ROSENTHAL SETS AND THE RADON-NIKODYM PROPERTY

PATRICK N. DOWLING

(Received 5 April 1991; revised 24 July 1991)

Communicated by P. G. Dodds

Abstract

Let \( X \) be a complex Banach space, \( G \) a compact abelian metrizable group and \( \Lambda \) a subset of \( \hat{G} \), the dual group of \( G \). If \( X \) has the Radon-Nikodym property and \( L^\infty(\Lambda; X) \) is separable, then \( L^\infty(G, X) \) has the Radon-Nikodym property. One consequence of this is that \( C(\Lambda, X) \) has the Radon-Nikodym property whenever \( X \) has the Radon-Nikodym property and the Schur property and \( \Lambda \) is a Rosenthal set. A partial stability property for products of Rosenthal sets is also obtained.


Keywords and phrases: Rosenthal sets, Radon-Nikodym property.

1. Introduction

For Banach spaces \( X \) and \( Y \) we denote by \( \mathcal{L}(X, Y) \) the Banach space of all bounded linear operators from \( X \) into \( Y \). In this set-up, Pettis's last theorem states: if \( \mathcal{L}(X, Y^*) \) is separable, then \( \mathcal{L}(X, Y^*) \) has the Radon-Nikodym property [3, p. 165]. Notice that if \( \mathcal{L}(X, Y^*) \) is separable, then \( X^* \) and \( Y^* \) are separable and so \( X^* \) and \( Y^* \) have the Radon-Nikodym property.

In this note, we will prove a result which is a variant of Pettis's last theorem. In our case, we will replace \( X \) by a particular quotient of \( L^1 \) and \( Y^* \) by an arbitrary Banach space with the Radon-Nikodym property. To be more specific, if \( G \) is a compact abelian metrizable group and \( \Lambda \) is a Rosenthal subset of \( \hat{G} \), then \( L^\infty(\Lambda; G) \) is a separable dual space. If we
let $\Lambda' = \{ y \in \widehat{G} : \overline{y} \notin \Lambda \}$, then $L_\Lambda^\infty(G)$ is the dual of $L_\Lambda^1(G)/L_\Lambda^1(G)$. We will show that $\mathcal{L}(L_1^1(G)/L_\Lambda^1(G), X)$ has the Radon-Nikodym property whenever $X$ has the Radon-Nikodym property and $\mathcal{L}(L_1^1(G)/L_\Lambda^1(G), X)$ is separable. In fact, a close analysis of Lemma 4 of [2, p. 62] yields that proving the above result is equivalent to proving that $L_\Lambda^\infty(G, X)$ has the Radon-Nikodym property whenever $X$ has the Radon-Nikodym property and $L_\Lambda^\infty(G, X)$ is separable. This is the first result we prove in the next section.

One consequence of these results is that if $\Lambda$ is a Rosenthal subset of $\widehat{G}$ and $X$ is a complex Banach space with the Radon-Nikodym property and the Schur property, then $C_\Lambda(G, X)$ has the Radon-Nikodym property. It is unknown, at the moment, if the restriction of $X$ having the Schur property is necessary. Actually, a related question, which is also still unanswered, is the following: if $G_1$ and $G_2$ are two compact abelian metrizable groups and $\Lambda_1$ and $\Lambda_2$ are two Rosenthal subsets of $\widehat{G}_1$ and $\widehat{G}_2$, respectively, is $\Lambda_1 \times \Lambda_2$ a Rosenthal subset of $\widehat{G}_1 \times \widehat{G}_2$? However, we will be able to show that if $\Lambda_1$ is a Rosenthal set and $\Lambda_2$ is such that $L_\Lambda^\infty(G_2)$ has the Schur property, then $\Lambda_1 \times \Lambda_2$ is a Rosenthal set. The question of whether the product of two Rosenthal sets is again a Rosenthal set was first raised by F. Lust-Piquard.

