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Note on duality of Kelleyspace products

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It has been shown by W.F. LaMartin that the Pontryagin dual of a k-space product of Hausdorff abelian k-groups is the coproduct of their duals. Here we offer a different proof, based partly on LaMartin's, that the dual of a product is the coproduct of the duals in a more general setting.

Introduction

Let $K = (K, 1, \times, \{-, -\}, ...)$ denote the cartesian closed category of compactly generated Hausdorff spaces and let $V = (V, I, \otimes, [-, -], ...)$ be the category of algebras for a finitary additive commutative algebraic theory on K (see Day [1], Example 4.3). Then V is symmetric monoidal closed, complete, cocomplete, and additive, while $V = V(I, -) : V \rightarrow K$ reflects isomorphisms and creates filtered colimits, which summarises the basic properties we require of V.

Duality of products

LEMMA 1. Let $\Omega \in V$ be an algebra with no small subalgebras and let $f : \prod_{\lambda \in \Lambda} M_{\lambda} \neq \Omega$ be a homomorphism. Then there exists a finite $F \subset \Lambda$ with $f(M_{\lambda}) = 0$ if $\lambda \notin F$.

Proof. See LaMartin [2], Lemma 8. //

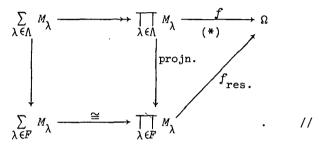
PROPOSITION 2. With the same hypotheses as in Lemma 1,

 $f:\prod_{\lambda\in\Lambda}M_{\lambda}\to\Omega \text{ factors through some projection onto a finite subproduct.}$

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Proof. The canonical map $\sum M_{\lambda} \rightarrow \prod M_{\lambda}$ is dense because, if $(m_{\lambda}) \in \prod M_{\lambda}$, then $\prod \{0, m_{\lambda}\}$ is compact and so has the product topology. Thus it follows that (m_{λ}) lies in the closure of $\sum M_{\lambda}$ in the ordinary product $\prod M_{\lambda}$ since $\sum M_{\lambda}$ is dense in the ordinary product $\prod M_{\lambda}$ by additivity of V. From this we see that (*) commutes and the result follows:



We now write $\prod_{\lambda \in \Lambda} M_{\lambda} = \lim_{\phi \in \Phi} N_{\phi}$ where the limit is cofiltered over the finite subsets of Λ . On considering the canonical map:

$$\begin{array}{c} \operatorname{colim} \left[\mathbb{N}_{\phi}, \Omega \right] \rightarrow \left[\operatorname{lim} \mathbb{N}_{\phi}, \Omega \right] \\ \phi \in \Phi \end{array}$$

we see that, provided Ω has no small subalgebras, it is a bijection by Proposition 2 and the fact that the colimit is filtered.

THEOREM 3. If $\Omega \in V$ has no small subalgebras then $\operatorname{colim}[\mathbb{N}_{\phi}, \Omega] \cong [\lim \mathbb{N}_{\phi}, \Omega]$.

Proof. It remains to prove that the spaces $\operatorname{colim}[\mathbb{N}_{\phi}, \Omega]$ and $[\lim \mathbb{N}_{\phi}, \Omega]$ admit the same morphisms from compact Hausdorff spaces. But, given any compact Hausdorff space C, $\{C, \Omega\}$ has no small subalgebras since the sub-basic open neighbourhood $W(C, V) = \{g \in K(C, \Omega); g(C) \subseteq V\}$ is a neighbourhood of $0 \in [C, \Omega]$ which contains no non-trivial subalgebras whenever V is an open neighbourhood of $0 \in \Omega$ containing no non-trivial subalgebras. Thus the result follows from Proposition 2. //

By additivity of V we see that colim $[N_{\phi}, \Omega] \cong \sum_{\lambda \in \Lambda} [M_{\lambda}, \Omega]$, as required.

References

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