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COMPACTNESS OF A LOCALLY COMPACT GROUP G AND GEOMETRIC PROPERTIES OF $A_p(G)$

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ABSTRACT. Let G be a locally compact topological group. A number of characterizations are given of the class of compact groups in terms of the geometric properties such as Radon-Nikodym property, Dunford-Pettis property and Schur property of $A_p(G)$, and the properties of the multiplication operator on $PF_p(G)$. We extend and improve several results of Lau and Ülger [17] to $A_p(G)$ and $B_p(G)$ for arbitrary p.

1. **Introduction.** Let G be a locally compact topological group. Let $A(G)$ and $B(G)$ be the Fourier and the Fourier-Stieltjes algebras of G, respectively. Taylor [22] showed that $A(G)$ has the Radon-Nikodym property if and only if $G \in [AR]$ and $B(G)$ has the Radon-Nikodym property if and only if $G \in [AU]$. Later, Lau and Ülger [17] studied extensively the geometric properties of $A(G)$ and $B(G)$. The main machinery for them is C^*-algebraic since, as is well known, the predual of $B(G)$ is the group C^*-algebra $C^*(G)$ and the dual of $A(G)$ is the von Neumann algebra $VN(G)$. In this paper, we will establish several sufficient conditions for G to be compact. We point out that the C^*-algebraic method, which worked for $p = 2$, does not work for arbitrary p since $A_p(G)$ and $B_p(G)$ are usually not related to any C^*-algebra. As we know, $A_2(G) = A(G)$ for an arbitrary locally compact group G, and if G is amenable then $B_2(G) = B(G)$ and $PF_2(G) = C^*(G)$, the group C^*-algebra of G. We extend and improve several results of Lau and Ülger [17] to $A_p(G)$ and $B_p(G)$ for $1 < p < \infty$ (see Lau and Ülger [17], page 321–322). We prove in this paper, among other things, the following: for any locally compact group G, the following are equivalent

(a) G is compact;

(b) G is amenable and each functional ϕ in $B_p(G)$ is almost periodic on $PF_p(G)$;

(c) Each functional ϕ in $A_p(G)$ is almost periodic on $PF_p(G)$;

(d) There exists a nonzero functional ϕ in $A_p(G)$ which is almost periodic on $PF_p(G)$;

(e) For each f in $PF_p(G)$, the multiplication operator $\tau_f \colon PF_p(G) \longrightarrow PF_p(G)$ (defined by $\tau_f(g) = f \cdot g$ for all $g \in PF_p(G)$) is compact;

(f) G is amenable and for each f in $PF_p(G)$, the multiplication operator $\tau_f \colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact;

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(g) G is amenable and the Banach algebra $PF_p(G)$ is an ideal of its second dual under either Arens product;

(h) For any $a \in A_p(G)$, or equivalently for a nonzero $a \in A_p(G)$, $\{_x a : x \in G\}$ is a compact subset of $A_p(G)$.

We also prove that each of the following conditions is sufficient for G to be compact:

(i) The space $A_p(G)$ has the Schur property;

(j) G is amenable and the space $B_p(G)$ has *RNP* and *DPP*;

(k) The algebra $PF_p(G)$ does not contain an isomorphic copy of ℓ^1 and has the *DPP*;

(l) Every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is compact.

The main tools we use to prove our results are Lemma 3.1, Lemma 4.5 and Lemma 5.1 in which various sequences approaching "infinity" will be considered. We organize this paper as follows. In Section 2, we collect the notations and definitions that will be used in this paper. In Section 3 we give various sufficient conditions for a locally compact group G to be compact in terms of some geometric properties of $A_p(G)$. In Section 4, by developing the technique used in Section 3 we characterize the compact groups in terms of almost periodic functionals on $PF_p(G)$ and $PM_p(G)$. In Section 5 we give some results relating the properties of the multiplication operator $\tau_f: PF_p(G) \longrightarrow PF_p(G)$ to the properties of the group G.

2. **Preliminaries and Some Notations.** Let G be a locally compact group with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated Lebesgue spaces ($1 \leq p \leq \infty$). For any $f: G \longrightarrow C$, and $x \in G$, the left [right] translation of f by x is defined by $_x f(y) = f(xy)$ [$f_x(y) = f(yx)$], and $\check{f}$ is defined by $\check{f}(y) = f(y^{-1})$, $y \in G$. We say that G is amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(_x f) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. Properties of amenable groups can be found in Greenleaf [13], Paterson [20] and Pier [21]. For any two measurable functions f and g on G, their convolution $f * g$ is defined by

$$f * g(x) = \int_G f(t)\, g(t^{-1}x)\, dt$$

whenever this makes sense. The Figà-Talamanca-Herz algebra $A_p(G)$ is the space of functions $u: G \longrightarrow C$ which can be represented, nonuniquely, as

$$u = \sum_{n=1}^{\infty} v_n * \check{u}_n \text{ for } u_n \in L^p(G) \text{ and } v_n \in L^q(G) \text{ with } \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$$

and $\|u\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q$, where the infimum is taken over all possible representations of u and $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $A_p(G)$ is a regular tauberian algebra under the pointwise multiplication and its Gelfand spectrum is G. Furthermore the algebra $A_p(G) \subset C_0(G)$ and it has a bounded approximate identity if and only if G is amenable. If $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of G. Each $f \in L^1(G)$ defines a bounded linear functional on $A_p(G)$ by

$$\langle f, u \rangle = \int_G f(x)\, u(x)\, dx \text{ for } u \in A_p(G),$$

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and it defines a bounded linear operator on $L^p(G)$ by $f(g) = f * g$, $g \in L^p(G)$. The norm of f as an element of $A_p(G)^*$ and the operator norm of f as an element of $\mathcal{B}\big(L^p(G)\big)$ are the same, that is,

$$\|f\|_{A_p(G)} = \sup_{u \in A_p(G), \|u\| \leq 1} |\langle f, u\rangle| \text{ and } \|f\|_{\mathcal{B}\left(L^p(G)\right)} = \sup_{g \in L^p(G), \|g\|_p \leq 1} \|f * g\|_p$$

are equal. It follows that $L^1(G)$ is a subspace of $A_p(G)^*$. Let $PF_p(G)$ and $PM_p(G)$ be the closures of $L^1(G)$ in $A_p(G)^*$ with respect to the norm topology and the *weak** topology respectively. It can be shown that $PM_p(G) = A_p(G)^*$. When $p = 2$, $PM_2(G)$ is usually denoted by $VN(G)$ while $PF_2(G)$ is $C^*_\rho(G)$, the reduced group C^*-algebra of G. When G is amenable, $C^*_\rho(G) = C^*(G)$, the group C^*-algebra of G.