2. The results

Throughout this section, $G$ will denote a compact abelian metrizable group, $\mathcal{B}(G)$ denotes the $\sigma$-algebra of Borel subsets of $G$ and $\lambda$ is normalised Haar measure on $G$. The dual group of $G$ is denoted by $\widehat{G}$. If $f$ is a function defined on $G$ and $\gamma \in \widehat{G}$, we define $\hat{f}(\gamma)$ by

$$\hat{f}(\gamma) = \frac{1}{\lambda(G)} \int_G \overline{\gamma(g)} f(g) d\lambda(g).$$

For a subset $\Lambda$ of $\widehat{G}$, we define

$$C_\Lambda(G) = \{ f \in C(G) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}$$

and

$$L_\Lambda^\infty(G) = \{ f \in L^\infty(G) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \}.$$

A subset $\Lambda$ of $\widehat{G}$ is called a Rosenthal set if $L_\Lambda^\infty(G) = C_\Lambda(G)$. A result of Lust-Piquard [5] says that $\Lambda$ is a Rosenthal set if and only if $C_\Lambda(G)$ has the Radon-Nikodym property.

For a complex Banach space $X$, we define

$$L_\Lambda^\infty(G, X) = \{ f \in L^\infty(G, X) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda \},$$
where $L^\infty(G, X)$ is the space of all (equivalence classes of) $X$-valued Bochner integrable functions defined on $G$ that are essentially bounded. $C^\Lambda(G, X)$ can be defined in a similar manner.

**Theorem.** Let $G$ be a compact abelian metrizable group, let $\Lambda$ be a subset of $\hat{G}$ and let $X$ be a complex Banach space. If $X$ has the Radon-Nikodym property and $L^\infty_\Lambda(G, X)$ is separable, then $L^\infty(G, X)$ has the Radon-Nikodym property.

**Proof.** To show that $L^\infty_\Lambda(G, X)$ has the Radon-Nikodym property it suffices to show that every bounded linear operator from $L^1[0, 1]$ into $L^\infty_\Lambda(G, X)$ is Bochner representable [2]. So, let $T: L^1[0, 1] \to L^\infty_\Lambda(G, X)$ be a bounded linear operator. For a Lebesgue measurable subset $A$ of $[0, 1]$ and $B \in \mathcal{B}(G)$, define

$$\mu(A \times B) = \int_B T(1_A)(g) d\lambda(g).$$

If we let $\mathcal{R}$ denote the algebra generated by sets of the form $A \times B$, where $A$ is a Lebesgue measurable subset of $[0, 1]$ and $B \in \mathcal{B}(G)$, then it is easily shown that $\mu$ is a finitely additive $X$-valued measure on $\mathcal{R}$. Also,

$$\|\mu(A \times B)\| \leq \int_B \|T(1_A)(g)\|_{X} d\lambda(g) \leq \int_B \|T(1_A)\|_{L^\infty_\Lambda(G, X)} d\lambda(g) = \lambda(B) \|T(1_A)\|_{L^\infty_\Lambda(G, X)} \leq \lambda(B) \|T\|_{L^1[0, 1]} = \lambda(B) \|T\|_{m(A)} = \|T\|(m \times \lambda)(A \times B),$$

where $m$ is Lebesgue measure on $[0, 1]$. Thus $\|\mu(R)\| \leq \|T\|(m \times \lambda)(R)$ for all $R \in \mathcal{R}$.

If we let $\mathcal{A}$ denote the $\sigma$-algebra of Lebesgue measurable subsets of $[0, 1]$, then the product $\sigma$-algebra, $\mathcal{A} \times \mathcal{B}(G)$, is generated by $\mathcal{R}$. Also, $m \times \lambda$ is a non-negative finite countably additive measure on $\mathcal{A} \times \mathcal{B}(G)$. For $E, F \in \mathcal{A} \times \mathcal{B}(G)$, define $d(E, F) = (m \times \lambda)(E \Delta F)$. Then $(\mathcal{A} \times \mathcal{B}(G), d)$ is a pseudo-metric space and $\mathcal{R}$ is dense in $(\mathcal{A} \times \mathcal{B}(G), d)$. For $E, F \in \mathcal{R}$, $\mu(E) - \mu(F) = \mu(E \setminus F) - \mu(F \setminus E)$. Therefore,

$$\|\mu(E) - \mu(F)\| \leq \|\mu(E \setminus F)\| + \|\mu(F \setminus E)\| \leq \|T\|(m \times \lambda)(E \setminus F) + \|T\|(m \times \lambda)(F \setminus E) = \|T\|(m \times \lambda)(E \Delta F) = \|T\|d(E, F).$$

