MEAN VALUE THEOREMS AND A TAYLOR THEOREM
FOR VECTOR VALUED FUNCTIONS

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Two mean value theorems and a Taylor theorem for functions with
values in a locally convex topological vector space are proved
without the use of the Hahn-Banach extension theorem.

1.   Introduction

It is well known that the classical mean value theorem does not hold
for vector valued functions. For example, if $f(x) = e^{ix}$,
$x \in [a, b] = [0, 2\pi]$ then the equation

(1) $f(b) - f(a) = f'(c)(b-a)$

cannot hold for any $c \in (0, 2\pi)$. $\big(f(b) = f(a)$, $|f'(c)| = 1$ for every
$c \in (a, b)$.$\big)$ In the mathematical literature there appear two kinds of
valid generalization of the mean value theorem for vector valued functions.

(a) The equation (1) is replaced by an inequality involving norms or
seminorms. For complex valued function this goes back to Darboux [3] who
proved: if $f : [a, b] \to \mathbb{C}$ has a continuous derivative then there exists
a $c \in (a, b)$ such that

(2) $|f(b)-f(a)| \leq |f'(c)|(b-a)$.

For linear normed spaces this generalization without the assumption of
continuity of f' is due to Aziz and Diaz [1], the sign $|\cdot|$ in
inequality (2) in this case must be interpreted as a norm. For further

development see [2], [4], and for history of the mean value theorem [10].
(Also concerning generalizations (b) to follow.)

(b) The equation (1) is replaced by the statement that $\dfrac{f(b)-f(a)}{b-a}$
lies in the closed convex hull of the range of the derivative. In the
sequel, the convex hull of A shall be denoted by co A and a closure by
a bar. Hence

(3) $\dfrac{f(b)-f(a)}{b-a} \in \overline{\text{co}}\{f'(t); \ a < t < b\}$.

This generalization for complex valued functions goes back to Weierstrass
[12, p. 58], for linear normed spaces is due to Ważewski [16]. There is a
nice example by Eberlein [8] showing that the bar cannot be omitted in (3)
in case of an infinite dimensional space. See also [17]. McLeod [13]
proved that the bar can be omitted if the space is finite dimensional. For
locally convex spaces see [6] and [13].

There is another feature to these generalizations, namely, the
occurrence of an exceptional set $E \subset (a, b)$ where the function is either
not required to be differentiable or the values of the derivative are not
taken into account (McLeod [13] actually works with two exceptional sets).
The classical mean value theorem is no longer true even if E is a single-
ton; however, in both generalizations mentioned above some infinite sets
are permitted.

There is yet another feature common to both generalizations, namely,
most of the proofs (an exception is, for example, [4]) employ the Hahn-
Banach theorem. It is one of the aims of this paper to prove a Cauchy
mean value theorem of Weierstrass' type for functions with values in a
locally convex topological vector space. By avoiding the Hahn-Banach
Theorem and hence also the axiom of choice (and by not using Solovey's
axiom either) we obtain a theorem of constructive functional analysis in
the sense of Garnir [11].

In a certain sense Weierstrass' mean value theorem is stronger than
Darboux mean value theorem [8], [15].

There exist very many generalizations of the Taylor theorem to vector
valued functions. Besides the integral form of the remainder (see, for
example, [7], [14, p. 188]) which is most often used there also exist

estimates of the remainder similar to Darboux type generalizations of the Mean Value Theorem [5], [9, Theorem 3]. In this paper we prove an estimate of the remainder in the spirit of Weierstrass generalization of the Mean Value Theorem. The advantage of our approach lies in the fact that we use only the mere existence of the $(n+1)$st derivative.

2. Mean Value Theorems

LEMMA. *If $\delta : [a, b] \to R_+$ (that is, δ is real valued and positive), $M \subseteq [a, b]$, $G = \{(\alpha_i, \beta_i)\}$ is a family (finite or infinite) of disjoint open intervals, $[\alpha_i, \beta_i] \subseteq [a, b]$ and if*

$$[a, b] \subseteq \bigcup_{\xi \in M} (\xi - \delta(\xi), \xi + \delta(\xi)) \cup \bigcup_i (\alpha_i, \beta_i)$$

then there exists a partition (P) *of $[a, b]$,*

(P) $a = x_0 < x_1 < x_2 < \ldots < x_n = b$

with the following property: for every j, $j = 0, 1, \ldots, n-1$, either (x_j, x_{j+1}) is one of the intervals $(\alpha_i, \beta_i) \in G$ or there is a $\xi_j \in M \cap [x_j, x_{j+1}]$ and

(4) $x_{j+1} - \xi_j < \delta(\xi_j)$,

                        $\xi_j - x_j < \delta(\xi_j)$.