$B_p(G)$ is the pointwise multiplier algebra of $A_p(G)$, consisting of the continuous functions v on G such that $uv \in A_p(G)$ for all $u \in A_p(G)$. The norm on $B_p(G)$ is defined by

$$\|v\|_{B_p(G)} = \sup\{\|vu\|_{A_p(G)} \text{ and } \|u\|_{A_p(G)} \leq 1\}.$$

Observe that $A_p(G) \subseteq B_p(G)$ and if $v \in A_p(G)$, $\|v\|_{B_p(G)} \leq \|v\|_{A_p(G)}$. It has been showed that $PF_p(G)^* = B_p(G)$ if and only if G is amenable (see Cowling [5]), in this case, for any $b \in B_p(G)$, $b \in PF_p(G)^*$ is defined by $\langle b,f\rangle = \int_G b(t)f(t)\,dt$ for $f \in L^1(G)$. When $p = 2$, $B_2(G) = B(G)$, the Fourier-Stieltjes algebra of G which is the dual of the group C^*-algebra $C^*(G)$.

For any $f \in PM_p(G)$ and $b \in A_p(G)$ or $b \in B_p(G)$, $f \circ b$ is a functional on $PM_p(G)$ or $PF_p(G)$ defined by $\langle f \circ b, g\rangle = \langle b, g \cdot f\rangle$ whenever this makes sense, where $g \cdot f$ is the product of g and f in $PM_p(G)$. Let $x \in G$, δ_x denotes the point measure at x.

Let X be a Banach space. We recall definitions of some geometric properties of X as follows.

Radon-Nikodym property (RNP). The Banach space X is said to have the RNP if each closed convex subset D of X is dentable, *i.e.* for any $\epsilon > 0$ there exists an $x \in D$ such that $x \notin \overline{Co}\big(D \sim B_\epsilon(x)\big)$, where $B_\epsilon(x) = \{y \in X : \|x - y\| < \epsilon\}$. It is known that X^* has the RNP if and only if every separable subspace of X has a separable dual (see Bourgin [3], Diestel [6] and Huff [16] for more information). Granirer [11] proved that $A_p(G)$ has the RNP if G is compact.

Dunford-Pettis property (DPP). The Banach space X is said to have the DPP if, for any Banach space Y, every weakly compact linear operator $u: X \longrightarrow Y$ sends weakly Cauchy sequences into norm convergent sequences. See Diestel [7] for more information on this property.

Schur property. The Banach space X is said to have the Schur property if every weakly convergent sequence is norm convergent.

Let A be an arbitrary Banach algebra. For any $a \in A$ and $f \in A^*$, $af \in A^*[fa]$ is defined by $\langle af, b\rangle = \langle f, ba\rangle[\langle fa, b\rangle = \langle f, ab\rangle]$. The functional f is said to be *weakly almost periodic* [*almost periodic*] on A if $\{af : a \in A, \|a\| \leq 1\}$ is relatively weakly compact [relatively compact]. By wap(A) [ap(A)] we denote the linear subspace of A^* consisting of all weakly almost periodic [almost periodic] functionals on A.

3. **The DPP, the RNP and the Schur property of $A_p(G)$ and $B_p(G)$.** In this section we will be concerned with geometric properties of $A_p(G)$ and $B_p(G)$. A number of sufficient conditions for G to be compact will be established. Our main tool is the following lemma. We give an elementary proof here.

LEMMA 3.1. *Let G be a locally compact group. If G is not compact then there is a sequence $\{x_n\}$ in G such that for any compact set K of G there is an N with $x_n \notin K$ for all $n > N$. Furthermore, for any $a \in A_p(G)$, ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$.*

PROOF. Since G is not compact we can choose an open and closed subgroup G_0 of G which is σ-compact. Let $G_0 = \bigcup_{n=1}^{\infty} K_n$, where K_n is an open set with compact closure and $K_n \subset K_{n+1}$ for all n. For each $n \geq 1$, choose an $x_n \in K_{n+1} \sim K_n$. Then for any compact set K of G we have $K \cap G_0 \subseteq \bigcup_{i=1}^{N} K_i \cap K$ for some natural number N. So $x_n \notin K$ for all $n > N$.

Let $a \in A_p(G)$. We now show that ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. Let $\epsilon > 0$ and let $F \in PM_p(G)$ be fixed. By the definition of $A_p(G)$ there are $f_n \in L^p(G)$ and $g_n \in L^q(G)$ such that $a = \sum_{n=1}^{\infty} g_n * \check{f}_n$ and $\sum_{n=1}^{\infty} \|g_n\|_q \|f_n\|_p < \infty$. Since continuous functions with compact supports are dense in $L^p(G)$ and $L^q(G)$, there exists an $u = \sum_{n=1}^{k} v_n * \check{u}_n$ in $A_p(G)$ such that $\|u - a\| < \epsilon$, where all u_n and v_n are continuous on G with compact supports. So $\|{}_{x_n}u - {}_{x_n}a\| = \|u - a\| < \epsilon$ for all n and

$$\begin{aligned}
|\langle F, {}_{x_n}a\rangle| &\leq |\langle F, {}_{x_n}a - {}_{x_n}u\rangle| + |\langle F, {}_{x_n}u\rangle| \\
&\leq \|F\| \, \|{}_{x_n}a - {}_{x_n}u\| + |\langle F, {}_{x_n}u\rangle| \\
&\leq \epsilon\|F\| + |\langle F, {}_{x_n}u\rangle|.
\end{aligned}$$