Consequently, $\mu: (\mathcal{R}, d) \to (X, \| \cdot \|)$ is a Lipschitz function and since $(\mathcal{R}, d)$ is dense in $(\mathcal{A} \times \mathcal{B}(G), d)$ there is a Lipschitz extension of $\mu$, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S1446788700037125.
\( \overline{\mu} : (\mathcal{A} \times \mathcal{B}(G), d) \to (X, \| \cdot \|) \), such that \( \overline{\mu}(R) = \mu(R) \) for all \( R \in \mathcal{B} \). Thus, \( \overline{\mu} \) is an \( X \)-valued countably additive measure on \( \mathcal{A} \times \mathcal{B}(G) \) and \( \| \overline{\mu}(S) \| \leq \| T \| (m \times \lambda)(S) \) for all \( S \in \mathcal{A} \times \mathcal{B}(G) \). Hence, \( \overline{\mu} \) is an \( X \)-valued measure of bounded average range and since \( X \) has the Radon-Nikodym property there exists \( F \in L^\infty([0, 1] \times G, X) \) such that

\[
\overline{\mu}(S) = \int_S F(t, g) d(m \times \lambda)(t, g)
\]

for all \( S \in \mathcal{A} \times \mathcal{B}(G) \).

In particular, for \( A \in \mathcal{A} \) and \( B \in \mathcal{B}(G) \),

\[
\int_B T(1_A)(g) d\lambda(g) = \mu(A \times B) = \overline{\mu}(A \times B) = \int_{A \times B} F(t, g) d(m \times \lambda)(t, g)
\]

\[
= \int_B \int_A F(t, g) dm(t) d\lambda(g).
\]

Now, for \( x^* \in X^* \), define

\[
T_{x^*} : L^1([0, 1]) \to L^\infty_A(G) \quad \text{by} \quad (T_{x^*} f)(g) = x^*((T f)(g))
\]

for \( g \in G \) and \( f \in L^1([0, 1]) \). \( T_{x^*} \) is a bounded linear operator and \( L^\infty_A(G) \) has the Radon-Nikodym property (this is so because \( L^\infty_A(G) \) is a dual space which is separable since it is a subspace of the separable space \( L^\infty(G, X) \)). Therefore, there is \( F_{x^*} \in L^\infty([0, 1], L^\infty_A(G)) \) such that

\[
T_{x^*}(f) = \int_{[0, 1]} f(t) F_{x^*}(t) dm(t)
\]

for all \( f \in L^1([0, 1]) \). In particular, if \( A \in \mathcal{A} \) and \( g \in G \), then

\[
x^*(T(1_A)(g)) = (T_{x^*}(1_A))(g) = \left[ \int_{[0, 1]} 1_A(t) F_{x^*}(t) dm(t) \right](g)
\]

\[
= \int_A (F_{x^*}(t))(g) dm(t).
\]

Thus for \( A \in \mathcal{A} \), \( B \in \mathcal{B}(G) \) and \( x^* \in X^* \)

\[
\int_B x^*(T(1_A)(g)) d\lambda(g) = \int_B \int_A (F_{x^*}(t))(g) dm(t) d\lambda(g).
\]

Consequently, for all \( A \in \mathcal{A} \), \( B \in \mathcal{B}(G) \) and \( x^* \in X^* \)

\[
\int_B \int_A x^*(F(t, g)) dm(t) d\lambda(g) = x^* \left[ \int_B \int_A F(t, g) dm(t) d\lambda(g) \right]
\]

\[
= x^*(\mu(A \times B))
\]

\[
= x^* \left[ \int_B T(1_A)(g) d\lambda(g) \right] = \int_B x^*(T(1_A)(g)) d\lambda(g)
\]

\[
= \int_B \int_A (F_{x^*}(t))(g) dm(t) d\lambda(g).
\]
The function $F_{x^*} \in L^\infty([0, 1], L_\infty^\infty(G))$, so by [4, p. 198], Theorem 17, there exists $H_{x^*} : [0, 1] \times G \to C$ which is $m \times \lambda$-measurable and such that $F_{x^*}(t) = H_{x^*}(t, \cdot)$ for $m$-almost all $t \in [0, 1]$. From this we get that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}(G)$

$$\int_B \int_A x^*(F(t, g))d\lambda(g) = \int_B \int_A H_{x^*}(t, g)d\lambda(g).$$

