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Some Results on the Domination Number of a Zero-divisor Graph

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Abstract. In this paper, we investigate the domination, total domination, and semi-total domination numbers of a zero-divisor graph of a commutative Noetherian ring. Also, some relations between the domination numbers of $\Gamma(R/I)$ and $\Gamma_I(R)$, and the domination numbers of $\Gamma(R)$ and $\Gamma(R[x,\alpha,\delta])$, where $R[x,\alpha,\delta]$ is the Ore extension of R, are studied.

1 Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph with a ring is a research subject in this field that has attracted considerable attention. In fact, research on this subject aims at exposing the relationship between ring theory and graph theory and at advancing applications of one to the other. The idea of associating graphs with rings goes back to a paper of Beck [4] in 1988, where he introduced the notion of a zero-divisor graph of a commutative ring R with identity. Let R be a commutative ring with identity. The *zero-divisor graph* $\Gamma(R)$ of a ring R is an undirected graph whose vertices are all elements of $Z(R)\setminus\{0\}$ and such that there is an edge between vertices a and b if and only if $a \neq b$ and $ab = 0$. The concept of zero-divisor graphs has been studied extensively by many authors. For more information about this graph the reader is referred to [2]. Also, for a survey and recent results concerning zero-divisor graphs, we refer the reader to [6]. For notation and definitions related to commutative rings the reader is referred to [11].

In graph theory, a *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is joined to at least one member of D by some edge. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for G. We call a dominating set of cardinality $\gamma(G)$ a γ-set. A *total dominating set* of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. We call a dominating set of cardinality $\gamma_t(G)$ a γ_t-set. In [5] the authors defined the *semi-total dominating set* in $\Gamma(R)$ as a subset $S \subseteq Z(R)$ such that S is a dominating set for $\Gamma(R)$ and for any $x \in S$ there is a vertex $y \in S$ (not necessarily distinct) such that $xy = 0$. The *semi-total domination number* $\gamma_{st}(\Gamma(R))$

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of $\Gamma(R)$ is the minimum cardinality of a semi-total dominating set in $\Gamma(R)$. Note that for all rings R, $\gamma(\Gamma(R)) \leq \gamma_{st}(\Gamma(R)) \leq 2\gamma(\Gamma(R))$. We call a semi-total dominating set of cardinality $\gamma_{st}(\Gamma(R))$ a γ_{st}-set.

In [5, 8] the authors studied the domination number of a zero-divisor graph. In Section 2, we follow their works and characterize all rings with finite domination numbers. Some results on the domination number of a zero-divisor graph of a Noethrian ring are given. Also, we study domination numbers of $\Gamma(R)$ and $\Gamma(R[x;\alpha,\delta])$, where $R[x,\alpha,\delta]$ is the Ore extension of R. We show that for an (α,δ)-compatible Noetherian ring R, either

$$\gamma\big(\Gamma\big(R[x;\alpha,\delta]\big)\big) = \gamma\big(\Gamma(R)\big) \quad \text{or} \quad \gamma\big(\Gamma\big(R[x;\alpha,\delta]\big)\big) = \gamma_t\big(\Gamma(R)\big).$$

In Section 3, some relations between domination numbers of $\Gamma(R/I)$ and $\Gamma_I(R)$ are given. Semi-total and total domination numbers are shown to be equivalent. Also, we find a necessary and sufficient condition under which $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$.

2 Domination Number of $\Gamma(R)$

In this section, we study the domination number of $\Gamma(R)$, when R is Noetherian.

We begin with the following proposition.

Proposition 2.1 *For every ring R, $\gamma(\Gamma(R))$ is finite if and only if $Z(R) = \bigcup_{i=1}^{n} \operatorname{Ann}(x_i)$, for some $n \in \mathbb{N}$ and $x_i \in R$.*

Proof Let $\gamma(\Gamma(R)) < \infty$ and $D = \{y_1, \dots, y_m\}$ be a dominating set of $\Gamma(R)$. Then $Z(R) = \bigcup_{i=1}^{m} \operatorname{Ann}(y_i) \cup D$. But for any $y_i \in D$ there exists $a_i \in Z(R)$ such that $y_i \in \operatorname{Ann}(a_i)$, and so $Z(R) = \bigcup_{i=1}^{m} (\operatorname{Ann}(y_i) \cup \operatorname{Ann}(a_i))$.