To prove that $F({}_{x_n}a) \longrightarrow 0$, it suffices to show that $\langle F, {}_{x_n}(v_i * \check{u}_i)\rangle \longrightarrow 0$ for each $1 \leq i \leq k$. Let i be fixed and let V be the support of v_i. Then V is compact and $|v_i(t)| \leq \|v_i\|_\infty 1_V(t)$ for all t. Recall that if T is in $PM_p(G)$ and $v * \check{u}$ is a simple element of $A_p(G)$ then $\langle T, v * \check{u}\rangle = \langle T(u), v\rangle$. Hence we have the following

$$\begin{aligned}
|\langle F, {}_{x_n}(v_i * \check{u}_i)\rangle| &= |\langle F, ({}_{x_n}v_i) * \check{u}_i\rangle| \qquad \text{(see Pier [21], pp. 14, (4))} \\
&= |\langle F(u_i), {}_{x_n}v_i\rangle| \\
&= \left|\int_G v_i(x_n t) F(u_i)(t)\, dt\right| \\
&\leq \int_G \|v_i\|_{\infty x_n} 1_V(t) |F(u_i)(t)|\, dt \\
&= \int_{x_n^{-1}V} \|v_i\|_\infty |F(u_i)(t)|\, dt \\
&\leq \|v_i\|_\infty \Big(\int_{x_n^{-1}V} 1^q\, dt\Big)^{\frac{1}{q}} \Big(\int_{x_n^{-1}V} |F(u_i)|^p\, dt\Big)^{\frac{1}{p}} \\
&= \|v_i\|_\infty \big(\lambda(V)\big)^{\frac{1}{q}} \Big(\int_{x_n^{-1}V} |F(u_i)|^p\, dt\Big)^{\frac{1}{p}} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ since $F(u_i) \in L^p(G)$ and for any compact subset K of G, $x_n^{-1}V$ is disjoint with K for sufficiently large n, where q is a real number so that $1/p + 1/q = 1$. ∎

REMARK 3.2. This lemma can also be derived from Lemma 8.4 of [17] as follows. Lemma 8.4 of [17] implies that we have not only the inclusion $A_p(G) \subseteq \mathrm{wap}\big(PF_p(G)\big)$ but also the inclusion $A_p(G) \subseteq \mathrm{wap}\big(PM_p(G)\big)$. It follows that, for $a \in A_p(G)$, the set $\{a_x : x \in G\}$ is relatively weakly compact ($a_x = \delta_x \circ a$; see our Proposition 4.4 below). The set $\{\delta_x : x \in G\}$, being the spectrum of the algebra $A_p(G)$, has the zero functional as the weak* cluster point if and only if G is not compact. Note that for any $f \in L^1(G)$ and $x \in G$, we have $a_x = \langle \delta_x \circ a, f\rangle = \big((\check{f}\check{\Delta})\big) * a(x)$, and $(\check{f}\check{\Delta}) * a \in A_p(G)$. It follows that if $\{x_n\}$ is any sequence in G that goes to "infinity", then $a_{x_n} \longrightarrow 0$ in the w^*-topology. So $a_{x_n} \longrightarrow 0$ weakly in $A_p(G)$ (hence ${}_{x_n}a = (\check{a}_{x_n^{-1}})^{\vee} \longrightarrow 0$ weakly; see Theorem 4.3).

COROLLARY 3.3. *Let G be a locally compact group. If $A_p(G)$ has the Schur property then G is compact.*

PROOF. Suppose G is not compact. Let $a \in A_p(G)$ be any nonzero element and let $\{x_n\}$ be a sequence as in Lemma 3.1. Then ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. But $\|{}_{x_n}a\| = \|a\| \not\to 0$. So $\{{}_{x_n}a\}$ does not converge in norm. Therefore $A_p(G)$ does not have the Schur property. ∎

COROLLARY 3.4. *Let G be a locally compact group. For any nonzero element $a \in A_p(G)$, $\{{}_xa : x \in G\}$ is a compact subset of $A_p(G)$ if and only if G is compact.*

PROOF. Let G be compact. Let $\{{}_{x_\alpha}a\}$ be any net in the orbit $\{{}_xa : x \in G\}$. Since G is compact, we can assume that $x_\alpha \longrightarrow x$ for some $x \in G$. Then ${}_{x_\alpha}a \longrightarrow {}_xa$ in $A_p(G)$ since translation is norm continuous (see Granirer and Leinert [12], pp. 466). Thus $\{{}_xa : x \in G\}$ is compact.

Conversely, suppose $\{{}_xa : x \in G\}$ is compact. If G is not compact, then we can choose a sequence $\{x_n\}$ from G as in Lemma 3.1. It follows from Lemma 3.1 that for any $a \in A_p(G)$, ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. Since $\|{}_{x_n}a\| = \|a\|$, for any non-zero $a \in A_p(G)$, $\{{}_{x_n}a\}$ does not have any subsequence which is convergent in norm. This is a contradiction. Thus G is compact. ∎

REMARK 3.5. This may fail if $a \in B_p(G) \sim A_p(G)$. For example, if G is a noncompact locally compact group then $1 \in B_p(G)$. So $\{{}_x1 : x \in G\} = \{1\}$ is compact, although G is not compact.

COROLLARY 3.6. *Let G be an amenable locally compact group. If $B_p(G)$ has the RNP and the DPP then G is compact.*

PROOF. Since $B_p(G) = PF_p(G)^*$ and $B_p(G)$ has the RNP and the DPP, it follows from Theorem 3 of [7] that $B_p(G)$ has the Schur property. Hence $A_p(G)$, the subspace of $B_p(G)$, also has the Schur property. Corollary 3.3 implies that G is compact. ∎

REMARK 3.7. Lau and Ülger [17] showed that $B(G)$ has the RNP and the DPP if and only if G is compact.

For any $x \in G$ and $f \in PM_p(G)$, define ${}_xf \in PM_p(G)$ by $\langle {}_xf, u\rangle = \langle f, {}_{x^{-1}}u\rangle$ for all $u \in A_p(G)$. It follows clearly that $\|f\| = \|{}_xf\|$ since $\|{}_{x^{-1}}u\| = \|u\|$ for all $u \in A_p(G)$.

