ON ORTHOMODULAR POSETS

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1. Introduction

Let S be a poset with a greatest element 1. We denote order in S by '$\leqq$' and, whenever they exist in S, l.u.b and g.l.b by '$\vee$' and '$\wedge$' respectively. An orthocomplementation of S is a bijection $\omega : S \to S$ such that $x \vee x\omega$ exists for each x in S and (i) $x\omega\omega = x$, (ii) $x \leqq y$ implies $y\omega \leqq x\omega$ and (iii) $x \vee x\omega = 1$. If a poset S admits an orthocomplementation ω we call the pair (S, ω) an orthoposet. If (S, ω) is an orthoposet then $0 = 1\omega$ is the least element of S and $x \wedge x\omega = 0$ for each x in S. Elements x and y in an orthoposet are orthogonal when $x \leqq y\omega$, in this case we sometimes write $x \perp y$. Note that $x \perp 0$ for any x and $x \perp y$ implies $y \perp x$. If X is a non-empty subset of S and x in S is orthogonal to each element of X we write $x \perp X$. A subset X of S is said to be orthogonal when $x \perp y$ for any $x \neq y$ in X. In particular the empty set $\square$ and a set which consists of a single element of S are orthogonal sets. An orthoposet which is in fact a lattice is called an ortholattice. An orthoposet (S, ω) is said to be an orthomodular poset when

(1.1) $$x \perp y \Rightarrow x \vee y \text{ exists in } S,$$

and

(1.2) $$x \geqq y\omega \quad \text{and} \quad x \wedge y = 0 \Rightarrow x = y\omega.$$

In an orthomodular poset any finite orthogonal subset has a l.u.b and $x \wedge y = (x\omega \vee y\omega)\omega$ exists whenever $x \geqq y\omega$. In the definition of an orthomodular poset one may replace the implication (1.2) by any one of the three following implications which, in the presence of (1.1), are each equivalent to (1.2).

$$x \leqq y\omega \quad \text{and} \quad x \vee y = 1 \Rightarrow x = y\omega,$$
$$x \leqq y\omega \Rightarrow (x \vee y) \wedge y\omega = x,$$
$$x \leqq y \Rightarrow x \vee (y \wedge x\omega) = y.$$

The asserted equivalence of these implications is contained essentially in Birkhoff [1] who, however, considers only the case of an ortholattice. An orthomodular poset S is said to be completely orthomodular when any orthogonal subset of S has a l.u.b in S. An orthomodular poset which is in fact a lattice is called an orthomodular lattice.

Interest in orthomodular posets and lattices is stimulated and, in part, motivated by the fact that in the Mackey-von Neumann formulation of non-relativistic quantum mechanics, Mackey [2], the quantum logic has the structure of a completely orthomodular poset. For some purposes it is convenient to assume that the quantum logic is a complete orthomodular lattice, and the convenience of this assumption leads one to enquire if a completely orthomodular poset can be embedded in a complete orthomodular lattice. For instance one can ask if the completion by cuts of an o.m poset is orthomodular. In answer to this question we show firstly that if the completion by cuts of an orthomodular poset is orthomodular then the poset has a certain property which we call the A-property. We then show that if a completely orthomodular poset does have the A-property then it is a complete orthomodular lattice. In particular since, as we show below, any orthomodular lattice has the A-property it follows that a completely orthomodular poset which is a lattice must be a complete lattice.

The axiomatic formulation of non-relativistic quantum mechanics leads to a number of related problems. In this connection A. Ramsay has asked if the completion by cuts of any orthomodular lattice is itself orthomodular (Birkhoff [1], problem 36). With this problem in mind we obtain a necessary and sufficient condition for the completion by cuts of an orthoposet to be orthomodular, although we make no use of the result in this paper.

2. The A-property

An orthomodular poset (S, ω) will be said to have the A-property when for any non-empty set $\{a_j : j \in J\}$ of elements of S with $\bigwedge \{a_j : j \in J\} = 0$ and any element b of S such that $b \perp a_j$, for each j in J, the set $\{a_j \vee b : j \in J\}$ has a g.l.b in S. In an orthomodular lattice the A-property always holds, in fact we have the stronger result.

Proposition (2.1). *Let (S, ω) be an orthomodular lattice. Let $\{a_j : j \in J\}$ be a non-empty set of elements of S such that $a = \bigwedge \{a_j : j \in J\}$ exists in S. If b is any element of S such that $b \perp a_j$ for each j in J then*

$$\bigwedge \{a_j \vee b : j \in J\} = a \vee b$$

exists in S.

Proof. If c is any lower bound to the set $\{a_j \vee b : j \in J\}$ then so is $c \vee b \vee a$. Since S is orthomodular

$$(c \vee b \vee a) \wedge b\omega \leqq (a_j \vee b) \wedge b\omega = a_j$$

for each j in J. Since $b\omega \geqq a_j \geqq a$ we have

$$a \leqq (c \vee b \vee a) \wedge b\omega \leqq \bigwedge a_j = a.$$

Thus $(c \vee b \vee a) \wedge b\omega = a$ and so, using orthomodularity again we have

$$b \vee a = b \vee \{(c \vee b \vee a) \wedge b\omega\} = c \vee b \vee a$$

whence $c \leqq b \vee a$. This shows that $b \vee a$ is the g.l.b of the set $\{a_j \vee b : j \in J\}$.