Therefore $x^*(F(t, g)) = H_{x^*}(t, g)$ for $m \times \lambda$-almost all $(t, g) \in [0, 1] \times G$ (where the exceptional set of measure zero may vary with $x^*$). In particular, for $m$-almost all $t \in [0, 1]$, $x^*(F(t, g)) = H_{x^*}(t, g)$ for $\lambda$-almost all $g \in G$. This yields that for $m$-almost all $t \in [0, 1]$, $x^*(F(t, g)) = F_{x^*}(t)(g)$ for $\lambda$-almost all $g \in G$. Therefore, $x^*(F(t, \cdot))$ is $\lambda$-measurable for $m$-almost all $t \in [0, 1]$, where again we note that the exceptional set of measure zero may vary with $x^*$. However, since $X$ is separable, there is a countable norming set $\{x^*_n\}_{n=1}^\infty$ in the unit ball of $X^*$. From this we have that for all $n \in \mathbb{N}$, $x^*_n(F(t, \cdot))$ is $\lambda$-measurable for $m$-almost all $t \in [0, 1]$. By Corollary 4 of [2, p. 42] we have that $F(t, \cdot)$ is $\lambda$-measurable for almost all $t \in [0, 1]$.

Define $K(t) = \begin{cases} F(t, \cdot) & \text{if } F(t, \cdot) \text{ is } \lambda\text{-measurable}, \\ 0 & \text{otherwise.} \end{cases}$

For each $t \in [0, 1]$, $K(t) : G \to X$ is $\lambda$-measurable and for $m$-almost all $t \in [0, 1]$, $x^*(K(t)(g)) = x^*(F(t, g)) = (F_{x^*}(t))(g)$ for $\lambda$-almost all $g \in G$.

$$\|F_{x^*} \|_{L_\infty([0, 1], L_\infty^\infty(G))} = \|T_{x^*_n}\| \leq \|x^*_n\| \| T \| \leq \| T \|.$$ 

Therefore, for $m$-almost all $t \in [0, 1]$ and for all $n \in \mathbb{N}$, $\|F_{x^*_n}(t)\|_{L_\infty^\infty(G)} \leq \| T \|$. Consequently, for $m$-almost all $t \in [0, 1]$ and for all $n \in \mathbb{N}$, $|F_{x^*_n}(g)| \leq \| T \|$ for $\lambda$-almost all $g \in G$. From this we get that for $m$-almost all $t \in [0, 1]$ and for all $n \in \mathbb{N}$, $|x^*_n(K(t))(g)| \leq \| T \|$ for $\lambda$-almost all $g \in G$. Since $\{x^*_n\}_{n=1}^\infty$ is a norming set we have that for $m$-almost all $t \in [0, 1]$, $\|K(t)(g)\|_X \leq \| T \|$ for $\lambda$-almost all $g \in G$. Thus, for $m$-almost all $t \in [0, 1]$, $\|K(t)\|_{L_\infty^\infty(G, X)} \leq \| T \|$.

Now define $K_1 : [0, 1] \to L_\infty^\infty(G, X)$ by

$$K_1(t) = \begin{cases} K(t) & \text{if } \| K(t) \|_{L_\infty^\infty(G, X)} \leq \| T \|, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \in [0, 1]$, $\gamma \notin \Lambda$ and $x^* \in X^*$

$$x^*\left(K_1(t)(\gamma)\right) = \int_G x^*(K_1(t)(g))\overline{\gamma(g)}d\lambda(g).$$

If $\|K(t)\|_{L_\infty^\infty(G, X)} \leq \| T \|$, then

$$x^*(K_1(t)(\gamma)) = \int_G x^*(K(t)(g))\overline{\gamma(g)}d\lambda(g) = \int_G (F_{x^*}(t))(g)\overline{\gamma(g)}d\lambda(g) = F_{x^*}(t)(\gamma) = 0.$$
since $F_x(t) \in L^\infty_A(G)$. If \( \|K(t)\|_{L^\infty(G, X)} > \|T\| \), then \( x^*(K_1(t)(\gamma)) = 0 \).