Conversely, suppose that $Z(R) = \bigcup_{i=1}^{n} \operatorname{Ann}(x_i)$. Then $\{x_1, \dots, x_n\}$ is a dominating set. Thus $\gamma(\Gamma(R)) < \infty$. ■

We have the following immediate corollary.

Corollary 2.2 *If R is a Noetherian ring, then $\gamma(\Gamma(R)) < \infty$.*

Proof The result follows from [11, Corollary 9.36], Proposition 2.1, and the fact that the number of associated prime ideals of a Noetherian ring is finite. ■

Before stating the next result, the following remark is needed.

Remark 2.3 Let $\gamma(\Gamma(R)) = 1$ and $S = \{x\}$ be a dominating set for $\Gamma(R)$. By [2, Theorem 2.5], either $Z(R) = \operatorname{Ann}(x)$ or $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain. If $Z(R) = \operatorname{Ann}(x)$, for some non-zero element x, then $\gamma(\Gamma(R)) = \gamma_{st}(\Gamma(R)) = 1$ and $\gamma_t(\Gamma(R)) = 2$. Also, if $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, then $\gamma_t(\Gamma(R)) = \gamma_{st}(\Gamma(R)) = 2$ and $\gamma(\Gamma(R)) = 1$.

Theorem 2.4 *Let R be a Noetherian ring and $\gamma(\Gamma(R)) \neq 1$. Then*

$$\gamma_t(\Gamma(R)) = \gamma_{st}(\Gamma(R)) = \big|\operatorname{Max}\{P \neq 0 \mid P \in \operatorname{Ass}(R)\}\big|.$$

Proof Let $A = \{P_1, \dots, P_k\}$ be the set of maximal associated primes of the Noetherian ring R. Then by [11, Corollary 9.36], $Z(R) = \bigcup_{i=1}^{k} P_i$. Let $P_i = \mathrm{Ann}(x_i)$, where x_i is a non-zero element of R, for $i = 1, \dots, k$. Set $S = \{x_1, \dots, x_k\}$. By assumption, $k > 1$. We claim that S is a total and semi-total dominating set of $\Gamma(R)$. Assume that $y \notin S$ is a vertex of $\Gamma(R)$. Then $y \in P_i$, for some i and so $x_i y = 0$. Also, it follows from [4, Lemma 3.6] that $x_i x_j = 0$, for all i, j such that $1 \leq i$, $j \leq k$, and $i \neq j$. Thus, S is a total and semi-total dominating set of $\Gamma(R)$, and the claim is proved. If $\gamma_{st}(\Gamma(R)) = n$, then there exist $a_1, \dots, a_n \in Z(R)$ such that $Z(R) = \bigcup_{i=1}^{n} \mathrm{Ann}(a_i)$. By the Prime Avoidance Theorem, for each $i \in \{1, \dots, n\}$, there exists $j \in \{1, \dots, k\}$ such that $\mathrm{Ann}(a_i) \subseteq P_j$. Hence, $Z(R) = \bigcup_{j=1}^{n} P_j$, and so if $P_l \in A$, then we have $P_l \subseteq \bigcup_{j=1}^{n} P_j$. Again by the Prime Avoidance Theorem and the maximality of elements of A, we conclude that $n = k$, as desired. ∎

The next theorem completely describes the relation between $\gamma(\Gamma(R))$ and $\gamma_t(\Gamma(R))$ in case that R is a Noetherian ring.