A dual form of this result is worth stating as the

COROLLARY. *If (S, ω) is an orthomodular lattice and $\bigvee c_j = c$ exists in S then*

$$\bigvee (c_j \vee x\omega) \wedge x = (c \vee x\omega) \wedge x$$

for any x in S.

We conclude this section with

LEMMA (2.1). *Let (S, ω) be an orthomodular poset with the A-property. Let a_j, $j \in J$ and b be as defined above, that is $\bigwedge a_j = 0$ and $b \perp a_j$, then*

$$\bigwedge \{a_j \vee b : j \in J\} = b.$$

PROOF. By the assumed A-property $c = \bigwedge \{a_j \vee b : j \in J\}$ exists in S. Since $c \geqq b$ the meet $c \wedge b\omega$ exists in S, using the orthomodularity of S and the fact that $a_j \perp b$ we have

$$c \wedge b\omega \leqq (a_j \vee b) \wedge b\omega = a_j, \qquad j \in J.$$

Since $\bigwedge a_j = 0$ we have $c \wedge b\omega = 0$, since $c \geqq b$ and S is orthomodular we have $c = b$.

3. The completion by cuts of an orthoposet

Let (S, ω) be an orthoposet. For any non-empty subset X of S let $X\sigma$ be its set of lower bounds in S and let $X\tau$ be its set of upper bounds in S. If $X = \{x\}$ consists of a single element of S we write $x\sigma$ and $x\tau$. For any non-empty subset X of S we write $X\omega = \{x\omega : x \in X\}$.

LEMMA (3.1). *Let (S, ω) be an orthoposet and write $\Omega = \omega\sigma$, then $0\Omega = S$, $S\Omega = 0$ and, for any non-empty subsets X, Y of S (1) $X\tau = X\omega\sigma\omega$, (2) $X \subseteqq Y$ implies $Y\Omega \subseteqq X\Omega$, (3) $X \subseteqq X\Omega\Omega$, (4) $X\Omega\Omega\Omega = X\Omega$ and (5) $X\tau\sigma = X\Omega\Omega$.*

We omit the proof of the statements in the lemma since they are straightforward and easily proved, for example Maclaren [3]. It follows, as shown by Maclaren, that $\mathscr{C} = \{X\tau\sigma : \square \subset X \subseteqq S\}$, the completion by cuts of S, is orthocomplemented by Ω, that is $(\mathscr{C}, \Omega)$ is a complete ortholattice. We prove

LEMMA (3.2). *Let (S, ω) be an orthomodular poset with the A-property,* *then for X in $\mathscr{C}$ and y in S*

$$X \supseteqq y\sigma \quad \text{and} \quad X \cap y\Omega = \{0\} \Rightarrow X = y\sigma.$$

Proof. Assume the antecedent in the implication above but suppose, contrary to what it is asserted to imply, that there is an x in X such that $x \nleq y$. Since x and y are in X and X in $\mathscr{C}$ the set $(x \cup y)\tau\sigma$ is a subset of X. Since $\{0\}$ is the least element of the lattice $\mathscr{C}$ and since lattice meet in $\mathscr{C}$ is set intersection

$$\{0\} \subseteqq (x \cup y)\tau\sigma \cap y\omega\sigma \subseteqq X \cap y\Omega = \{0\}.$$

Thus

$$(x \cup y)\tau\sigma \cap y\omega\sigma = \{0\}.$$

If z is in $(x \cup y)\tau$ then $z \geqq y$ and so $z \wedge y\omega$ exists in S. But

$$\{z \wedge y\omega : z \in (x \cup y)\tau\}\sigma \subseteqq (x \cup y)\tau\sigma \cap y\omega\sigma$$

and so

$$\bigwedge \{z \wedge y\omega : z \in (x \cup y)\tau\} = 0.$$

By orthomodularity

$$y \vee (z \wedge y\omega) = z, \forall z \in (x \cup y)\tau$$

and $y \leqq z\omega \vee y = (z \wedge y\omega)\omega$ for each z in $(x \cup y)\tau$, we have then $y \perp z \wedge y\omega$ for each z in $(x \cup y)\tau$. Thus from lemma (2.1)

$$\begin{aligned} y &= \bigwedge \{y \vee (z \wedge y\omega) : z \in (x \cup y)\tau\} \\ &= \bigwedge \{z : z \in (x \cup y)\tau\}. \end{aligned}$$

It follows that $x \leqq y$ contrary to assumption. This contradiction establishes the desired result.

