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GROWTH CONDITIONS FOR OPERATORS WITH SMALLEST SPECTRUM

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Dedicated to the memory of professor Mirabbas Gasymov

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Abstract. Let A be an invertible operator on a complex Banach space X. For a given $\alpha \geq 0$, we define the class $\mathcal{D}_A^\alpha(\mathbb{Z})$ (resp. $\mathcal{D}_A^\alpha(\mathbb{Z}_+)$) of all bounded linear operators T on X for which there exists a constant $C_T > 0$, such that

$$\|A^n T A^{-n}\| \leq C_T (1 + |n|)^\alpha ,$$

for all $n \in \mathbb{Z}$ (resp. $n \in \mathbb{Z}_+$). We present a complete description of the class $\mathcal{D}_A^\alpha(\mathbb{Z})$ in the case when the spectrum of A is real or is a singleton. If $T \in \mathcal{D}_A(\mathbb{Z}) \left(= \mathcal{D}_A^0(\mathbb{Z})\right)$, some estimates for the norm of $AT - TA$ are obtained. Some results for the class $\mathcal{D}_A^\alpha(\mathbb{Z}_+)$ are also given.

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1. Introduction. Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. As usual, $K(X)$ will denote the ideal of compact operators on X. By $\sigma(T)$, $r(T)$, and $R(z, T) := (zI - T)^{-1}$ $(z \notin \sigma(T))$, respectively, we denote the spectrum, the spectral radius, and the resolvent of $T \in B(X)$. Throughout, $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$.

Let H be a separable Hilbert space and let A be an invertible operator on H. In [**6**], Deddens introduced the set

$$\mathcal{B}_A := \left\{ T \in B(H) : \sup_{n \geq 0} \left\| A^n T A^{-n} \right\| < \infty \right\}.$$

Notice that $\mathcal{B}_A$ is an algebra (not necessarily closed) with identity which contains the commutant $\{A\}'$ of A. In [**6**], Deddens showed that if A is a positive operator with the spectral measure $E(\cdot)$, then $\mathcal{B}_A$ coincides with the nest algebra associated with the nest $\{E[0, \lambda] : \lambda \geq 0\}$ (recall that every nest algebra arises in this manner). In the same paper, Deddens conjectured that in the infinite dimensional Hilbert case, the equality $\mathcal{B}_A = \{A\}'$ holds if the spectrum of A is reduced to $\{1\}$. In [**16**], Roth gave a negative answer to Deddens conjecture. He showed the existence of a quasinilpotent operator V (the Volterra integration operator) for which $\mathcal{B}_{I+V} \neq \{I + V\}'$. In [**18**], Williams proved that if the spectrum of $A \in B(X)$ is reduced to $\{1\}$ and if $T \in B(X)$ satisfies the condition $\sup_{n \in \mathbb{Z}} \|A^n T A^{-n}\| < \infty$, then $AT = TA$. In [**7**], Drissi and Mbekhta improved Williams result by replacing his condition on A^{-1} by the weaker condition $\|A^{-n}TA^n\| = o(e^{\varepsilon\sqrt{n}})$ $(n \to \infty)$, for every $\varepsilon > 0$ (see also [**8**] and [**12**]).

In this paper, for an invertible operator $A \in B(X)$ and $\alpha \geq 0$, we define the class $\mathcal{D}_A^\alpha(\mathbb{Z})$ (resp. $\mathcal{D}_A^\alpha(\mathbb{Z}_+)$) of all operators $T \in B(X)$ for which the growth of $\|A^nTA^{-n}\|$ is at most polynomial in $n \in \mathbb{Z}$ (resp. $n \in \mathbb{Z}_+$), explicitly, there exists a constant $C_T > 0$, such that

$$\|A^nTA^{-n}\| \leq C_T(1 + |n|)^\alpha,$$

for all $n \in \mathbb{Z}$ (resp. $n \in \mathbb{Z}_+$). Clearly, both $\mathcal{D}_A^\alpha(\mathbb{Z})$ and $\mathcal{D}_A^\alpha(\mathbb{Z}_+)$ contains the commutant of A. In the case when $\alpha = 0$, instead of $\mathcal{D}_A^0(\mathbb{Z})$ and $\mathcal{D}_A^0(\mathbb{Z}_+)$ we will use the notations $\mathcal{D}_A(\mathbb{Z})$ and $\mathcal{D}_A(\mathbb{Z}_+)$, respectively. Notice also that $\mathcal{D}_A(\mathbb{Z})$ and $\mathcal{D}_A(\mathbb{Z}_+)$ are algebras (not necessarily closed) with identity.

The main results of the paper can be summarized as follows.

In Section 2, we give a complete characterization (Theorem 2.1) of the class $\mathcal{D}_A^\alpha(\mathbb{Z})$ in the case when the spectrum of A is real or is a singleton. It is shown (Theorem 2.8) that if $\sigma(A) = \{\lambda\}$ and $K(X) \subset \mathcal{D}_A^\alpha(\mathbb{Z}_+)$, then $A = \lambda I + N$, where N is nilpotent of degree $\leq [\alpha] + 1$. It is shown (Theorem 2.9) also that if $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $K(X) \subset \mathcal{D}_A^\alpha(\mathbb{Z}_+)$ $(0 \leq \alpha < 1)$, then $|\lambda_1| = \cdots = |\lambda_n|$ and there exist pairwise disjoint (bounded) projections $P_1, \ldots, P_n$ such that $P_1 + \cdots + P_n = I$ and $A = \lambda_1 P_1 + \cdots + \lambda_n P_n$.

In Section 3, in the case when $T \in D_A(\mathbb{Z})$, some estimates for the norm of $AT - TA$ are given (Theorem 3.2).

2. The class $\mathcal{D}_A^\alpha(\mathbb{Z})$.

The first main result of this section is the following.

