SIX AND SEVEN DIMENSIONAL NON-LATTICE SPHERE PACKINGS

John Leech

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The densest lattice packings of equal spheres in Euclidean spaces E_n of n dimensions are known for $n \leqslant 8$. However, it is not known for any $n \geqslant 3$ whether there can be any non-lattice sphere packing with density exceeding that of the densest lattice packing. W. Barlow described [1] a non-lattice packing in E_3 with the same density as the densest lattice packing, and I described [6] three non-lattice packings in E_5 which also have this property. In this note I describe a non-lattice packing in E_6 and two in E_7 which are also as dense as the densest lattice packings; these packings are all obtained by a uniform construction.

Consider a lattice packing in E_n, not necessarily the densest possible, and define a <u>layer</u> of spheres in E_{n+1} to be a set of spheres whose centres form a lattice in E_n, so that their cross section in the flat containing their centres is the chosen lattice packing in E_n. We consider packings in E_{n+1} made by stacking such layers. To make these as dense as possible, we require to have the centre flats of the layers as close together as possible, so we place the spheres of each layer opposite to the points of the adjacent layer most distant from the centres of spheres, unless this would result in the spheres of two layers adjacent to an intermediate layer overlapping each other. However, the spheres may not be opposite to all the points at this maximum distance, and when they are not, it may be possible to construct non-lattice packings by suitably stacking the layers. Since the layers are always lattice packed, if one sphere is opposite to a point at maximum distance from centres in the adjacent layer, so are all spheres in its layer.

This construction is clearly illustrated in the case $n = 2$. First we consider layers packed in square lattices. The points most distant from the vertices of the lattice are the centres of squares, so we place each layer with its spheres opposite to the squares of the adjacent layer. As the spheres and the squares are equally numerous, there is no choice here, and we obtain the densest lattice packing in E_3. Now consider layers packed in triangular lattices. For maximum density we place the spheres of each layer opposite to the centres of triangles of the adjacent layer, but now we have a choice, as the triangles are twice as numerous as the spheres. If we regard the triangles as coloured black and white alternately, the spheres of each layer can be placed opposite to either all the black triangles or all the white triangles of the adjacent layer. If the layers are so stacked that the spheres in the two layers adjacent to any layer are opposite to triangles of different colours, then the densest lattice packing is obtained. But if the adjacent spheres on both sides of each layer are opposite to triangles of the same colour, then the uniform non-lattice packing is obtained. Clearly these packings are of the same density. If we require our packings to be uniform, there is no further choice, as all layers have to be fitted alike, and this construction gives precisely these two packings.

We now consider packings in E_{n+1} made by stacking layers whose centres form the lattice $h\delta_{n+1}$ in E_n, i.e. the alternate vertices of the regular cubic lattice δ_{n+1}. (For details and bibliography see [2, pp. 395,414] and [3, pp. 236-238].) These themselves give the densest lattice packings in E_n for $n = 3, 4, 5$, but not for other values of n. For all values of n, each sphere touches $2n(n-1)$ others in each layer. For $n \geq 3$ the cells of the lattice are of two kinds: each omitted vertex of the cubic lattice is the centre of a cell β_n, while inscribed in each cube of the lattice is a cell $h\gamma_n$. The latter cells are twice as numerous as the former, since they are inscribed in all the cubes while the former are centred in only half of the equally numerous vertices of the cubic lattice. We shall regard the cells $h\gamma_n$ as being coloured black and white alternately, corresponding to "chessboard" colouring of the cubes.

For $n = 3$ the cells β_3 (octahedra) are larger than the cells $h\gamma_3 = \alpha_3$ (tetrahedra), so to stack such layers for maximum density we place the spheres of each layer opposite to octahedra of the adjacent layer. As the octahedra and spheres are equally numerous, there is no choice here, and we arrive uniquely at the densest lattice packing in E_4.

For $n = 4$, the cells β_4 and $h\gamma_4$ are the same, the lattice $h\delta_5$ being the regular honeycomb $\{3, 3, 4, 3\}$. So in this case we have a threefold choice for placing each layer: relatively to $h\delta_5$ the spheres of each layer may be placed opposite to the omitted vertices

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or the black cells or the white cells. I have analysed this case in [6], finding three distinct uniform non-lattice packings of the same density as the densest lattice packing. In each of these, each sphere touches 40 others.

For $n > 4$, the cells $h\gamma_n$ are larger than the cells β_n, so for maximum density in E_{n+1} we stack the layers with the spheres of each layer opposite to the cubes of the adjacent layers. As these cubes are twice as numerous as the spheres, we have a choice at each stage whether we place the spheres opposite to black or white cubes. Thus the centres of the spheres in any one layer may or may not lie on the joins of centres of spheres in the layers adjacent to it. If every layer is such that they do so, a lattice packing will be obtained, but if they are such that this is not so, a non-lattice packing will be obtained. In a uniform packing, all layers must be alike in this respect, so we have uniquely determined a lattice packing and a uniform non-lattice packing.