We prove now

Proposition (3.1). *Let (S, ω) be an orthomodular poset. If $(\mathscr{C}, \Omega)$ is orthomodular then (S, ω) has the A-property.*

Proof. Let $\{a_j : j \in J\}$ be a non-empty subset of S whose g.l.b is 0. Let b in S be such that $b \perp a_j$ for each j in J. We have to show that the set $\{a_j \vee b : j \in J\}$ has a g.l.b in S. Each set $a_j\sigma$ is in $\mathscr{C}$ and $\bigcap \{a_j\sigma : j \in J\} = \{0\}$. The set $b\sigma$ is in $\mathscr{C}$ and since $b \leqq a_j\omega$ we have

$$b\sigma \subseteqq a_j\omega\sigma = a_j\sigma\omega\sigma = a_j\sigma\Omega,$$

that is $b\sigma$ is orthogonal in $\mathscr{C}$ to each of the $a_j\sigma$. Since $\mathscr{C}$ is an orthomodular lattice by assumption it has the A-property and so

$$\bigcap \{(a_j \cup b)\tau\sigma : j \in J\} = b\sigma.$$

If c in S is a lower bound to the set $\{a_j \vee b : j \in J\}$ then c belongs to each of the sets $(a_j \cup b)\tau\sigma$ and so $c \leqq b$. But b is itself a lower bound to the set

$\{a_j \vee b : j \in J\}$ and is therefore its g.l.b. This establishes the desired result. Let (S, ω) be an orthoposet, let X be a subset of S and let Y be an orthogonal subset of X. The set union of a non-decreasing chain of orthogonal subsets of X each of which contains Y is itself an orthogonal subset of X containing Y. It follows that there exist maximal orthogonal subsets of X which contain the orthogonal set Y. We say that an orthoposet (S, ω) has the B-property when for any non-empty subset X of S and any maximal orthogonal subset M of $X\tau\sigma$ one has $M\tau\sigma = X\tau\sigma$. We prove that

PROPOSITION (3.2). *The completion by cuts of an orthoposet is orthomodular if and only if the orthoposet has the B-property.*

PROOF. Let (S, ω) be an orthoposet and let $(\mathscr{C}, \Omega)$ be its completion by cuts. We show that the B-property in S implies orthomodularity in $\mathscr{C}$ by showing that if (S, ω) does have the B-property then

$$X, Y \in \mathscr{C}, X \subseteqq Y\Omega \ \& \ (X \cup Y)\tau\sigma = S \Rightarrow X = Y.$$

To do so suppose that (S, ω) has the B-property and assume the antecedent in the implication above. Let M be a maximal orthogonal subset of X and let P be a maximal orthogonal subset of Y. Then

$$S = (X \cup Y)\tau\sigma = (M\tau\sigma \cup P\tau\sigma)\tau\sigma = (M \cup P)\tau\sigma.$$

It follows that $M \cup P$ is a maximal orthogonal subset of S for

$$x \perp (M \cup P) \Rightarrow x \in (M \cup P)\Omega = \{0\}.$$

Let $N \supseteqq M$ be a maximal orthogonal subset of $Y\Omega$ which contains M then $N \perp P$ and

$$(N \cup P)\tau\sigma = (Y\Omega \cup Y)\tau\sigma = S.$$

Since $M \cup P$ is maximal and $M \cup P \subseteqq N \cup P$ we must have $M = N$ that is $X = M\tau\sigma = N\tau\sigma = Y\Omega$. This establishes the orthomodularity of $\mathscr{C}$. Now assume that $\mathscr{C}$ is orthomodular, we show that S has the B-property. Let X be any non-empty subset of S and let M be a maximal orthogonal subset of $X\tau\sigma$. Since M is maximal we have $M\Omega \cap X\tau\sigma = \{0\}$, but $X\tau\sigma \supseteqq M\tau\sigma = M\Omega\Omega$ and so, by orthomodularity in $\mathscr{C}$, $X\tau\sigma = M\tau\sigma$. This is the desired result.

4. Completely orthomodular posets with the A-property

We prove

LEMMA (4.1). *Let (S, ω) be an orthomodular poset with the A-property. Let $X \subseteqq S$ be in $\mathscr{C}$, the completion by cuts of S, and let M be a maximal orthogonal subset of X. If $x = \bigvee M$ exists in S then $X = x\sigma$.*

PROOF. Let X and M be as defined in the statement of the lemma. Since $M \subseteqq X$ and X is in $\mathscr{C}$ the element x is in X and so $x\sigma \subseteqq X$. If y is in $X \cap x\Omega$ then $y \leqq x\omega$ and so y is orthogonal to each element of M. But y is in X and M is a maximal orthogonal subset of X thus $y = 0$. We have then both $X \supseteqq x\sigma$ and $X \cap x\Omega = 0$ and, since S has the A-property it follows from lemma (3.2) that $X = x\sigma$. This is the desired result.

We are now able to establish our final result, namely

PROPOSITION (4.1). *A completely orthomodular poset with the A-property is a complete orthomodular lattice.*

PROOF. Let (S, ω) be a completely orthomodular poset with the A-property. Since $\bigvee Y$ exists in S for any orthogonal subset Y of S, it follows from lemma (4.1) that each element X of $\mathscr{C}$ is of the form $x\sigma$ for a unique x in S. That is S is isomorphic with its completion by cuts and so, in addition to being orthomodular, is also a complete lattice.

References

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