THEOREM 2.1. *Assume that the spectrum of an invertible operator $A \in B(X)$ lies on the real line and $0 \notin \sigma(A) + \sigma(A)$. Then,*

$$\mathcal{D}_A^\alpha(\mathbb{Z}) = \left\{ T \in B(X) : \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i}TA^{-k+i} = 0 \right\},$$

where $k = [\alpha] + 1$. In particular, if $0 \leq \alpha < 1$, then $\mathcal{D}_A^\alpha(\mathbb{Z}) = \{A\}'$.

For the proof, we need some preliminary results.

For arbitrary $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighbourhood O_λ of λ with $u(z)$ analytic on O_λ having values in X, such that $(zI - T)u(z) = x$ for all $z \in O_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the *local spectrum* of T at x, denoted by $\sigma_T(x)$, is the complement of $\rho_T(x)$, so it is a compact subset of $\sigma(T)$. This object is most tractable if the operator T has the *single-valued extension property* (in abbreviation SVEP), i.e., for every open set U in $\mathbb{C}$, the only analytic function $f : U \to X$ for which the equation $(zI - T)f(z) = 0$ holds, is the constant function $f \equiv 0$. In that case, for every $x \in X$, there exists a maximal analytic extension of $R(z, T)x$ to $\rho_T(x)$. It follows that if T has SVEP, then $\sigma_T(x) \neq \emptyset$, whenever $x \neq 0$. It is easy to see that an operator $T \in B(X)$ having spectrum without interior points has the SVEP (see, [**5**] and [**13**]).

Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers with $\omega_n \geq 1$ and $\omega_{n+m} \leq \omega_n\omega_m$ for all $n, m \in \mathbb{Z}$. We say then that ω is a *weight* on $\mathbb{Z}$. The *Beurling algebra* $\mathcal{A}_\omega$, defined by

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the weight ω, is the set of all functions

$$f(\zeta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,\zeta^n \;\; (|\zeta| = 1), \text{ with } \|f\|_\omega = \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| \omega_n < \infty.$$

Notice that $\mathcal{A}_\omega$ is a commutative, semisimple Banach algebra with respect to pointwise multiplication. For arbitrary $\varphi \in \mathcal{A}_\omega^*$, we will write $\varphi = \{\widehat{\varphi}(n)\}_{n\in\mathbb{Z}}$, where $\widehat{\varphi}(n) = \varphi(\zeta^n)$ $(n \in \mathbb{Z})$. We have

$$\|\varphi\|_\omega := \sup_{n\in\mathbb{Z}} \frac{|\widehat{\varphi}(n)|}{\omega_n} < \infty.$$

The duality being implemented by the formula

$$\langle\varphi, f\rangle = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n)\widehat{f}(n) \;\; \left(\varphi \in \mathcal{A}_\omega^*,\; f \in \mathcal{A}_\omega\right).$$

We say, the weight ω is *regular* if

$$\sum_{n\in\mathbb{Z}} \frac{\log \omega_n}{1+n^2} < \infty.$$

For example, the weight $\omega_n = (1+|n|)^\alpha$ $(\alpha \geq 0)$ is regular and it is called *polynomial weight*. If ω is a regular weight, then

$$\lim_{n\to\infty} \omega_n^{\frac{1}{n}} = \lim_{n\to\infty} \omega_{-n}^{\frac{1}{n}} = 1. \tag{1}$$

Consequently, the maximal ideal space of the algebra $\mathcal{A}_\omega$ can be identified with $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ [**10**, Chapter III]. Moreover, the algebra $\mathcal{A}_\omega$ is regular in the Shilov sense [**10**, Chapter III and **1**, Chapter XII] if and only if the weight ω is regular. Below, we will assume that ω is a regular weight.

If I is a closed ideal of $\mathcal{A}_\omega$, the *hull* of I is the set

$$\text{hull}(I) = \{\xi \in \Gamma : f(\xi) = 0,\; \forall f \in I\}.$$

If $\varphi \in \mathcal{A}_\omega^*$, then

$$I_\varphi := \{f \in \mathcal{A}_\omega : \varphi \cdot f = 0\}$$

is a closed ideal of $\mathcal{A}_\omega$, where $\varphi \cdot f$ is a functional on $\mathcal{A}_\omega$, defined by

$$\langle\varphi \cdot f, g\rangle = \langle\varphi, fg\rangle,\; g \in \mathcal{A}_\omega.$$

Recall that the *support* of $\varphi \in \mathcal{A}_\omega^*$ is defined as follows. For $\xi \in \Gamma$, we let $\xi \notin \text{supp}\varphi$ iff there is a neighbourhood O_ξ of ξ such that $\langle\varphi, f\rangle = 0$ for all $f \in \mathcal{A}_\omega$ with $\text{supp} f \subset O_\xi$. An equivalent definition for $\text{supp}\varphi$ is that $\xi \in \text{supp}\varphi$ iff $\varphi \cdot f = 0$ implies $f(\xi) = 0$. It follows that

$$\text{supp}\varphi = \text{hull}\left(I_\varphi\right),\; \forall\varphi \in \mathcal{A}_\omega^*.$$

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Given a closed subset S of Γ, there are two distinguished closed ideals of $\mathcal{A}_\omega$ with hull equal to S, namely

$$J_\omega(S) := \overline{\{f \in \mathcal{A}_\omega : \operatorname{supp} f \cap S = \emptyset\}}$$

is the smallest closed ideal whose hull is S and

$$I_\omega(S) := \{f \in \mathcal{A}_\omega : f(\xi) = 0, \ \forall \xi \in S\}$$

is the largest closed ideal whose hull is S. The set S is a set of *synthesis* for $\mathcal{A}_\omega$ if $J_\omega(S) = I_\omega(S)$. This is equivalent to the existence of a unique closed ideal I of $\mathcal{A}_\omega$ whose hull is S. It is well known [**10**, Chapter VI, Section 41] that if $\omega = (\omega_n)_{n\in\mathbb{Z}}$, where $\omega_n = (1+|n|)^\alpha$ $(0 \le \alpha < 1)$, then each point of Γ is a set of synthesis for $\mathcal{A}_\omega$.

