducible representations are the extensions of irreducible representations of $G$.

Let $a \in L \setminus \{0\}$; if $a^2 \neq 0$, then (cf. the proof of the corollary to Theorem 1) $(a, a^2) \in \rho$ and $(a, a^2) \in \rho$, so that $(a, a^2) \in \rho$. Thus $L/\rho$ is a completely 0-simple semigroup in which each element is either idempotent or nilpotent. Hence [2] $L/\rho$ is isomorphic to a regular Rees matrix semigroup over a group-with-zero $G^0$; further, since each element of $L/\rho$ is either idempotent or nilpotent, it can be verified by direct calculation that $G$ has only one element.

Suppose that $L/\rho \cong \mathfrak{M}(\{e\}; I, \Lambda; P)$, where $\{e\}$ is a one element group. Let $\Phi$ be a field and let $\Omega$ be the $\Lambda \times I$ matrix over $\Phi$ where $\Omega_{\lambda i} = 1, 0, -1$ according as $p_{\lambda i}$ is greater than, is equal to, is less than $p_{\lambda 1}p_{1 i}$; $\{e\}$ is partially ordered by $e > 0$. If $\Omega$ has finite rank over $\Phi$, then we say that $S$ has finite rank over $\Phi$; if $L = \{0\}$, then rank $S$ is zero.

Since $\{e\}$ has only one member, every irreducible representation of $\{e\}$ over $\Phi$ is of degree one. Hence, by Clifford’s results, mentioned above, $L/\rho$ has an irreducible representation over $\Phi$ if and only if $\Omega$ has finite rank.

The above results are gathered together in the following proposition.

**PROPOSITION 7.** Let $S = S^0$ be a semigroup that obeys $C_2$, and let $\Phi$ be a field. If $S$ has a 0-restricted representation over $\Phi$, then $S$ has nonzero rank over $\Phi$. Conversely, if $S$ has finite nonzero rank over $\Phi$, then $S$ has a 0-restricted representation over $\Phi$.

Finally, we point out that, if $S = S^0$ is an inverse semigroup or a weakly regular semigroup in which the idempotents commute, it can be shown that the criterion of Proposition 7 is not only sufficient but is also necessary; cf. [10] for the inverse case. In this case it takes the form: $S$ has a 0-restricted representation if and only if $L/\rho$ is finite with at least two members.

**REFERENCES**