COROLLARY 3.8. *Let G be a locally compact group. If there is a sequence $\{x_n\}$ in G as in Lemma 3.1 then for any $f \in PM_p(G)$, ${}_{x_n}f \longrightarrow 0$ in the w^*-topology of $PM_p(G)$. Consequently, if there is a nonzero $f \in PM_p(G)$ such that the right multiplication operator $\gamma_f\colon PM_p(G) \longrightarrow PM_p(G)$ defined by $\gamma_f(g) = g \cdot f$ for $g \in PM_p(G)$ is compact then G is a compact group.*

PROOF. To prove the first claim, if $\{x_n\}$ in G is as in Lemma 3.1 then it follows from Lemma 3.1 that for any $u \in A_p(G)$, $\langle {}_{x_n}f, u\rangle = \langle f, {}_{x_n^{-1}}u\rangle \longrightarrow 0$.

For the second claim, suppose G is noncompact, and let γ_f be a compact operator as defined above. Note that for any $u \in A_p(G)$ and $x \in G$ we have $u\delta_x = {}_xu \in A_p(G)$, since for any $h \in L^1(G) \subseteq PF_p(G)$,

$$\langle u\delta_x, h\rangle = \langle u, \delta_x \cdot h\rangle = \langle u, {}_{x^{-1}}h\rangle = \langle {}_xu, h\rangle.$$

It follows that $\gamma_f(\delta_x) = {}_{x^{-1}}f$ since

$$\langle \gamma_f(\delta_x), u\rangle = \langle \delta_x \cdot f, u\rangle = \langle f, u\delta_x\rangle = \langle f, {}_xu\rangle = \langle {}_{x^{-1}}f, u\rangle.$$

Thus if ${}_{x_n^{-1}}f \longrightarrow 0$ in w^*-topology, then $\{\gamma_f(\delta_{x_n})\}$ has a subsequence $\{\gamma_f(\delta_{x_{n_k}})\}$ which converges to 0 in norm by the compactness of the operator γ_f. This contradicts $\|{}_{x_{nk}^{-1}}f\| = \|f\| > 0$ for all k. ∎

That $A_p(G)$ has the RNP is close to $A_p(G)$ being a dual space (see Theorem 4.1 of Taylor [22]). Recall that for any $T \in PM_p(G)$, the support of T, denoted by $\mathrm{supp}(T)$, is a subset of G defined by $x \in \mathrm{supp}(T)$ if and only if for all $u \in A_p(G)$ we have $u \cdot T = 0$ implies $u(x) = 0$. Let E be a subset of G. Let $PM_p(G)|_E$ denote the subset of $PM_p(G)$ consisting all the elements $T \in PM_p(G)$ such that $\mathrm{supp}(T)$ is contained in E.

PROPOSITION 3.9. *Let G be a locally compact group and let G_0 be an open subgroup of G. If $A_p(G) = X^*$ for some Banach space X, then*

$$X \text{ is a subspace of } PM_p(G) \text{ and } A_p(G_0) = \big(X \cap PM_p(G_0)\big)^*.$$

PROOF. Observe that X is a subspace of $X^{**} = PM_p(G)$ and $A_p(G_0)$ can be identified with the subalgebra of $A_p(G)$ consisting of functions which vanish outside G_0 (see Herz [14], Proposition 5). Thus, $PM_p(G_0) \cong PM_p(G)|_{G_0}$ and every element in $A_p(G_0)$ is a bounded linear functional on $X \cap PM_p(G_0)$. Let $u \in \big(X \cap PM_p(G_0)\big)^*$. Then by Hahn-Banach Theorem, u can be extended to an element $\tilde{u}$ in X^*. It follows that $\tilde{u} \in X^* = A_p(G)$. Since for any $f \in X \cap PM_p(G_0)$, $\tilde{u}(f) = \tilde{u}|_{G_0}(f)$, we have $u = \tilde{u}|_{X \cap PM_p(G_0)} = \tilde{u}|_{G_0}$ and u is an element of $A_p(G_0)$ since $\tilde{u}|_{G_0} \in A_p(G_0)$ (see Herz [14], Proposition 5). ∎

COROLLARY 3.10. *Let G be a metrizable locally compact group. If $A_p(G)$ is a dual space then $A_p(G)$ has the RNP.*

PROOF. It suffices to show that any separable subspace of $A_p(G)$ has the RNP (see Huff [16], page 77). Since any separable subspace of $A_p(G)$ is contained in $A_p(G_0)$ for some open σ-compact subgroup of G, we only show that $A_p(G_0)$ has the RNP. If $A_p(G)$ is a dual space then it follows from Proposition 3.9 that $A_p(G_0)$ is a separable dual space. Hence $A_p(G_0)$ has the RNP. ∎

REMARK 3.11. Let G be a locally compact group. If $p = 2$ then $A_2(G)$ has the RNP implies that $A_2(G)$ is a dual space (see Taylor [22] Theorem 3.5).

4. **Almost periodic functionals on $PF_p(G)$.** In this section we investigate when elements of $A_p(G)$ or $B_p(G)$ are almost periodic functionals on $PF_p(G)$. When $\ell^1(G) \subseteq PF_p(G)$, we will see that it is easy to prove our main result Theorem 4.7 by applying Lemma 3.1. We like to know exactly for which groups that $\ell^1(G) \subseteq PF_p(G)$ holds, that is, $PF_p(G)$ has an identity. The following is a consequence of results of Granirer [10].

PROPOSITION 4.1. *Let G be a locally compact group. $\ell^1(G) \subseteq PF_p(G)$ if and only if G is discrete.*

PROOF. It follows from the definition that if G is discrete then $\ell^1(G) = L^1(G) \subseteq PF_p(G)$.

Conversely, let $\ell^1(G) \subseteq PF_p(G)$. Then $\delta_e \in PF_p(G)$. Assume that G is separable first. So $PF_p(G)$ is separable. By the fact that δ_e is the identity of $PF_p(G)$ together with the Corollary of Granirer [10] on page 128, which says that $PF_p(G) \subset UC_p(\hat{G}) \subset C_p(\hat{G})$ for any locally compact group G, we have that $PF_p(G) = UC_p(\hat{G}) = C_p(\hat{G})$, where $UC_p(\hat{G}) = \overline{A_p(G) \cdot PM_p(G)}$ and $C_p(\hat{G}) = \{T \in PM_p(G) : Tf_1 + f_2T \in PF_p(G) \text{ if } f_1, f_2 \in PF_p(G)\}$. Theorem 16 of Granirer [10] implies that G is discrete if we take $PF_p(G)$ as X in that Theorem.