Let $\varphi \in \mathcal{A}_\omega^*$ be given. Since $|\widehat{\varphi}(n)| \le \|\varphi\|_\omega \omega_n$ $(n \in \mathbb{Z})$, it follows from (1) that

$$\overline{\lim}_{n\to\infty} |\widehat{\varphi}(n)|^{\frac{1}{n}} \le 1 \text{ and } \overline{\lim}_{n\to\infty} |\widehat{\varphi}(-n)|^{\frac{1}{n}} \le 1.$$

Recall that the *Carleman transform* of φ is defined as the analytic function $\Phi(z)$ on $\mathbb{C}\setminus\Gamma$ given by

$$\Phi(z) = \begin{cases} \sum_{n=0}^{\infty} \frac{\widehat{\varphi}(n)}{z^n}, & |z| > 1; \\ -\sum_{n=1}^{\infty} \widehat{\varphi}(-n) z^n, & |z| < 1. \end{cases}$$

We know (see, [**2**, Theorem 3.3] and [**17**, Lemma 3]) that $\xi \in \operatorname{supp}\varphi$ if and only if the Carleman transform $\Phi(z)$ of φ has no analytic extension to a neighbourhood of ξ.

Let T be an invertible operator on a Banach space X and let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a weight on $\mathbb{Z}$. We put

$$E_T^\omega := \left\{x \in X : \exists C > 0, \ \left\|T^n x\right\| \le C\omega_n, \ \forall n \in \mathbb{Z}\right\}.$$

Clearly, E_T^ω is a linear (in general, non-closed) subspace of X. If $x \in E_T^\omega$, then for arbitrary $f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \zeta^n \in \mathcal{A}_\omega$, we can define $x_f \in X$ by

$$x_f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) T^n x.$$

Then, $f \mapsto x_f$ is a bounded linear map from $\mathcal{A}_\omega$ into X;

$$\|x_f\| \le C \|f\|_\omega, \ \forall f \in \mathcal{A}_\omega.$$

Further, from the identity

$$T^m x_f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) T^{n+m} x,$$

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we can write

$$\begin{aligned}\|T^m x_f\| &\le \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| \|T^{n+m}x\| \\ &\le C\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right| \omega_{n+m} \\ &\le C\|f\|_\omega \omega_m, \forall m \in \mathbb{Z}.\end{aligned}$$

This shows that $x_f \in E_T^\omega$ for every $f \in \mathcal{A}_\omega$. It is easy to check that if $x \in E_T^\omega$, then

$$(x_f)_g = x_{fg} \text{ for all } f, g \in \mathcal{A}_\omega.$$

It follows that if $x \in E_T^\omega$, then

$$I_x := \{f \in \mathcal{A}_\omega : x_f = 0\}$$

is a closed ideal of $\mathcal{A}_\omega$.

For a given $x \in E_T^\omega$, consider the function

$$u(z) := \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{T^n x}{z^{n+1}}, & |z| > 1; \\ \displaystyle -\sum_{n=1}^{\infty} z^{n-1}T^{-n}x, & |z| < 1. \end{cases}$$

It follows from (1) that $u(z)$ is an analytic function on $\mathbb{C}\diagdown\Gamma$ and

$$(zI - T)u(z) = x \ (|z| \ne 1). \tag{2}$$

It follows that $\sigma_T(x) \subset \Gamma$. Now, assume that T has SVEP. We claim that $\sigma_T(x)$ consists of all $\xi \in \Gamma$ for which the function $u(z)$ has no analytic extension to a neighbourhood of ξ. Assume that $v(z)$ is the analytic extension of $u(z)$ to a neighbourhood O_ξ of $\xi \in \Gamma$. It follows from the identity (2) that the function $w(z) := (zI - T)v(z) - x$ vanishes on $O_\xi^+ := \{z \in O_\xi : |z| > 1\}$ and on $O_\xi^- := \{z \in O_\xi : |z| < 1\}$. By uniqueness theorem, $w(z) = 0$ for all $z \in O_\xi$. So we have $(zI - T)v(z) = x$ for all $z \in O_\xi$. This shows that $\xi \in \rho_T(x)$. Now, assume that $\xi \in \rho_T(x) \cap \Gamma$. Then, there exists a neighbourhood O_ξ of ξ with $v(z)$ analytic on O_ξ having values in X such that $(zI - T)v(z) = x$ for all $z \in O_\xi$. In view of the identity (2), $(zI - T)(u(z) - v(z)) = 0$ for all $z \in O_\xi^+$ and $z \in O_\xi^-$. Since T has SVEP, we have $u(z) = v(z)$ for all $z \in O_\xi^+$ and $z \in O_\xi^-$. This shows that the function $u(z)$ can be analytically extended to a neighbourhood of ξ.