Theorem 2.5 *Let R be a Noetherian ring. Then either $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R))$ or $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) - 1$.*

Proof If $\gamma(\Gamma(R)) = 1$, then by the previous remark we have $\gamma_t(\Gamma(R)) = 2$. So, suppose that $\gamma(\Gamma(R)) > 1$. Let $\gamma(\Gamma(R)) \neq \gamma_t(\Gamma(R))$ and $D = \{x_1, \dots, x_k\}$ be the γ-set of $\Gamma(R)$. Then $Z(R) = \bigcup_{i=1}^{k} \mathrm{Ann}(x_i) \cup D$. Also, by Theorem 2.4, $Z(R) = \bigcup_{i=1}^{n} P_i$, where $A = \{P_1, \dots, P_n\}$ is the set of maximal associated primes of R and $\gamma_{st}(\Gamma(R)) = n$. Let t be the maximum number of elements of D such that their product is not zero. Since $k < n$, we have $t \geq 2$. Hence, there exist $x_{i1}, \dots, x_{it} \in D$ such that $\prod_{j=1}^{t} x_{ij} \neq 0$. By the Prime Avoidance Theorem, there exists $P \in A$ such that $\mathrm{Ann}(\prod_{j=1}^{t} x_{ij}) \subseteq P$, and so $\mathrm{Ann}(x_{ij}) \subseteq P$, for each $j \in \{1, \dots, t\}$. On the other hand, any product of $t + 1$ elements of D is zero. Therefore, $k - t$ elements of D are contained in the ideal P. So $\bigcup_{i=1}^{t} \mathrm{Ann}(x_{ij}) \cup D$ is a subset of union of at most $t + 1$ elements of A. Again the Prime Avoidance Theorem implies that A has at most $k + 1$ elements. But $|A| = \gamma_t(\Gamma(R))$ and $\gamma_t(\Gamma(R)) > \gamma(\Gamma(R))$. This means that $\gamma_t(\Gamma(R)) = \gamma(\Gamma(R)) + 1$. Now the proof is complete. ∎

The following corollary is a generalization of [8, Theorem 11], in case where R is a Noetherian ring.

Corollary 2.6 *Let $R \cong R_1 \times \cdots \times R_k$ be a Noetherian ring and $n = |\,\mathrm{Max}\{P \neq 0 \mid P \in \mathrm{Ass}(R)\}|$. Then the following hold:*

(i) *If $k = 1$, then $n - 1 \leq \gamma(\Gamma(R)) \leq n$.*

(ii) *If $k \geq 2$ and $\mathbb{Z}_2 \notin \{R_1, \dots, R_k\}$, then $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = n$.*

(iii) *If $k \geq 2$, $\mathbb{Z}_2 \in \{R_1, \dots, R_k\}$ and $R \cong \mathbb{Z}_2 \times \acute{R}$, then:*

  (a) *$\gamma(\Gamma(R)) = 1$, if $\acute{R}$ is an integral domain;*

  (b) *$\gamma(\Gamma(R)) = |\,\mathrm{Max}\{P \neq 0 \mid P \in \mathrm{Ass}(\acute{R})\}| + 1$, if $\acute{R}$ is not an integral domain.*

Proof (i) The result follows from Theorems 2.4 and 2.5.

(ii) The result is proved by Theorem 2.4 and [5, Corollary 2.7].

(iii) To prove (a), see [5, Proposition 2.2]. Part (b) follows from Theorem 2.4 and [5, Proposition 2.5]. ∎

One extension of an arbitrary ring R is the Ore extension. Assume that $\alpha\colon R \to R$ is a ring endomorphism and $\delta\colon R \to R$ is an α-derivation of R; that is, δ is an additive map such that $\delta(ab) = \alpha(a)\delta(b)+\delta(a)b$, for all $a, b \in R$. The Ore extension $R[x;\alpha,\delta]$ of R is the ring obtained by giving the polynomial ring (with indeterminate x) over R with the multiplication

$$xr = \alpha(r)x + \delta(r),$$

for all $r \in R$. Recall that R is reversible if $ab = 0$ implies that $ba = 0$ for $a, b \in R$.

Definition 2.7 (*cf.* [3, 9]) Let α be a ring endomorphism and let δ be an α-derivation of a ring R.

(i) R is said to be α-compatible whenever, for every $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$.

(ii) The ring R is δ-compatible if $ab = 0$ implies that $a\delta(b) = 0$.

(iii) We say that R is (α, δ)-compatible if R is both α-compatible and δ-compatible.