Let G be an arbitrary locally compact group with $\ell^1(G) \subseteq PF_p(G)$. Take any open-closed compactly generated subgroup G_0 of G. Then $\ell^1(G_0) \subseteq PF_p(G_0)$. In fact, $PM_p(G_0) \cong PM_p(G) \mid G_0$ (see Herz [14], Proposition 5). So $PF_p(G_0) \cong PF_p(G) \mid G_0$. For any $x \in G_0$, $\delta_x \in PF_p(G)$ implies $\delta_x \in PF_p(G_0)$. Therefore $\ell^1(G_0) \subseteq PF_p(G_0)$. If G_0 is discrete we are done. Otherwise there is a compact normal subgroup K in G_0 such that $\lambda(K) = 0$ and G_0/K is separable (see Theorem 8.7 of Hewitt and Ross [15]). Now we prove that $\ell^1(G_0/K) \subseteq PF_p(G_0/K)$. Let $x \in G_0$. So $\delta_x \in PF_p(G_0)$. Let $f_n \in L^1(G_0)$ and $\|\delta_x - f_n\| \longrightarrow 0$ in $PM_p(G_0)$. Let $A_p(G_0)_K$ be the subalgebra of $A_p(G_0)$ consisting of functions which are constant on cosets of K. Then $A_p(G_0/K) \cong A_p(G_0)_K$ by Proposition 6 of Herz [14]. If we consider δ_x and f_n as elements of $PM_p(G_0/K)$, $\|f_n - \delta_x\| \longrightarrow 0$ since $\|(f_n - \delta_x)|_{A_p(G_0)_K}\| \leq \|f_n - \delta_x\|$. So $\delta_{xK} \in PF_p(G_0/K)$ and $\ell^1(G_0/K) \subseteq PF_p(G_0/K)$. Thus, G_0/K is discrete. It follows that K is open. This is a contradiction since $\lambda(K) = 0$. ∎

REMARK 4.2. (a) If $A_p(G)^* = PF_p(G)$ then G is discrete since $\ell^1(G) \subseteq A_p(G)^*$ and so $\ell^1(G) \subseteq PF_p(G)$ always holds. This was proved by Forrest [9].

(b) Let G be a discrete group. Then $\ell^1(G) \subseteq PF_p(G)$. By Lemma 3.1, it is easy to prove that $\mathrm{ap}\big(PF_p(G)\big) = B_p(G)$ implies that G is compact. In fact, if G is not compact, let $\{x_n\}$ in G be a sequence as in Lemma 3.1 and $a \in A_p(G)$ be such that $a(e) = 1$. Then $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$ (see Proposition 4.4 below). But $\|\delta_{x_n} \circ a\| = \|a_{x_n}\| \geq a(e) \neq 0$. So $\{\delta_{x_n} \circ a\}$ does not have any subsequence converging in norm. Hence $a \notin \mathrm{ap}\big(PF_p(G)\big)$.

THEOREM 4.3. *Let G be a locally compact group. For any real numbers $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, the operator $T: A_p(G) \longrightarrow A_q(G)$ defined by $T(u) = \check{u}$ is an isometric isomorphism of Banach algebras $A_p(G)$ onto $A_q(G)$. Hence $T^*: PM_q(G) \longrightarrow PM_p(G)$ is a linear isometry from $PM_q(G)$ onto $PM_p(G)$. Furthermore, the restriction T_R^* is a linear isometry from $PF_q(G)$ onto $PF_p(G)$.*

PROOF. By definition, $\|u\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q$, where the infimum is taken over all possible representations $u = \sum_{n=1}^{\infty} v_n * \check{u}_n$ for $u_n \in L^p(G)$ and $v_n \in L^q(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$. Hence $T(u) = \check{u} = \sum_{n=1}^{\infty} u_n * \check{v}_n$ for $u_n \in L^p(G), v_n \in L^q(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$ (see Pier [21], pp. 14 (3)). So $T(u) \in A_q(G)$ and $\|T(u)\|_{A_q(G)} = \|u\|_{A_p(G)}$ by the definition. It is clear that T is an algebraic isomorphism and T^* is a linear isometry from $PM_q(G)$ onto $PM_p(G)$.

Now we show T_R^* is a linear isometry from $PF_q(G)$ onto $PF_p(G)$. Since $L^1(G)$ is dense in $PF_q(G)$ and T^* is a linear isometry of Banach algebras, it is sufficient to show that $T^*(f) \in PF_p(G)$ for all $f \in L^1(G) \subseteq PF_q(G)$. Let $f \in L^1(G)$ and $u \in A_p(G)$, then

$$\langle T^*(f), u\rangle = \int_G u(t^{-1}) f(t)\, dt = \int_G u(t) f(t^{-1}) \Delta(t^{-1})\, dt = \langle \check{f}\check{\Delta}, u\rangle.$$

So $T^*(f) = \check{f}\check{\Delta}$ is in $L^1(G) \subseteq PF_p(G)$. ∎

PROPOSITION 4.4. *Let G be a locally compact group. Then $\delta_x \circ u = u_x$ for all $x \in G$ and $u \in A_p(G)$. Furthermore, if $\{x_n\}$ is as in Lemma 3.1 and $a \in A_p(G)$ is any element, then $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$.*

PROOF. We show that $\delta_x \circ u = u_x$ first. For all $f \in L^1(G) \subseteq PF_p(G)$, by the definition,

$$\langle \delta_x \circ u, f\rangle = \langle u, f * \delta_x\rangle = \langle u, \Delta(x^{-1}) f_{x^{-1}}\rangle = \langle u_x, f\rangle.$$

Thus, $\delta_x \circ u = u_x$ since both $\delta_x \circ u$ and u_x are elements of $PF_p(G)^*$.