Let $x \in E_T^\omega$ be given. For arbitrary $\varphi \in X^*$, define a functional φ_x on $\mathcal{A}_\omega$, by

$$\langle \varphi_x, f\rangle = \langle \varphi, x_f\rangle.$$

We have

$$\left|\langle \varphi_x, f\rangle\right| \le C\|\varphi\| \|f\|_\omega, \ \forall f \in \mathcal{A}_\omega,$$

and $\widehat{\varphi_x}(n) = \varphi(T^n x)$ $(n \in \mathbb{Z})$. Consequently, we can write

$$z\langle \varphi, u(z)\rangle = \begin{cases} \sum_{n=0}^{\infty} \frac{\widehat{\varphi_x}(n)}{z^n}, & |z| > 1; \\ -\sum_{n=1}^{\infty} z^n \widehat{\varphi_x}(-n), & |z| < 1. \end{cases}$$

This shows that the function $z \to z\langle \varphi, u(z)\rangle$ $(|z| \neq 1)$ is the Carleman transform of φ_x. It follows that

$$\overline{\bigcup_{\varphi \in X^*} \mathrm{supp}\varphi_x} \subseteq \sigma_T(x).$$

To show the reverse inclusion, assume that $\xi_0 \in \Gamma$ and

$$\xi_0 \notin \overline{\bigcup_{\varphi \in X^*} \mathrm{supp}\varphi_x}.$$

Then, there exists $f \in \mathcal{A}_\omega$ such that $f(\xi_0) \neq 0$ and f vanishes in a neighbourhood of $\mathrm{supp}\varphi_x$, for every $\varphi \in X^*$. Consequently, there exists a neighbourhood O_{ξ_0} of ξ for which $f(\xi) \neq 0$ for all $\xi \in O_{\xi_0}$ and $\varphi_x \cdot f = 0$. Therefore, $O_{\xi_0} \subset \Gamma \backslash \mathrm{supp}\varphi_x$. This shows that the function $z \to \langle \varphi, u(z)\rangle$ can be analytically extended to O_{ξ_0} for every $\varphi \in X^*$. It follows that $u(z)$ can be analytically extended to O_{ξ_0}. Consequently, $\xi_0 \notin \sigma_T(x)$. Thus, we obtain

$$\overline{\bigcup_{\varphi \in X^*} \mathrm{supp}\varphi_x} = \sigma_T(x).$$

On the other hand, from the identity

$$\langle \varphi_x \cdot f, g\rangle = \langle \varphi_x, fg\rangle = \langle \varphi, x_{fg}\rangle \ (f, g \in \mathcal{A}_\omega),$$

we can deduce that

$$I_x = \bigcap_{\varphi \in X^*} I_{\varphi_x}.$$

Now, it follows from the general theory of Banach algebras that

$$\mathrm{hull}(I_x) = \overline{\bigcup_{\varphi \in X^*} \mathrm{hull}\left(I_{\varphi_x}\right)} = \overline{\bigcup_{\varphi \in X^*} \mathrm{supp}\varphi_x} = \sigma_T(x).$$

Hence, we have the following.

PROPOSITION 2.2. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a regular weight on $\mathbb{Z}$ and let T be an invertible operator on a Banach space X with the SVEP. If $x \in X$ satisfies the condition $\|T^n x\| \leq C\omega_n$ for all $n \in \mathbb{Z}$ and for some constant $C > 0$, then*

$$\sigma_T(x) = \mathit{hull}(I_x).$$

As a consequence of Proposition 2.2, we have the following.

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PROPOSITION 2.3. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a regular weight on $\mathbb{Z}$ and let T be an invertible operator on a Banach X with the SVEP. Assume that $x \in X$ satisfies the condition $\|T^n x\| \leq C\omega_n$ for all $n \in \mathbb{Z}$ and for some constant $C > 0$. Then, the following assertions hold for $f \in \mathcal{A}_\omega$:*

(a) If $x_f = 0$, then f vanishes on $\sigma_T(x)$.
(b) If f vanishes in a neighbourhood of $\sigma_T(x)$, then $x_f = 0$.
(c) If $f = 1$ in a neighbourhood of $\sigma_T(x)$, then $x_f = x$.
(d) $\sigma_T(x_f) \subset \sigma_T(x) \cap \text{supp} f$.
(e) $\sigma_T(x) \cap \{\xi \in \Gamma : f(\xi) \neq 0\} \subset \sigma_T(x_f)$.

Proof. By Proposition 2.2, we can write

$$J_\omega(\sigma_T(x)) \subset I_x \subset I_\omega(\sigma_T(x)).$$

The assertions (a) and (b) follows from this relation.

(c) Since $f - 1$ vanishes in a neighbourhood of $\sigma_T(x)$, by (b), $x_f = x$.

(d) If $g \in I_x$, then $x_g = 0$. As

$$(x_f)_g = x_{fg} = (x_g)_f = 0,$$

we have $g \in I_{x_f}$. Hence, $I_x \subset I_{x_f}$ which implies hull$(I_{x_f}) \subset$hull(I_x). By Proposition 2.2, $\sigma_T(x_f) \subset \sigma_T(x)$. On the other hand, if $g \in \mathcal{A}_\omega$ vanishes on supp$f$, then $fg = 0$. This implies

$$(x_f)_g = x_{fg} = 0.$$

Consequently, $I_\omega(\text{supp} f) \subset I_{x_f}$, so that hull $(I_{x_f}) \subset$supp$f$. By Proposition 2.2, $\sigma_T(x_f) \subset$supp$f$. Thus, we have $\sigma_T(x_f) \subset \sigma_T(x) \cap$supp$f$.

(e) Assume that $\xi \in \sigma_T(x)$, $f(\xi) \neq 0$, and $\xi \notin \sigma_T(x_f)$. Since the algebra $\mathcal{A}_\omega$ is regular, there exists $g \in \mathcal{A}_\omega$ such that $g(\xi) \neq 0$ and g vanishes in a neighbourhood of $\sigma_T(x_f)$. Consequently, g belongs to the smallest closed ideal of $\mathcal{A}_\omega$ whose hull is $\sigma_T(x_f)$. By Proposition 2.2, $g \in I_{x_f}$ and so

$$x_{fg} = (x_f)_g = 0.$$

By (a), fg vanishes on $\sigma_T(x)$. It follows that $f(\xi) = 0$ which contradicts $f(\xi) \neq 0$. □

Next, we have the following.