Hence \( K_1 : [0, 1] \to L^\infty_A(G, X) \).

At this stage, we need to show \( K_1 \) is \( m \)-measurable. Since \( L^\infty_A(G, X) \) is separable, it suffices to show that \( K_1 \) is scalarly measurable on a total set of linear functionals on \( L^\infty_A(G, X) \) [1]. Note that for \( \gamma \in \widehat{G} \) and \( x^* \in X^* \), the map \( x^* \otimes \gamma : L^\infty_A(G, X) \to \mathbb{C} \) defined by \( (x^* \otimes \gamma)(f) = x^*(f(\cdot))(\gamma) \) is a bounded linear functional. The total set we will use is the set \( \{x_n^* \otimes \gamma : n \in \mathbb{N}, \gamma \in \Lambda\} \) where \( \{x_n^*\} \) is a norming set in \( X^* \). Note that this set is countable since \( \Lambda \) is countable.

For \( n \in \mathbb{N} \) and \( \gamma \in \Lambda \),

\[
(x_n^* \otimes \gamma)(K_1(t)) = \int_G x_n^*(K_1(t)(g))\overline{\gamma(g)}d\lambda(g) = \int_G x_n^*(F(t, g))\overline{\gamma(g)}d\lambda(g).
\]

However, since \( x_n^*(F(t, g)) \) is a measurable function of \( t \), so is the above integral and hence \( (x_n^* \otimes \gamma)(K_1(t)) \) is a measurable function of \( t \). Therefore, \( K_1 \) is an \( m \)-measurable function.

To complete the proof we must show that for all \( A \in \mathcal{A} \), \( T(1_A) = \int_A K_1(t)dm(t) \). For \( A \in \mathcal{A} \) and \( B \in \mathcal{B}(G) \),

\[
\int_BT(1_A)(g)d\lambda(g) = \int_B \int_AF(s, t)dm(t)d\lambda(g) = \int_B \int_AK_1(t)(g)dm(t)d\lambda(g).
\]

Thus, for \( \lambda \)-almost all \( g \in G \),

\[
T(1_A)(g) = \int_AK_1(t)(g)dm(t)
\]

and so as elements of \( L^\infty_A(G, X) \), \( T(1_A) = \int_A K_1(t)dm(t) \). Hence \( L^\infty_A(G, X) \) has the Radon-Nikodym property.

**Remark.** The converse of the theorem is also true: if \( L^\infty_A(G, X) \) has the Radon-Nikodym property and \( X \) is separable, then \( L^\infty_A(G, X) \) is separable. The proof can essentially be found in [5].

**Corollary 1.** Let \( G \) be a compact abelian metrizable group, \( \Lambda \) a subset of \( \widehat{G} \) and \( X \) a complex Banach space with the Radon-Nikodym property. If \( \mathcal{L}(L^1(G)/L^1_A(G), X) \) is separable, then it has the Radon-Nikodym property \( (\Lambda' = \{\gamma : \overline{\gamma} \notin \Lambda\}) \).

**Proof.** Since \( X \) has the Radon-Nikodym property, \( \mathcal{L}(L^1(G)/L^1_A(G), X) \) is isometrically isomorphic to \( L^\infty_A(G, X) \) [2, p. 63]. Now apply the Theorem.

**Remark.** A close analysis of the proof of Theorem 1 allows one to generalise Corollary 1 to obtain the following:
Let \((\Omega, \Sigma, \mu)\) be a probability space such that \(L^1(\mu)\) is separable. Let \(Y\) be a closed subspace of \(L^1(\mu)\). If \(X\) is a Banach space with the Radon-Nikodym property and if \(\mathcal{L}(L^1(\mu)/Y, X)\) is separable, then \(\mathcal{L}(L^1(\mu)/Y, X)\) has the Radon-Nikodym property.