For any $F_p \in PM_p(G)$, by Theorem 4.3, there is an $F \in PM_q(G)$ such that $T^*(F) = F_p$. Hence, by applying Lemma 3.1 to $A_q(G)$, we have

$$\langle F_p, \delta_{x_n} \circ a\rangle = \langle T^*(F), a_{x_n}\rangle = \langle F, {}_{x_n^{-1}}\check{a}\rangle \longrightarrow 0.$$ ∎

To prove our main theorem for general G, we need the following lemma which is a nondiscrete version of Lemma 3.1.

LEMMA 4.5. *Let G be a locally compact group. If there is a sequence $\{x_n\}$ in G as in Lemma 3.1 then $h_n \circ a \in A_p(G)$ and $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$, where $a \in A_p(G)$ is any element and $h_n = {}_{x_n}1_W \in PF_p(G)$ for any compact subset W of G with positive measure.*

PROOF. For all $f \in L^1(G) \subseteq PF_p(G)$, we have

$$\begin{aligned} \langle h_n \circ a, f\rangle &= \langle a, f * h_n\rangle \\ &= \int_G a(t) f(y)\, h_n(y^{-1}t)\, dy\, dt \\ &= \int_G a(t) f(y)\, \check{h}_n(t^{-1}y)\, dt\, dy \\ &= \langle a * \check{h}_n, f\rangle. \end{aligned}$$

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Hence $h_n \circ a = a * \check{h}_n$. To show that $h_n \circ a \in A_p(G)$, let $a_k \in A_p(G)$ be with compact support and $\|a_k - a\| \longrightarrow 0$. Then $h_n \circ a_k \in A_p(G)$ and $\|h_n \circ a - h_n \circ a_k\| \leq \|h_n\| \, \|a_k - a\| \longrightarrow 0$ uniformly for n as $k \longrightarrow \infty$. Hence $h_n \circ a \in A_p(G)$. To prove that $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$, we can assume that the support of a is compact. So $a \in L^q(G)$, where q is a real number such that $1/p + 1/q = 1$. For any $F_p \in PM_p(G)$, by Theorem 4.3, there is an $F \in PM_q(G)$ such that $T^*(F) = F_p$. Thus we have

$$\begin{aligned}
|\langle F_p, h_n \circ a\rangle| &= |\langle T^*(F), a * \check{h}_n\rangle| \\
&= |\langle F, h_n * \check{a}\rangle| \qquad \text{(see Pier [21], pp. 14, (4))} \\
&= |\langle F(a), {}_{x_n}1_W\rangle| \\
&= \left|\int_{x_n^{-1}W} F(a)(t)\,dt\right| \\
&\leq \left(\int_{x_n^{-1}W} 1^p\,dt\right)^{\frac{1}{p}} \left(\int_{x_n^{-1}W} |F(a)|^q\,dt\right)^{\frac{1}{q}} \\
&= \big(\lambda(W)\big)^{\frac{1}{p}} \left(\int_{x_n^{-1}W} |F(a)|^q\,dt\right)^{\frac{1}{q}} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ since $F(a) \in L^q(G)$, and for any compact subset K of G, $x_n^{-1}W$ and K are disjoint for sufficiently large n. ∎

REMARK 4.6. This lemma can also be derived from our Corollary 3.8 and Corollary 8.5 of Lau and Ülger in [17].

The following theorem improves some of the results of Lau and Ülger [17]. The equivalence (1) $\Leftrightarrow$ (2) is a generalization of the equivalence (a) $\Leftrightarrow$ (j) in Theorem 4.5 of Lau and Ülger [17]. It also improves Corollary 8.7 of Lau and Ülger [17]. Each of the equivalences (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4) improves (a) $\Leftrightarrow$ (b) in Theorem 8.8 of Lau and Ülger [17]. The equivalence (1) $\Leftrightarrow$ (5) is in Theorem 8.8 of Lau and Ülger [17]; here we give a different proof for (5) $\Rightarrow$ (1).

THEOREM 4.7. *Let G be a locally compact group. Then the following are equivalent*
(1) *G is compact.*
(2) $\mathrm{ap}\big(PF_p(G)\big) = B_p(G)$
(3) $A_p(G) \subseteq \mathrm{ap}\big(PF_p(G)\big)$
(4) $A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big) \neq \{0\}$
(5) $A_p(G) \subseteq \mathrm{ap}\big(PM_p(G)\big)$.

PROOF. (1) $\Rightarrow$ (2) In Corollary 8.7 of Lau and Ülger [17], it is proved that compactness of G implies $\mathrm{ap}\big(PF_p(G)\big) = B_p(G)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It is clear since $A_p(G) \subseteq B_p(G)$ and $A_p(G) \neq \{0\}$.

(4) $\Rightarrow$ (1) Let G be noncompact and let $a \in A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big)$ be nonzero. Let W be a compact subset of G such that $1_W * \check{a}(e) = \int_G 1_W(t)\check{a}(t)\,dt \neq 0$ and let h_n be defined as in Lemma 4.5. Then $h_n \circ a \in A_p(G)$ and $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$. Since $a \in A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big)$, we assume that $\|h_n \circ a\| \longrightarrow 0$ in the norm topology of $A_p(G)$ by passing to a subsequence. As $h_n \circ a = a * \check{h}_n$ (see the proof of Lemma 4.5) and $\delta_x \in A_p(G)^*$

has norm 1 for any $x \in G$, we have $\|h_n \circ a\| \geq \delta_{x_n}(a * \check{h}_n) = (h_n * \check{a})(x_n{}^{-1}) = 1_W * \check{a}(e) > 0$ for all n. This is a contradiction.

(1) $\Rightarrow$ (5) It was proved in Theorem 8.8 of Lau and Ülger [17].

(5) $\Rightarrow$ (1) If G is not compact, we choose $\{x_n\}$ from G as in Lemma 3.1 and $a \in A_p(G)$ with $a(e) > 0$. It follows from Proposition 4.4 that $\delta_{x_n} \circ a = a_{x_n}$ and $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$. Thus, $\delta_{x_n} \circ a \longrightarrow 0$ in the w^*-topology of $PM_p(G)^*$ as well. By (5), we assume that $\delta_{x_n} \circ a \longrightarrow 0$ in norm by passing to a subsequence. But $\|\delta_{x_n} \circ a\| = \|a_{x_n}\| \geq a(e) > 0$ for all n. This is impossible. Therefore, G is compact. ∎

REMARK 4.8. It is interesting to compare (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4) of our Theorem 4.6 with Corollary 8.5 of Lau and Ülger [17], where it has been showed that $A_p(G) \subseteq \mathrm{wap}\big(PF_p(G)\big)$ for all locally compact groups G.