(iv) The ring R is left (α, δ)-McCoy if for non-zero polynomials $f(x), g(x) \in R[x;\alpha,\delta]$ with $f(x)g(x) = 0$, there exists a non-zero element $r \in R$ such that $rg(x) = 0$. Similarly, R is right (α, δ)-McCoy, if for non-zero polynomials $f(x)$, $g(x) \in R[x;\alpha,\delta]$ with $f(x)g(x) = 0$, there exists a non-zero element $s \in R$ such that $f(x)s = 0$. Also if a ring R satisfied in both left and right (α, δ)-McCoy, we say that it is (α, δ)-McCoy. In the special case, when δ is a zero map, we say that R is α-McCoy, and if, in addition, α is an identity map, then we say that R is McCoy.

Theorem 2.8 *Let R be an (α, δ)-compatible commutative Noetherian ring. Then either $\gamma(\Gamma(R[x;\alpha,\delta])) = \gamma(\Gamma(R))$ or $\gamma(\Gamma(R[x;\alpha,\delta])) = \gamma_t(\Gamma(R))$.*

Proof By [1, Theorem 2.4], R is an (α, δ)-McCoy ring. Hence, for any non-zero polynomial $f(x) \in Z(R[x;\alpha,\delta])$ there exists $r \in R$ such that $rf(x) = 0 = f(x)r$. Let $f(x) = \sum_{i=1}^{n} a_i x^i$. Then $ra_i = 0$ and so $r \in Z(R)$ and $Z(R) \subseteq Z(R[x;\alpha,\gamma])$. Thus $\gamma(\Gamma(R)) \leq \gamma(\Gamma(R[x;\alpha,\delta]))$.

Since R is a Noetherian ring, we conclude that $Z(R) = \bigcup_{i=1}^{t} \mathrm{Ann}(x_i)$, for some $t \in \mathbb{N}$, such that $\mathrm{Ann}(x_i)$ is a maximal associated prime of R, for every $i \in \{1, \dots, t\}$. Therefore, there exists $j \in \{1, \dots, t\}$ such that $\mathrm{Ann}(r) \subseteq \mathrm{Ann}(x_j)$, and so $rf(x) = 0$ implies that $x_j f(x) = 0$. Hence, S is a dominating set of $\Gamma(R[x;\alpha,\delta])$. By Theorem 2.4, $S = \{x_1, \dots, x_t\}$ is a γ_t set of $\Gamma(R)$, and so by Theorem 2.5, either

$$\gamma\big(\Gamma\big(R[x;\alpha,\delta]\big)\big) = \gamma\big(\Gamma(R)\big) \quad \text{or} \quad \gamma\big(\Gamma\big(R[x;\alpha,\delta]\big)\big) = \gamma_t\big(\Gamma(R)\big). \qquad ∎$$

3 Relation Between $\gamma(\Gamma(R/I))$ and $\gamma(\Gamma_I(R))$

Let R be a commutative ring with non-zero identity and let I be an ideal of R. The ideal-based zero-divisor graph of R is an undirected graph, denoted by $\Gamma_I(R)$, with vertices $\{x \in R \setminus I : xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. Therefore, if $I = 0$, then $\gamma(\Gamma_I(R)) = \gamma(\Gamma(R))$. For more details, see [7, 10].

In this section, we study domination numbers of $\gamma(\Gamma(R/I))$ and $\gamma(\Gamma_I(R))$. Also, a necessary and sufficient condition is given under which $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$.

Proposition 3.1 *Let R be a ring and let I be an ideal of R. Then the following hold:*

(i) $\gamma_{st}(\Gamma(R/I)) = \gamma_{st}(\Gamma_I(R))$.
(ii) $\gamma_t(\Gamma(R/I)) = \gamma_t(\Gamma_I(R))$.

Proof To prove (i), let $D = \{x_1, \ldots, x_k\}$ be a semi-total dominating set of $\Gamma_I(R)$. Clearly, $x_i \in R\backslash I$ for every $1 \leq i \leq k$. For every $y \in V(\Gamma_I(R))$, there exists $x_j \in D$ such that $yx_j \in I$. Thus $(x_j + I)(y + I) \in I$ and so $D + I = \{x + I \mid x \in D\}$ is a semi-total dominating set of $\Gamma(R/I)$. Therefore, $\gamma_{st}(\Gamma(R/I)) \leq \gamma_{st}(\Gamma_I(R))$.