PROPOSITION 2.4. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$, where $\omega_n = (1 + |n|)^\alpha$ $(\alpha \geq 0)$. Assume that an invertible operator T on a Banach space X and $x \in X$ satisfies the following conditions:*

(i) $\|T^n x\| \leq C\omega_n$ for all $n \in \mathbb{Z}$ and for some constant $C > 0$.
(ii) T has SVEP.
If $\sigma_T(x) = \{\xi\}$, then for every $f \in \mathcal{A}_\omega$, we have

$$x_f = f(\xi)x + \frac{f'(\xi)}{1!}(T - \xi I)x + \cdots + \frac{f^{(k)}(\xi)}{k!}(T - \xi I)^k x,$$

where $k = [\alpha]$. In particular, we have $(T - \xi I)^{k+1} x = 0$.

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Proof. We know [**10**, Chapter, Section 41] that if $f \in \mathcal{A}_\omega$, then the first k derivatives of f exist and

$$J_\omega(\{\xi\}) = \left\{f \in \mathcal{A}_\omega : f(\xi) = f'(\xi) = \cdots = f^{(k)}(\xi) = 0\right\},$$

where $k = [\alpha]$. Recall that $J_\omega(\{\xi\})$ is the smallest closed ideal of $\mathcal{A}_\omega$ whose hull is $\{\xi\}$. On the other hand, by Proposition 2.2, hull$(I_x) = \{\xi\}$. Therefore, we have $J_\omega(\{\xi\}) \subset I_x$. Now, for a given $f \in \mathcal{A}_\omega$, consider the function

$$h(\zeta) = f(\zeta) - f(\xi) - \frac{f'(\xi)}{1!}(\zeta - \xi) - \cdots - \frac{f^{(k)}(\xi)}{k!}(\zeta - \xi)^k.$$

As

$$h(\xi) = h'(\xi) = \cdots = h^{(k)}(\xi) = 0,$$

we have $h \in J_\omega(\{\xi\})$, so that $h \in I_x$. Thus, we obtain $x_h = 0$ and so

$$x_f = f(\xi)x + \frac{f'(\xi)}{1!}(T - \xi I)x + \cdots + \frac{f^{(k)}(\xi)}{k!}(T - \xi I)^k x.$$

By taking in the preceding identity $f(\zeta) = (\zeta - \xi)^{k+1}$, we get

$$(T - \xi I)^{k+1} x = 0.$$

□

For a given $A \in B(X)$, by L_A and R_A, respectively, we denote the left and right multiplication operators on $B(X)$;

$$L_A T = AT, \ R_A T = TA, \ T \in B(X).$$

By Lumer–Rosenblum theorem [**15**, Theorem 10], for arbitrary $A, B \in B(X)$,

$$\sigma(L_A R_B) = \{\lambda\mu : \lambda \in \sigma(A), \ \mu \in \sigma(B)\}.$$

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. If $T \in \mathcal{D}_A^\alpha(\mathbb{Z})$, then we can write

$$\left\|(L_A R_{A^{-1}})^n T\right\| \leq C(1 + |n|)^\alpha, \quad \forall n \in \mathbb{Z}.$$

As we have noted above, in that case

$$\sigma_{L_A R_{A^{-1}}}(T) \subset \Gamma.$$

On the other hand, the Lumer–Rosenblum theorem mentioned above and the condition $0 \notin \sigma(A) + \sigma(A)$ implies that

$$\sigma_{L_A R_{A^{-1}}}(T) \subset \sigma(L_A R_{A^{-1}}) \subset \mathbb{R} \setminus \{-1\}.$$

Consequently, the operator $L_A R_{A^{-1}}$ has SVEP and $\sigma_{L_A R_{A^{-1}}}(T) \subset \{1\}$. Since $L_A R_{A^{-1}}$ has SVEP, $\sigma_{L_A R_{A^{-1}}}(T) \neq \emptyset$. So we have $\sigma_{L_A R_{A^{-1}}}(T) = \{1\}$. Applying now Proposition

2.4 to the operator $L_A R_{A^{-1}}$ on the space $B(X)$, we get

$$(L_A R_{A^{-1}} - I)^k T = 0,$$

where $k = [\alpha] + 1$. This clearly implies

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} A^{k-i} T A^{-k+i} = 0.$$

For the reverse inclusion, assume that $T \in B(X)$ satisfies the last equation. Since

$$(L_A R_{A^{-1}} - I)^k T = 0 \ (k \geq 1),$$

we can write

$$\begin{aligned} \|A^n T A^{-n}\| &= \|(L_A R_{A^{-1}})^n T\| \\ &= \left\| T + \binom{n}{1} (L_A R_{A^{-1}} - I) T + \cdots + \binom{n}{k-1} (L_A R_{A^{-1}} - I)^{k-1} T \right\| \\ &= O(1+n)^{k-1}. \end{aligned}$$

As $(L_{A^{-1}} R_A - I)^k T = 0$, similarly we have $\|A^{-n} T A^n\| = O(1+n)^{k-1}$. Hence, $\|A^n T A^{-n}\| = O(1+|n|)^{k-1}$ $(n \in \mathbb{Z})$. □

Corollary 2.5. *If the spectrum of an invertible operator $A \in B(X)$ consists of one point, then the conclusion of Theorem 2.1 remains true.*

Proof. Assume that $\sigma(A) = \{\lambda\}$, where $\lambda \neq 0$. If $T \in \mathcal{D}_A^\alpha(\mathbb{Z})$, then $T \in \mathcal{D}_B^\alpha(\mathbb{Z})$, where $B = \frac{A}{\lambda}$. Since $\sigma(B) = \{1\}$, by Theorem 2.1, we obtain as required. □