For $n = 5$ we obtain the densest lattice packing in E_6 when the spheres of the two layers adjacent to each layer are opposite to cubes of different colours, and the uniform non-lattice packing when they are opposite to cubes of the same colour. In both cases each sphere touches 40 others in its own layer and 16 in each adjacent layer, a total of 72.

For $n = 6$ the roles are reversed: the lattice packing is that in which the spheres of the two layers adjacent to each layer are opposite to cubes of the same colour. This has a remarkable consequence. In this case the spheres of each layer touch the flats of centres of adjacent layers, and consequently in the lattice packing, but not in the non-lattice packing, the spheres in the two layers adjacent to a given layer touch each other in the flat of centres of the given layer. Thus although in both packings each sphere touches 60 others in its own layer and 32 others in each adjacent layer, a total of 124, in the lattice packing each sphere also touches one in each of the two layers two away from it, bringing the total to 126.

It is also possible to stack the layers so that each sphere touches 125 others, each sphere touching one in a layer two away on one side only. This packing, however, is not uniform. Half of the layers are such that their flats of centres contain points of contact of spheres of adjacent layers, while the others contain no such points, and the spheres in layers of opposite kinds are not equivalent.

For $n = 7$, the spheres of each layer cut the flats of centres of the adjacent layers, so we can obtain a high density packing in E_8 only by placing the two layers adjacent to a given layer with the spheres opposite to cubes of different colours. We thus arrive uniquely at the densest lattice packing. Each sphere is opposite to a β_7 cell in the

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layer two away from it, and so it touches 84 others in its own layer, 64 in each adjacent layer, and 14 in each layer two away from it, a total of 240.

For $n = 8$, the spheres of a layer can be fitted exactly into the interstices between the spheres of an "adjacent" layer, producing the densest lattice in E_8 again instead of a packing in E_9. Clearly such layers will not produce dense packings for any $n > 8$.

As remarked above, the lattice packing $h\delta_{n+1}$ is not the densest in E_n for any $n \geq 6$, and we now examine the use of layers of spheres based on the densest lattice packings. It happens that in E_6, as in E_2, the cells of the densest lattice are all the same and are twice as numerous as the vertices, so there are two ways of stacking such layers in E_7, as in E_3. This may be seen as follows. Basing our coordinates for the densest lattice in E_6 on its construction from layers based on $h\delta_6$ in E_5, we may take the vertices of the lattice to have coordinates $(x_1, x_2, x_3, x_4, x_5, x_6\sqrt{3})$, where $x_1, \ldots, x_6$ are integers, all even or all odd, with their sum divisible by 4. Thus the 72 vertices adjacent to the origin comprise 40 with two of $x_1, \ldots, x_5$ being ± 2 and the others and x_6 being 0, and 32 with $x_1, \ldots, x_6$ being ± 1 with an odd number of each sign. The cells each have 27 vertices, and the centres of the cells are points which have the same coordinates as the vertices except for adding or subtracting $4/\sqrt{3}$ to the last coordinate. The two sets of cells correspond to whether $4/\sqrt{3}$ was added or subtracted. Deeming these to be black and white respectively, we obtain the lattice packing when the adjacent layers have their spheres opposite to cells of different colours, otherwise the uniform non-lattice packing. In this case the spheres do not cut or touch the flats of centres of adjacent layers, so in both packings each sphere touches 72 in its own layer and 27 in each adjacent layer, a total of 126. We thus have two uniform non-lattice packings in E_7 of the same density as the densest lattice packing, each sphere of one touching 126 others while each sphere of the other touches only 124 others.

The densest lattice in E_7 has cells of two kinds, the larger of which have 56 vertices, and these cells are as numerous as the vertices. So the densest packing in E_8 which can be made from these layers is uniquely determined, and is the densest lattice packing in E_8 again. Each sphere touches the flats of centres of the adjacent layers, and so each sphere touches 126 others in its own layer, 56 in each adjacent layer, and one in each layer two away from it, totalling 240.

The cells of the densest lattice in E_8 are α_8's and β_8's, both of which are more numerous than vertices, so lattice and non-lattice packings of equal density can be constructed in E_9 from layers of this type. These are the densest known, but have not been proved to be the densest possible even for lattice packings. Similarly the densest known packings in E_{10} can be obtained from these E_9 layers, but the densest known packing in E_{11} is not obtainable from the densest known in E_{10} in this way. These and other packings for $9 \leqslant n \leqslant 24$ and for $n = 2^m$ are discussed in [4;5].

It is only for $n \leqslant 8$ that the densest lattice packings are known, and we have now constructed equally dense non-lattice packings for $n = 3, 5, 6, 7$, i.e. whenever n is not a power of 2. It remains an interesting field for speculation and study whether these are possible for $n = 4$ or 8, and whether denser non-lattice packings are possible for any of these values of n.

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University of Stirling
Stirling, Scotland

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