**Corollary 2.** Let \(G\) be a compact abelian metrizable group, let \(\Lambda\) be a subset of \(\hat{G}\) and let \(X\) be a complex Banach space. If \(\Lambda\) is a Rosenthal set and \(X\) has the Radon-Nikodym property and the Schur property, then \(C_{\Lambda}(G, X)\) has the Radon-Nikodym property.

**Proof.** It suffices to show that every separable subspace of \(C_{\Lambda}(G, X)\) has the Radon-Nikodym property. Notice that each separable subspace of \(C_{\Lambda}(G, X)\) is a subspace of \(C_{\Lambda}(G, Y)\) for some separable subspace \(Y\) of \(X\). We will be finished if we can show that \(C_{\Lambda}(G, Y)\) has the Radon-Nikodym property for every separable subspace \(Y\) of \(X\).

Let \(Y\) be a separable subspace of \(X\). We will show that \(C_{\Lambda}(G, Y)\) is isomorphic to \(L^\infty_{\Lambda}(G, Y)\). By Theorem 5 of [2, p. 63], \(L^\infty_{\Lambda}(G, Y)\) is isomorphic to \(\mathcal{L}(L^1(G)/L^1_{\Lambda}(G), Y)\). However, since \(L^\infty_{\Lambda}(G)\) is a separable dual, \(L^1(G)/L^1_{\Lambda}(G)\) does not contain a copy of \(\ell^1\). Hence every bounded linear operator from \(L^1(G)/L^1_{\Lambda}(G)\) to \(Y\) is compact because \(Y\) has the Schur property. \(L^\infty_{\Lambda}(G)\) has the approximation property so the space of all compact operators from \(L^1(G)/L^1_{\Lambda}(G)\) to \(Y\) is isomorphic to \(L^\infty_{\Lambda}(G)\otimes Y\). But \(L^\infty_{\Lambda}(G)\otimes Y = C_{\Lambda}(G)\otimes Y\), which in turn is isomorphic to \(C_{\Lambda}(G, Y)\). Thus we have shown that \(C_{\Lambda}(G, Y)\) is isomorphic to \(L^\infty_{\Lambda}(G, Y)\). \(C_{\Lambda}(G, Y)\) is separable since \(Y\) is separable and so \(L^\infty_{\Lambda}(G, Y)\) has the Radon-Nikodym property by the Theorem. Hence \(C_{\Lambda}(G, Y)\) has the Radon-Nikodym property and so the proof is complete.

**Remark.** In [6], Lust-Piquard shows that if \(G\) is a compact abelian metrizable group and \(\Lambda\) is a subset of \(\hat{G}\) such that \(L^\infty_{\Lambda}(G)\) has the Schur property, then \(\Lambda\) is a Rosenthal set.

**Corollary 3.** Let \(G_1\) and \(G_2\) be two compact abelian metrizable groups and let \(\Lambda_1\) and \(\Lambda_2\) be subsets of \(\hat{G}_1\) and \(\hat{G}_2\), respectively. If \(\Lambda_1\) is a Rosenthal set and \(L^\infty_{\Lambda_1}(\hat{G}_1)\) has the Schur property, then \(\Lambda_1 \times \Lambda_2\) is a Rosenthal subset of \(\hat{G}_1 \times \hat{G}_2\).

**Proof.** To show that \(\Lambda_1 \times \Lambda_2\) is a Rosenthal set, it suffices to show that \(C_{\Lambda_1 \times \Lambda_2}(G_1 \times G_2)\) has the Radon-Nikodym property [5]. \(C_{\Lambda_1 \times \Lambda_2}(G_1 \times G_2)\) is isomorphic to \(C_{\Lambda_1}(G_1, C_{\Lambda_2}(G_2))\). Since \(L^\infty_{\Lambda_2}(G_2)\) has the Schur property,
\( \Lambda_2 \) is a Rosenthal set, by the preceding Remark, so \( C_{\Lambda_2}(G_2) \) has the Radon-Nikodym property. An application of Corollary 2 completes the proof.

**Remark.** Corollary 3 was proved independently by F. Lust-Piquard in [8, Chapter 4, Theorem 5].

**Acknowledgement**

The author wishes to thank the referee for many insightful comments and suggestions and for providing him a copy of [8].

**References**


Miami University
Oxford, Ohio 45056
U.S.A.