It follows from Corollary 2.5 that if $\sigma(A)$ consists of one point and $0 \leq \alpha < 1$, then $\mathcal{D}_A^\alpha(\mathbb{Z}) = \{A\}'$. Note that if $\alpha \geq 1$, then $\mathcal{D}_A^\alpha(\mathbb{Z}) \neq \{A\}'$, in general. To see this, let $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be 2×2 matrices on 2−dimensional Hilbert space. We have $\sigma(A) = \{1\}$ and

$$A^n T A^{-n} = [I + n(A-I)]\, T\, [I - n(A-I)] = \begin{pmatrix} 0 & 0 \\ -n & 1 \end{pmatrix} \ (n \in \mathbb{N}).$$

Similarly, $A^{-n} T A^n = \left(\begin{smallmatrix} 0 & 0 \\ n & 1 \end{smallmatrix}\right)$ $(n \in \mathbb{N})$. So we have

$$\|A^n T A^{-n}\| = \left(1 + |n|^2\right)^{\frac{1}{2}} \ (n \in \mathbb{Z}).$$

This shows that $T \in \mathcal{D}_A^1(\mathbb{Z})$, but $AT \neq TA$.

Recall that an invertible operator T acting on a Banach space is called *doubly power bounded* if $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Well-known Gelfand's theorem **[9]** states that if T is doubly power bounded with $\sigma(T) = \{1\}$, then $T = I$.

We include here the following result, which seems to be unnoticed.

Proposition 2.6. *Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$, where $\omega(n) = (1+|n|)^\alpha$ $(0 \leq \alpha < 1)$. Assume that an invertible operator T on a Banach space X and $x \in X$ satisfies the following conditions:*

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(*i*) $\|T^n x\| \leq C\omega(n)$ *for all* $n \in \mathbb{Z}$ *and for some constant* $C > 0$.
(*ii*) *T has SVEP.*
If $\sigma_T(x) = \{\xi_1, \ldots, \xi_k\}$ $(\xi_i \neq \xi_j,\ i \neq j,\ i,j = 1,\ldots,k)$, *then*

$$x \in \ker(T - \xi_1 I) \oplus \cdots \oplus \ker(T - \xi_k I).$$

Proof. Let $U_1, \ldots, U_k$ be disjoint neighbourhoods of $\xi_1, \ldots, \xi_k$, respectively. Let V_i be a neighbourhood of ξ_i such that $\overline{V_i} \subset U_i$ $(i = 1, \ldots, k)$. Since the algebra $\mathcal{A}_\omega$ is regular, there exist functions $f_1, \ldots, f_k$ in $\mathcal{A}_\omega$ such that $f_i = 1$ on V_i and $f_i = 0$ outside U_i $(i = 1, \ldots, k)$. We put $f := f_1 + \cdots + f_k$. Since $f = 1$ in a neighbourhood of $\sigma_T(x)$, by Proposition 2.3 (c), $x_f = x$. So we have $x = x_1 + \cdots + x_k$, where $x_i = x_{f_i}$ $(i = 1, \ldots, k)$. Further, it follows from Proposition 2.3 (d) and (e) that

$$\{\xi_i\} \subset \sigma_T(x_i) \subset \sigma_T(x) \cap \operatorname{supp} f_i = \{\xi_i\}.$$

Consequently, we have $\sigma_T(x_i) = \{\xi_i\}$. Now, it remains to show that if $x \in E_T^\omega$ with $\sigma_T(x) = \{\xi\}$, then $Tx = \xi x$. By Proposition 2.2, $\operatorname{hull}(I_x) = \{\xi\}$. Since $\{\xi\}$ is a set of synthesis for $\mathcal{A}_\omega$, we have $I_x = I_\omega(\{\xi\})$, so that

$$\{f \in \mathcal{A}_\omega : x_f = 0\} = \{f \in \mathcal{A}_\omega : f(\xi) = 0\}.$$

If we put in this identity $f(\zeta) = \zeta - \xi$, then we have $Tx = \xi x$. □

Let $\omega = (\omega_n)_{n \in \mathbb{Z}}$ be a regular weight. Assume that an invertible operator T on a Banach space satisfies the condition $\|T^n\| \leq C\omega(n)$ for all $n \in \mathbb{Z}$ and for some constant $C > 0$. As we have noted above, in that case $\sigma(T) \subset \Gamma$ and therefore T has SVEP.

The following result is an immediate consequence of the preceding proposition.

COROLLARY 2.7. *Assume that* $0 \leq \alpha < 1$ *and* $T \in B(X)$ *satisfies the condition* $\|T^n\| \leq C(1 + |n|)^\alpha$ *for all* $n \in \mathbb{Z}$ *and for some constant* $C > 0$. *If*

$$\sigma(T) = \{\xi_1, \ldots, \xi_k\}\ (\xi_i \neq \xi_j,\ i \neq j,\ i,j = 1, \ldots, k),$$

then there exist pairwise disjoint (bounded) projections $P_1, \ldots, P_k$ *such that* $P_1 + \cdots + P_k = I$ *and*

$$T = \xi_1 P_1 + \cdots + \xi_k P_k.$$

(*in fact,* $P_i = \frac{1}{2\pi i}\int_{\Gamma_i} R(z, T)\,dz$, *where* Γ_i *is an appropriate contour around* $\{\xi_i\}$).

Another application of Proposition 2.4 is the following.

THEOREM 2.8. *Assume that the spectrum of* $A \in B(X)$ *consists of one point* $\lambda \neq 0$. *If* $K(X) \subset \mathcal{D}_A^\alpha(\mathbb{Z}_+)$, *then the operator* A *has the form* $A = \lambda I + N$, *where* N *is nilpotent of degree* $\leq [\alpha] + 1$.