The latter intervals will be referred to as ξ-intervals.

Proof. If G is empty an indirect proof based on bisecting $[a, b]$ and on the nested interval theorem is easy. The general case can be reduced to the case of finitely many intervals for which G is empty. This is done by applying Borel's covering theorem to the system of open intervals consisting of intervals of G and of intervals $(\xi - \delta(\xi), \xi + \delta(\xi))$, $\xi \in M$.

THEOREM 1. *If*

(i) X is a real locally convex topological vector space,

(ii) $f : [a, b] \to X$ is continuous at a and b from the right and left respectively,

(iii) $g : [a, b] \to R$ *is strictly increasing, continuous at* a
and b *, from the right and from the left, respectively,*

(iv) E *is a subset of* (a, b) *and* f *and* g *are*
differentiable on $(a, b) - E$ *; moreover* g' *is positive*
on $(a, b) - E$ *,*

(v) C *is convex and closed,* $\left\{\dfrac{f'(t)}{g'(t)}; \ t \in (a, b)-E\right\} \subset C$ *,*

(vi) *for every neighbourhood of zero* U *there exists a family*
of disjoint open intervals $\left(\alpha_i, \ \beta_i\right)$ *such that*

(5)
$$E \subset \bigcup_i \left(\alpha_i, \ \beta_i\right) \ ,$$

and for any finite set of i*'s ,*

(6)
$$\sum \left[f\left(\beta_i\right)-f\left(\alpha_i\right)\right] \in U \ ,$$

(vii) $0 \in C$ *,*

then

$$\frac{f(b)-f(a)}{g(b)-g(a)} \in C \ .$$

Proof. Denote $q = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ and let V be a convex, symmetric
neighbourhood of zero. The theorem will be proved if we show that there
exists an element $c \in C$ such that $q-c \in V$. There exist positive $\delta(a)$
and $\delta(b)$ such that

(7a)
$$f(x) - f(a) \in \frac{g(b)-g(a)}{4} \cdot V$$

and

(7b)
$$f(x) - f(b) \in \frac{g(b)-g(a)}{4} \cdot V$$

for $a \leq x \leq a+\delta(a)$ and $b-\delta(b) \leq x \leq b$, respectively. Further, for
every $\xi \in (a, b)-E$ there exists a positive $\delta(\xi)$ such that

(8)
$$\frac{f(x)-f(\xi)}{g(x)-g(\xi)} - \frac{f'(\xi)}{g'(\xi)} \in \tfrac{1}{4}V \ ,$$

for $0 < |x-\xi| \leq \delta(\xi)$. Let $\left(\alpha_i, \ \beta_i\right)$ be the system of intervals from

(vi). Let (P) be the partition from Lemma for $M = (a, b) - E$. If $\left(x_j, x_{j+1}\right)$ is a ξ-interval then it follows from (8) using strict monotonicity of g and convexity of V that

(9)
$$\frac{f\left(x_{j+1}\right)-f\left(x_j\right)}{g\left(x_{j+1}\right)-g\left(x_j\right)} - \frac{f'\left(\xi_j\right)}{g'\left(\xi_j\right)} \in \tfrac{1}{4}V .$$

We denote by $\sum'$ the sum which is extended over those j for which $\left(x_j, x_{j+1}\right)$ is a ξ-interval and in $\sum''$ the summation is understood over the remaining indices. We define $\lambda_j = \dfrac{g\left(x_{j+1}\right)-g\left(x_j\right)}{g(b)-g(a)}$, $c = \sum' \lambda_j \dfrac{f'\left(\xi_j\right)}{g'\left(\xi_j\right)}$ (if there is no ξ-interval in (P) we define $c = 0$). Since $0 \leq \lambda_i \leq 1$, $\sum' \lambda_i \leq 1$ we have, by using *(vii)*, that $c \in C$. Now we estimate $q - c$;

$$q - c = \sum' \lambda_j \frac{f\left(x_{j+1}\right)-f\left(x_j\right)}{g\left(x_{j+1}\right)-g\left(x_j\right)} - c + \frac{1}{g(b)-g(a)} \sum'' \left[f\left(x_{j+1}\right)-f\left(x_j\right)\right]$$
$$\in \tfrac{1}{4}V + \tfrac{3}{4}V ,$$

by using *(vi)* with $U = \tfrac{1}{4}V$, (7) and (9). Hence we established $q-c \in V$ and the theorem is proved.

REMARK 1. The hypothesis $0 \in C$ cannot be omitted without affecting the validity of Theorem 1. For a counterexample see [6, Remark 0, pp. 295-296].