5. **Compactness of G and some operators.** For any Banach algebra A, Arens [1], [2] introduced two natural Banach algebra products on A^{**}, each of which is an extension of the original product in A when A is canonically embedded in A^{**}. Duncan and Hosseiniun showed in Lemma 3 in [8] that A is an ideal under either Arens product if and only if all the multiplication operators on $PF_p(G)$ are weakly compact (see also Ülger [23] and Forrest [9]). Brian Forrest [9] proved that $A_p(G)$ is an ideal in its second dual if and only if G is discrete. The main result of this section is Theorem 5.2, which shows that $PF_p(G)$ is an ideal in its second dual if and only if G is compact. It also generalizes some results of Lau and Ülger [17]. For its proof we need the following lemma.

LEMMA 5.1. *Let G be a locally compact group. If $\{x_n\}$ is a sequence as in Lemma 3.1, then $\tau_f({}_{x_n}1_V) \longrightarrow 0$ and $\tau_f\big((1_V)_{x_n}\Delta(x_n)\big) \longrightarrow 0$ in the w^*-topology of $PM_p(G)$, where $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is defined by $\tau_f(h) = f \cdot h$ for all $h \in PF_p(G)$, V is any compact subset of G, and $f \in L^1(G)$.*

PROOF. Let $u \in A_p(G)$. By applying Corollary 3.8, we have

$$\langle \tau_f({}_{x_n}1_V), u\rangle = \langle {}_{x_n}1_V, (\check{\Delta f}) * u\rangle \longrightarrow 0 \text{ as } (\check{\Delta f}) * u \in A_p(G).$$

Note that $\langle \tau_f\big((1_V)_{x_n}\Delta(x_n)\big), u\rangle = \langle f * 1_V, u_{x_n^{-1}}\rangle$. By Theorem 4.3, let $f * 1_V = T_R^*(h)$ for some $h \in PF_q(G)$. Then $\langle \tau_f({}_{x_n}1_V), u\rangle = \langle T_R^*(h), u_{x_n^{-1}}\rangle = \langle h, {}_{x_n}\check{u}\rangle \longrightarrow 0$ by Lemma 3.1. ∎

In the following theorem, the equivalence (1) $\Leftrightarrow$ (3) generalizes Theorem 6.7 (a) of Lau and Ülger [17], and the equivalence (1) $\Leftrightarrow$ (5) is a parallel result of Forrest [9].

THEOREM 5.2. *Let G be a locally compact group. Then the following are equivalent:*
(1) G is compact.
(2) $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is compact for all $f \in PF_p(G)$, where $\tau_f(g) = f \cdot g$.

Furthermore, if G is amenable, then all the conditions from (1) to (5) are equivalent.
(3) $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact for all $f \in PF_p(G)$, where $\tau_f(g) = f \cdot g$.
(4) $\gamma_f\colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact for all $f \in PF_p(G)$, where $\gamma_f(g) = g \cdot f$.
(5) $PF_p(G)$ is an ideal in its second dual under either Arens product.

PROOF. (1) $\Rightarrow$ (2) This follows from Lemma 8.6 of Lau and Ülger [17].

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) Let G be noncompact. Then we can have a sequence $\{x_n\}$ in G as in Lemma 5.1 (see Lemma 3.1). Let U and V be compact subsets of G with positive measures and $f = 1_U$.

Let (2) holds. As $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is compact and

$$\|(1_V)_{x_n}\Delta(x_n)\| \leq \|(1_V)_{x_n}\Delta(x_n)\|_1 = \lambda(V)$$

for all n, we may assume that $\tau_f\big((1_V)_{x_n}\Delta(x_n)\big) \longrightarrow 0$ in the norm topology of $PF_p(G)$ by passing to a subsequence since, by Lemma 5.1, it converges to 0 in the w^*-topology of $A_p(G)^*$. Hence we get

$$\begin{aligned}\Big\|\tau_f\big((1_V)_{x_n}\Delta(x_n)\big)\Big\| &= \Big\|f * \big((1_V)_{x_n}\Delta(x_n)\big)\Big\| \\ &= \|f * (1_V * \delta_{x_n^{-1}})\| \qquad \text{(see Pier [21], pp. 14(2))} \\ &\geq \|f * (1_V * \delta_{x_n^{-1}}) * \delta_{x_n}\| \\ &= \|f * 1_V\| > 0 \text{ for all } n.\end{aligned}$$

This is a contradiction.

Assume (3) is true. By a similar argument as above, we can assume that $\tau_f({}_{x_n}1_V) \longrightarrow 0$ in the weak topology of $PF_p(G)$. Then since $1 \in B_p(G) = PF_p(G)^*$,

$$\begin{aligned}\langle 1, \tau_f({}_{x_n}1_V)\rangle &= \int_G f(t)({}_{x_n}1_V)(t^{-1}y)\,dt\,dy \\ &= \int_G 1_U(t)({}_{x_n}1_V)(t^{-1}y)\,dy\,dt \\ &= \lambda(U)\lambda(V) > 0.\end{aligned}$$

This is a contradiction.

(1) $\Rightarrow$ (3) is trivial since (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4) follows from Lemma 8.6 of Lau and Ülger [17].

(4) $\Rightarrow$ (1) Let $T_R^*\colon PF_p(G) \longrightarrow PF_q(G)$ be the linear isometry as in Theorem 4.3. We prove that $T_R^*(f)\cdot g = T_R^*\big((\Delta\check{g})\cdot f\big)$ for any $f \in PF_p(G)$ and $g \in PF_q(G)$. Assume that $f, g \in L^1(G)$ and $u \in A_q(G)$. Then, since $g \circ u = u * \check{g}$ (see the proof of Lemma 4.5),

$$\begin{aligned}\langle T_R^*(f)\cdot g, u\rangle &= \langle T_R^*(f), g \circ u\rangle \\ &= \langle f, T_R(g \circ u)\rangle \\ &= \langle f, T_R(u * \check{g})\rangle \\ &= \langle f, g * \check{u}\rangle\end{aligned}$$

$$\begin{aligned}\langle T_R^*\big((\Delta\check{g})\cdot f\big), u\rangle &= \langle (\Delta\check{g}) * f, \check{u}\rangle \\ &= \langle f, g * \check{u}\rangle. \qquad \blacksquare\end{aligned}$$

That $\{g\cdot f : \|g\| \leq 1, g \in PF_p(G)\}$ is weakly compact implies that $\{T_R^*(f)\cdot g : \|g\| \leq 1, g \in PF_q(G)\}$ is weakly compact. So G is compact by (3) $\Leftrightarrow$ (1).