Proof. We have

$$\|A^n T A^{-n}\| = \|(L_A R_{A^{-1}})^n T\| \leq C_T (1 + n)^\alpha,$$

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for all $T \in K(X)$ and $n \in \mathbb{N}$. Applying uniform boundedness principle to the sequence of operators

$$B_n := \frac{1}{(1+n)^{\alpha}} (L_A R_{A^{-1}})^n ,$$

we obtain that there exists a constant $C > 0$ such that

$$\|A^n T A^{-n}\| \leq C(1+n)^{\alpha} \|T\|,$$

for all $T \in K(X)$ and $n \in \mathbb{N}$. For a given $x \in X$ and $\varphi \in X^*$, let $x \otimes \varphi$ be the one dimensional operator on X defined by

$$x \otimes \varphi : y \mapsto \varphi(y) x \ (y \in X).$$

As $x \otimes \varphi \in \mathcal{D}_A^{\alpha}(\mathbb{Z}_+)$, we have

$$\|A^n x\| \|A^{*-n}\varphi\| \leq C(1+n)^{\alpha} \|x\| \|\varphi\|,$$

for all $x \in X$ and $\varphi \in X^*$. This implies

$$\|A^n\| \|A^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

Further if $B := \frac{1}{\lambda} A$, then

$$\|B^n\| \|B^{-n}\| = \|A^n\| \|A^{-n}\| \leq C(1+n)^{\alpha}.$$

On the other hand, as $\sigma(B) = \sigma(B^{-1}) = \{1\}$, we have $\|B^n\| \geq 1$ and $\|B^{-n}\| \geq 1$. Consequently,

$$\|B^n\| \leq \|B^n\| \|B^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

Similarly, we have

$$\|B^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

Thus, we obtain

$$\|B^n\| \leq C(1+|n|)^{\alpha}, \quad \forall n \in \mathbb{Z}.$$

By Proposition 2.4, $(B-I)^k = 0$, where $k = [\alpha] + 1$. It follows that $(A - \lambda I)^k = 0$. If we put $N := A - \lambda I$, then $A = \lambda I + N$, where N is nilpotent of degree $\leq k$. □

The following result is an application of Proposition 2.6.

THEOREM 2.9. *Let $A \in B(X)$ be such that $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, where $0 \neq \lambda_i \neq \lambda_j$ $(i \neq j)$ $(i, j = 1, \ldots, k)$. If $K(X) \subset \mathcal{D}_A^{\alpha}(\mathbb{Z}_+)$ $(0 \leq \alpha < 1)$, then $|\lambda_1| = \cdots = |\lambda_k|$ and there exist pairwise disjoint (bounded) projections $P_1, \ldots, P_k$ such that $P_1 + \cdots + P_k = I$ and*

$$A = \lambda_1 P_1 + \cdots + \lambda_k P_k.$$

Proof. As in the proof of Theorem 2.8, we have

$$\|A^n\| \|A^{-n}\| \leq C(1+n)^{\alpha}, \forall n \in \mathbb{N}.$$

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It follows that $r(A)r(A^{-1}) \leq 1$. Since $r(A)r(A^{-1}) \geq 1$, we obtain

$$r(A)r(A^{-1}) = 1.$$

Consequently, $|\lambda_1| = \cdots = |\lambda_k| = a$ for some $a > 0$. Further if $B := \frac{1}{a}A$, then

$$\|B^n\|\|B^{-n}\| = \|A^n\|\|A^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

On the other hand, as

$$\sigma(B) = \left\{\frac{\lambda_1}{a}, \ldots, \frac{\lambda_k}{a}\right\} \quad \text{and} \quad \sigma(B^{-1}) = \left\{\frac{a}{\lambda_1}, \ldots, \frac{a}{\lambda_k}\right\},$$

we have $\|B^n\| \geq 1$ and $\|B^{-n}\| \geq 1$. This implies

$$\|B^n\| \leq \|B^n\|\|B^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

Similarly, we have

$$\|B^{-n}\| \leq C(1+n)^{\alpha}, \quad \forall n \in \mathbb{N}.$$

Thus, we obtain

$$\|B^n\| \leq C(1+|n|)^{\alpha}, \quad \forall n \in \mathbb{Z}.$$

By Corollary 2.7, there exist pairwise disjoint projections $P_1, \ldots, P_k$ such that $P_1 + \cdots + P_k = I$ and

$$B = \frac{\lambda_1}{a}P_1 + \cdots + \frac{\lambda_k}{a}P_k.$$

So we have $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$. □

3. The norm of $AT - TA$. In this section, we give some estimates for the norm of $AT - TA$ in the case when $T \in \mathcal{D}_A(\mathbb{R})$.

Recall that an entire function f is said to be of *order* ρ if

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r},$$

where

$$M_f(r) = \sup\{|f(z)| : |z| \leq r\} \ \ (r > 0).$$

An entire function f of finite order ρ is said to be of *type* σ if

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{r^{\rho}}.$$

If the entire function f is of order at most one and type less than or equal to σ, we say f is of *exponential type* σ [**3**, p. 8].