To prove that $\gamma_{st}(\Gamma(R/I)) \geq \gamma_{st}(\Gamma_I(R))$, if $S = \{x_1 + I, \ldots, x_t + I\}$ is a semi-total dominating set of $\Gamma(R/I)$, then we want to show that $S' = \{x_1, \ldots, x_t\}$ is a semi-total dominating set of $\Gamma_I(R)$. Obviously, for every $y \in V(\Gamma(R/I))$, there exists $i \in \{1, \ldots, t\}$ such that $y + I$ is adjacent to $x_i + I$ or $x_i + I = y + I$. Thus there exists $x_j + I \in S$ such that $(x_j + I)(y + I) = 0$, and so y is adjacent to x_j in $\Gamma_I(R)$. Hence, $\gamma_{st}(\Gamma(R/I)) \geq \gamma_{st}(\Gamma_I(R))$, and so the proof is complete.

The proof of (ii) is similar to the proof of (i). ■

Theorem 3.2 *Let R be a ring and I be an ideal of R. Then $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$ if and only if $\gamma(\Gamma(R/I)) = \gamma_{st}(\Gamma(R/I))$.*

Proof Suppose that $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$. We have only to prove $\gamma(\Gamma(R/I)) > \gamma_{st}(\Gamma(R/I))$. Let $D = \{x_1, \ldots, x_k\}$ be a γ-set of $\Gamma_I(R)$. Then $D + I = \{x + I \mid x \in D\}$ is a dominating set of $\Gamma(R/I)$, and so it is a γ-set of $\Gamma(R/I)$. Also, it is not hard to see that, for any $a \in I\backslash\{0\}$ and $x \in D$, we have $x + a \notin (D \cup I)$. Since $x \in V(\Gamma_I(R))$ and $a \in I\backslash\{0\}$, we deduce that $x + a$ is also a vertex of $\Gamma_I(R)$. Hence, there exists $x_i \in D$ such that $x + a$ and x_i are adjacent in $\gamma(\Gamma_I(R))$. Therefore, $xx_i \in I$. Thus $D + I$ is also a γ_{st}-set, and we are done.

By Proposition 3.1 and the inequality $\gamma(\Gamma(R/I)) \leq \gamma(\Gamma_I(R)) \leq \gamma_{st}(\Gamma_I(R))$, the converse is clear. ■

The following example shows that, in the previous theorem, we cannot replace $\gamma_{st}(\Gamma(R/I))$ with $\gamma_t(\Gamma(R/I))$.

Example 3.3 Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $I = \mathbb{Z}_2 \times 0$. Then it is not hard to see that $\gamma(\Gamma(R/I)) = \gamma_{st}(\Gamma(R/I)) = \gamma(\Gamma_I(R)) = 1$ and $\gamma_t(\Gamma(R/I)) = 2$.

In light of the proof of Theorem 3.2, we have the following corollary.

Corollary 3.4 *Let R be a ring and I be an ideal of R such that $\sqrt{I} = I$. Then the following hold:*

(i) $\gamma_{st}(\Gamma(R/I)) = \gamma_t(\Gamma(R/I))$.
(ii) $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$ *if and only if* $\gamma(\Gamma(R/I)) = \gamma_t(\Gamma(R/I))$.

Proof (i) Let $S = \{x_i + I \mid x_i \in R \setminus I\}$ be a γ_{st} set of $\Gamma(R/I)$. If $(x_i + I)$ is an arbitrary element of S, then either $(x_i + I)$ is adjacent to $(x_j + I)$, for some $j \neq i$, or $(x_i + I)^2 \in I$. The latter case leads to a contradiction. Thus S is a γ_t set of $\Gamma(R/I)$, as desired.

Part (ii) follows from (i) and Theorem 3.2. ■

Corollary 3.5 *Let $R \cong R_1 \times \cdots \times R_k$, where $k \geq 2$, and let $I = I_1 \times \cdots \times I_k$ be a proper ideal of R. If at least two of I_i's are proper ideals, then $\gamma(\Gamma(R/I)) = \gamma(\Gamma_I(R))$.*

Proof The result is proved by Theorem 3.2 and [5, Corollary 2.7]. ■

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