(1) $\Leftrightarrow$ (5) follows from (3), (4) and Lemma 3 of Duncan and Hosseiniun [8] (p. 318). ■

COROLLARY 5.3. *Let G be a locally compact group. If every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is compact then G is compact. Conversely, if G is compact then every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is weakly compact.*

PROOF. Suppose that every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is compact and G is not compact. Let $h_n \in PF_p(G)$ and $a \in A_p(G)$ be as in Lemma 4.5. Define $T: PF_p(G) \longrightarrow B_p(G)$ by $T(f) = f \circ a$ for all $f \in PF_p(G)$. Then T is a bounded linear operator. So it is a compact operator. Thus, we assume $h_n \circ a \longrightarrow u$ in norm for some $u \in B_p(G)$ by passing to a subsequence. By Lemma 4.5, $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$. Since $\|u\|_{B_p(G)} \leq \|u\|_{A_p(G)}$ if $u \in A_p(G)$, $h_n \circ a \longrightarrow 0$ weakly in $B_p(G)$ as well. So $h_n \circ a \longrightarrow 0$ in the norm of $B_p(G)$. But $\|h_n \circ a\| \geq \delta_{x_n}(a * \check{h}_n) = (h_n * \check{a})(x_n{}^{-1}) = 1_W * \check{a}(e) > 0$ for all n if we take W and a properly. This is a contradiction.

Conversely, let G be compact. Suppose $T: PF_p(G) \longrightarrow B_p(G)$ is any bounded linear operator. To show that T is weakly compact, let $f_n \in PF_p(G)$ with $\|f_n\| \leq 1$. Since G is compact, $B_p(G)$ has the RNP by Theorem 2 of Granirer [11]. So the predual $PF_p(G)$ has no copy of ℓ^1. By applying Rosenthal's ℓ^1-theorem, we can assume that $\{f_n\}$ is a weak Cauchy sequence by passing to a subsequence. Next, we show that $\{T(f_n)\}$ is a weak Cauchy sequence. Let $\alpha \in B_p(G)^*$. Define $\tilde{\alpha} \in PF_p(G)^*$ by $\tilde{\alpha}(f) = \langle \alpha, T(f) \rangle$. Then $\tilde{\alpha}$ is bounded since both T and α are bounded. So $\{\tilde{\alpha}(f_n)\}$ is a Cauchy sequence, *i.e.* $\{\langle \alpha, T(f_n) \rangle\}$ is a Cauchy sequence. Since G is compact, $A_p(G) = B_p(G)$ is weakly sequentially complete by Lemma 18 of Granirer [10]. Thus there is an $b \in B_p(G)$ such that $T(f_n) \longrightarrow b$ weakly in $B_p(G)$. ∎

COROLLARY 5.4. *Let G be a discrete group. Then every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is compact if and only if G is finite.*

REMARK 5.5. For discrete groups, this generalizes the equivalence $(a) \Leftrightarrow (h)$ in Theorem 4.5 of Lau and Ülger [17], where this result for $p = 2$ was proved by using C^*-algebraic techniques.

REFERENCES

1. R. Arens, *Operations induced in function classes*, Monatsh. Math. **55**(1951), 1–19.

2. ———, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2**(1951), 839–848.

3. R. Bourgin, *Geometric aspectsof convex sets with the Radon-Nikodym property*, Lecture Notes in Math. **993**, Springer-Verlag, 1983.

4. L. J. Bunce, *The Dunford-Pettis property in the predual of a Von Neumann algebra*, Proc. Amer. Math. Soc. **116**(1992), 99–100.

5. M. Cowling, *An application of Littlewood-Paley theory in harmonic analysis*, Math. Ann. **241**(1979), 83–96.

6. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Math., Springer-Verlag, New York, 1984.

7. ———, *A survey of results related to the Dunford-Pettis property*, Contemporary Math. **2**(1980), 15–60.

8. J. Duncan and S. A. R. Hosseiniun, *The second dual of Banach algebra*, Proc. Roy. Soc. Edinburgh **84A**(1979), 309–325.

9. B. Forrest, *Arens regularity and discrete groups*, Pacific J. Math. **151**(1991), 217–227.

10. E. E. Granirer, *On some spaces of linear functionals on the algebras $A_p(G)$ for locally compact groups*, Colloq. Math. **52**(1987), 119–132.

11. ———, *An application of the Radon-Nikodym property in harmonic analysis*, Bull. Un. Mat. Ital. B(5)**18** (1981), 663–671.
12. E. E. Granirer and M. Leinert, *On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra $B(G)$ and of the measure algebra $M(G)$*, Rocky Mountain J. Math. **11**(1981), 459–472.
13. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, New York, 1969.
14. C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23**(1973), 91–123.
15. E. Hewitt and K. Ross, *Abstract harmonic analysis, Vol. I*, Springer-Verlag, Berlin, 1963.
16. R. Huff, *The Radon-Nikodym property*, Contemporary Math. **2**(1980), 75–89.
17. A. T. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. **337**(1993), 321–359.
18. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, New York, 1984.
19. T. W. Palmer, *Classes of nonabelian, noncompact locally compact groups*, Rocky Mountain J. Math. **8**(1973), 683–741.
20. A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, Rhode Island, 1988.
21. J. P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984.
22. K. Taylor, *Geometry of Fourier algebras and locally compact groups with atomic representations*, Math. Ann. **262**(1983), 183–190.
23. A. Ülger, *Arens regularity sometimes implies the RNP*, Pacific J. Math. **143**(1990), 377–399.

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