However, if the hypothesis *(v)* of Theorem 1 is replaced by the following stronger hypothesis,

(viS) E *is a subset of* (a, b) *such that, for every* $\varepsilon > 0$
and every neighbourhood of zero U , *there is a family of*
disjoint open intervals $\left(\alpha_i, \beta_i\right)$, $\left[\alpha_i, \beta_i\right] \subset [a, b]$
such that (5), (6) *hold and moreover*

(10)
$$\sum \left[g\left(\beta_i\right)-g\left(\alpha_i\right)\right] < \varepsilon ;$$

then the hypothesis *(vii)* can be omitted. More precisely, we have

THEOREM 2. *If hypotheses (i)-(v) and (viS) hold then*

$$\frac{f(b)-f(a)}{g(b)-g(a)} \in \overline{\mathrm{co}}\left\{\frac{f'(t)}{g'(t)};\ t \in (a,\ b)\text{-}E\right\}\ .$$

Proof. We keep the notation from the proof of Theorem 1, $\left(\alpha_i,\ \beta_i\right)$ now satisfy *(viS)* rather than *(vi)*. We choose a positive ε satisfying $\varepsilon < \frac{1}{2}\{g(b)\text{-}g(a)\}$, $\frac{2\varepsilon}{\{g(b)\text{-}g(a)\}} q \in V$. It follows that $\sum' \lambda_j \geq \frac{1}{2}$, in particular there are ξ-intervals and $c = \sum' \lambda_j \dfrac{f'\left(\xi_j\right)}{g'\left(\xi_j\right)}$. Define $a = \sum' \lambda_j$, $d = \frac{1}{a} c$. Then $d \in \mathrm{co}\left\{\dfrac{f'(\xi)}{g'(\xi)};\ \xi \in (a,\ b)\text{-}E\right\}$. Finally, we have $q - d = q - \frac{1}{a} q + \frac{1}{a} (q\text{-}c) \in V + \frac{1}{a} V \subset 3V$. This completes the proof.

REMARK 2. Some examples when hypothesis *(viS)* is satisfied are:

(a) f and g continuous throughout $[a,\ b]$ and E is at most
      countable;

(b) f and g are absolutely continuous and E is of measure
      zero;

(c) f is absolutely continuous with respect to g and E is
      of g-measure zero;

(d) f is α-Hölder continuous, $g(x) = x$, and E is of
      α-Hausdorff measure zero.

REMARK 3. One can obtain a mean value theorem for a mapping F from one locally convex topological vector space to another by using either Theorem 1 or Theorem 2 on the function f of a real variable defined by $f(t) = F\left(a\text{+}t(b\text{-}a)\right)$.

3.   Taylor's Theorem

Let X and Y be locally convex topological spaces, $a \in X$, $a\text{+}x \in X$, $f : X \to Y$, $n + 1$ times Gateaux differentiable at all points of the segment $[a,\ a\text{+}x]$. Let further $g : [0,\ 1] \to R$ be continuous and $g'(t) > 0$ for $0 < t < 1$. Then

(11) $\quad f(a+x) - \sum\limits_{i=0}^{n} f^{(i)}(a)\left(x^i\right)$

$$\epsilon \; \frac{g(1)-g(0)}{n!} \; \overline{co}\left\{\frac{(1-t)^n f^{n+1}(a+tx)\left(x^{n+1}\right)}{g'(t)}, \; 0 < t < 1\right\} \; .$$

In particular, for $g(t) = -(1-t)^n$,

(12) $\quad f(a+x) - \sum\limits_{i=0}^{n} f^{(i)}(a)\left(x^i\right) \; \epsilon \; \overline{co}\left\{\frac{f^{(n+1)}(a+tx)\left(x^{n+1}\right)}{(n+1)!}, \; 0 < t < 1\right\} \; .$

REMARK 4. For convenience of exposition we assumed the existence of $f^{(n+1)}(a+tx)$ for all $t \in [0, 1]$. One can, however, weaken this hypothesis and allow an exceptional set E , similarly as in paragraph 2.

Proof. Let us define

$$H(t) = f(a+tx) + \frac{(1-t)}{1!} \; f'(a+tx)(x) + \ldots + \frac{(1-t)^n}{n!} \; f^{(n)}(a+tx)\left(x^n\right) \; .$$

An easy calculation shows $H'(t) = \frac{(1-t)^n}{n!} \; f^{(n+1)}(a+tx)\left(x^{n+1}\right)$. Applying the mean value theorem to H, g and the interval $[0, 1]$, we obtain (11). The relation (12) is then immediate.

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