ON PEARL'S PAPER "A DECOMPOSITION THEOREM FOR MATRICES"*

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Let A be an $m \times n$ matrix of complex numbers. Let A^{τ} and A^* denote the transpose and conjugate transpose, respectively, of A. We say A is diagonal if A contains only zeros in all positions (i, j) with $i \neq j$. In a recently published paper [4], M.H. Pearl established the following fact: *There exist real orthogonal matrices* O_1 *and* O_2 (O_1 *m-square*, O_2 *n-square*) *such that* O_1AO_2 *is diagonal, if and only if both* AA^* *and* A^*A *are real.* It is the purpose of this paper to show that a theorem substantially stronger than this result of Pearl's is included in the real case of a theorem of N.A. Wiegmann [2]. (For other papers related to Wiegmann's, see [1;3].)

THEOREM. *Let* $A_1, \ldots, A_k$ *be a set of* $m \times n$ *matrices of complex numbers. Then real orthogonal matrices* O_1 *and* O_2 (O_1 *m-square*, O_2 *n-square*) *exist such that simultaneously all matrices* $O_1A_iO_2$ *are diagonal*, $1 \leq i \leq k$, *if and only if the matrices*

$$(1) \qquad A_iA_j^*,\ A_iA_j^{\tau},\ A_i^*A_j,\ A_i^{\tau}A_j, \quad 1 \leq i, j \leq k$$

are all symmetric.

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Here a (real or complex) matrix M is said to be symmetric if $M = M^{\tau}$. It is easy to see that $A_1 A_1^*$ and $A_1^* A_1$ are real if and only if they are symmetric. Thus the case $k = 1$ of the Theorem yields Pearl's result.

Proof. Let $A_j = B_j + \sqrt{-1}\, C_j$ where B_j and C_j are real. Upon separating the four equations

$$(2) \quad \begin{aligned} A_i A_j^* &= (A_i A_j^*)^{\tau}, \quad A_i A_j^{\tau} = (A_i A_j^{\tau})^{\tau} \\ A_i^* A_j &= (A_i^* A_j)^{\tau}, \quad A_i^{\tau} A_j = (A_i^{\tau} A_j)^{\tau} \end{aligned}$$

into real and imaginary parts, we obtain eight equations:

$$(3) \quad \begin{aligned} B_i B_j^{\tau} + C_i C_j^{\tau} &= B_j B_i^{\tau} + C_j C_i^{\tau}, \quad B_i B_j^{\tau} - C_i C_j^{\tau} = B_j B_i^{\tau} - C_j C_i^{\tau}, \\ -B_i C_j^{\tau} + C_i B_j^{\tau} &= B_j C_i^{\tau} - C_j B_i^{\tau}, \quad B_i C_j^{\tau} + C_i B_j^{\tau} = B_j C_i^{\tau} + C_j B_i^{\tau}, \\ B_i^{\tau} B_j + C_i^{\tau} C_j &= B_j^{\tau} B_i + C_j^{\tau} C_i, \quad B_i^{\tau} B_j - C_i^{\tau} C_j = B_j^{\tau} B_i - C_j^{\tau} C_i, \\ B_i^{\tau} C_j - C_i^{\tau} B_j &= -B_j^{\tau} C_i + C_j^{\tau} B_i, \quad B_i^{\tau} C_j + C_i^{\tau} B_j = B_j^{\tau} C_i + C_j^{\tau} B_i. \end{aligned}$$

By addition and subtraction we see that the equations (3) are equivalent to:

$$(4) \quad \begin{aligned} &B_i B_j^{\tau} = B_j B_i^{\tau}, \; C_i C_j^{\tau} = C_j C_i^{\tau}, \; B_i C_j^{\tau} = C_j B_i^{\tau}, \; C_i B_j^{\tau} = B_j C_i^{\tau}, \\ &B_i^{\tau} B_j = B_j^{\tau} B_i, \; C_i^{\tau} C_j = C_j^{\tau} C_i, \; B_i^{\tau} C_j = C_j^{\tau} B_i, \; C_i^{\tau} B_j = B_j^{\tau} C_i. \end{aligned}$$

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However, by the real analogue of Wiegmann's first theorem in [2], the validity of the equations (4) for all i, j, $1 \leq i, j \leq k$, is exactly the necessary and sufficient condition for the existence of real orthogonal O_1 and O_2 such that the matrices $O_1B_iO_2$, $O_1C_iO_2$, $1 \leq i \leq k$, are all diagonal. This completes the proof of the Theorem. For completeness, we now sketch a proof of Wiegmann's theorem in the real case. Our proof is somewhat shorter than Wiegmann's.

LEMMA. <u>Let</u> $M_1, \ldots, M_r$ <u>be a set of</u> $m \times n$ <u>real matrices. Then real orthogonal matrices</u> O_1 <u>and</u> O_2 <u>exist such that all of</u> $O_1M_iO_2$ <u>are diagonal,</u> $1 \leq i \leq r$, <u>if and only if all matrices</u> $M_iM_j^\tau$ <u>and</u> $M_i^\tau M_j$ <u>are symmetric.</u>

<u>Proof of Lemma.</u> The necessity is trivial since then all $M_iM_j^\tau$ and all $M_i^\tau M_j$ are orthogonally similar to real diagonal matrices. Let p, q, r, s be integers. Using the properties $M_iM_j^\tau = M_jM_i^\tau$ and $M_i^\tau M_j = M_j^\tau M_i$, we see that $(M_pM_q^\tau)(M_rM_s^\tau) = M_pM_r^\tau M_qM_s^\tau$ $= M_rM_p^\tau M_sM_q^\tau = (M_rM_s^\tau)(M_pM_q^\tau)$. Hence the symmetric matrices $M_iM_j^\tau$, $1 \leq i, j \leq k$ are commutative. Therefore we may find an orthogonal matrix O_1 such that $O_1M_iM_j^\tau O_1^\tau$ are all diagonal. Without loss of generality we may assume that $O_1M_1M_1^\tau O_1^\tau$ has $(\alpha_1^2, 0, \ldots, 0)$ as its top row, with $\alpha_1 > 0$. Thus the top row of O_1M_1 has norm α_1. We may find orthogonal O_2 mapping this top row of O_1M_1 to $(\alpha_1, 0, 0, \ldots, 0)$. Changing notation and replacing $O_1M_iO_2$ throughout with M_i, we now see that

$$M_i = \begin{bmatrix} \alpha_i & x_i \\ y_i & M'_i \end{bmatrix}, \quad 1 \leq i \leq k,$$

with $\alpha_1 > 0$, $x_1 = 0$. The diagonal form of $M_1M_1^\tau$ forces $y_1 = 0$; the diagonal form of $M_1M_i^\tau$ forces $y_i = 0$ $(i > 1)$, and then the normality of $M_i^\tau M_1$ forces $x_i = 0$ $(i > 1)$. Thus $M_i = (\alpha_i) \dotplus (M_i')$,

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$1 \leq i \leq k$. By an obvious induction on the size of the matrices, we may now diagonalize $M_1', \ldots, M_k'$ by an orthogonal equivalence, and hence complete the diagonalization of $M_1, \ldots, M_k$.

For use elsewhere, observe that if all $M_i M_j^\tau$ are positive semidefinite then we may find O_1 and O_2 such that $O_1 M_i O_2$ are all diagonal with nonnegative diagonal entries. This follows by using the fact that $\alpha_1 \alpha_i \geq 0$ $(i > 1)$ denies the possibility that $\alpha_i < 0$.

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