For a given $\sigma > 0$, we denote by B_σ the set of all bounded on the real line entire functions f of exponential type $\leq \sigma$. Recall [**11**] that B_σ is a Banach space under the norm given by

$$\|f\|_\sigma := \sup_{z\in\mathbb{C}} \left[e^{-\sigma|\mathrm{Im}\, z|}\, |f(z)|\right].$$

It follows from the Phragmen–Lindelöf theorem that

$$\|f\|_\sigma = \sup_{t\in\mathbb{R}} |f(t)|, \qquad \forall f \in B_\sigma.$$

The following inequality of Bernstein type is well known [**11**]: If $f \in B_\sigma$, where $0 \le \sigma h \le \frac{\pi}{2}$, then

$$\sup_{t\in\mathbb{R}} |f(t+h) - f(t-h)| \le 2\sin\sigma h\, \|f\|_\sigma .$$

It follows that for every $f \in B_\sigma$,

$$|f(1) - f(0)| \le 2\sin\frac{\sigma}{2}\, \|f\|_\sigma \quad (\sigma \le \pi),$$

$$|f(1) - f(-1)| \le 2\sin\sigma\, \|f\|_\sigma \quad \left(\sigma \le \frac{\pi}{2}\right).$$

On the other hand, by Cartwright theorem (see, [**3**, Chapter 10] and [**11**]), the inequality

$$\|f\|_\sigma \le \frac{1}{\cos\frac{\sigma}{2}} \sup_{n\in\mathbb{Z}} |f(n)|$$

holds for every $f \in B_\sigma$ $(\sigma < \pi)$. Hence, we have

$$|f(1) - f(0)| \le 2\tan\frac{\sigma}{2}\left(\sup_{n\in\mathbb{Z}} |f(n)|\right), \ \forall f \in B_\sigma \ (\sigma < \pi), \tag{3}$$

$$|f(1) - f(-1)| \le 4\sin\frac{\sigma}{2}\left(\sup_{n\in\mathbb{Z}} |f(n)|\right), \ \forall f \in B_\sigma \ \left(\sigma \le \frac{\pi}{2}\right). \tag{4}$$

We will need the following.

LEMMA 3.1. *Assume that $T \in B(X)$ and $x \in X$ satisfies the following conditions,*

(i) $\sigma(T) \subset \mathbb{C}\diagdown\mathbb{R}_-$,
(ii) $\sup_{n\in\mathbb{Z}} \|T^n x\| \le C$ *for some* $C > 0$.
If $\tau_T := \sup\{|\log z| : z \in \sigma(T)\}$, *then the following assertions hold:*
(a) If $\tau_T < \pi$, *then*

$$\|Tx - x\| \le 2C\tan\frac{\tau_T}{2}.$$

(b) If $\tau_T \le \frac{\pi}{2}$, *then*

$$\|Tx - T^{-1}x\| \le 4C\sin\frac{\tau_T}{2}.$$

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Proof. By condition (i), we can write $T = e^S$, where $S = \log T$ [**4**, Chapter I, Section 7]. For arbitrary functional $\varphi \in X^*$ with norm one, consider the entire function

$$f(z) := \langle \varphi, e^{zS}x \rangle.$$

From the inequality,

$$|f(z)| \leq e^{|z|\|S\|} \|x\|,$$

we deduce that f is an entire function of order

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log r} \leq \lim_{r\to\infty} \frac{\log(r\|S\| + \log\|x\|)}{\log r} = 1.$$

Notice also that the nth derivative of f at zero is $\varphi(S^n x)$. By Levin's Theorem [**14**, p. 84], the type of f is less than or equal to

$$\begin{aligned} \overline{\lim_{n\to\infty}} |\varphi(S^n x)|^{\frac{1}{n}} \leq \lim_{k\to\infty} \|S^n\|^{\frac{1}{n}} &= r(S) \\ &= \sup\{|\log z| : z \in \sigma(T)\} = \tau_T. \end{aligned}$$

Consequently, f is an entire function of exponential type τ_T. Further, since

$$\sup_{n\in\mathbb{Z}} \|e^{nS}x\| \leq C,$$

from the identity $t = n + r$, where $n \in \mathbb{Z}$, $|r| < 1$, and $|n| \leq |t|$, we can write

$$\sup_{t\in\mathbb{R}} \|e^{tS}x\| \leq Ce^{\|S\|}.$$

Hence, f is bounded on $\mathbb{R}$. Thus, we obtain that $f \in B_{\tau_T}$. Now, taking into account that

$$\sup_{n\in\mathbb{Z}} |f(n)| \leq C,$$

in the case when $\tau_T < \pi$, from the inequality (3), we can write

$$|f(1) - f(0)| \leq 2C \tan\frac{\tau_T}{2}.$$

It follows that

$$\left\|e^S x - x\right\| \leq 2C \tan\frac{\tau_T}{2},$$

which means that

$$\|Tx - x\| \leq 2C \tan\frac{\tau_T}{2}.$$

Similarly, from the inequality (4), we can deduce that if $\tau_T \leq \frac{\pi}{2}$, then

$$\|Tx - T^{-1}x\| \leq 4C \sin\frac{\tau_T}{2}.$$

□

The following theorem gives us another generalization of Williams result [**18**].

THEOREM 3.2. *Let A be an invertible operator on a Banach space X and let $T \in B(X)$. Assume that the following conditions are satisfied:*

(i) $\{\lambda\mu^{-1} : \lambda, \mu \in \sigma(A)\} \subset \mathbb{C} \setminus \mathbb{R}_-$,
(ii) $\sup_{n\in\mathbb{Z}} \|A^n T A^{-n}\| \le C_T$ *for some* $C_T > 0$.
If $\tau_A := \sup\{|\log(\lambda\mu^{-1})| : \lambda, \mu \in \sigma(A)\}$, *then the following assertions hold:*

(*a*) *If* $\tau_A < \pi$, *then*

$$\|AT - TA\| \le 2C_T \|A\| \tan\frac{\tau_A}{2}.$$

(*b*) *If* $\tau_A \le \frac{\pi}{2}$, *then*

$$\|A^2T - TA^2\| \le 2C_T \|A\|^2 \sin\frac{\tau_A}{2}.$$

Proof. We have

$$\sup_{n\in\mathbb{Z}} \|(L_A R_{A^{-1}})^n T\| \le C_T.$$

By Lumer–Rosenblum theorem mentioned above, we also have

$$\sigma(L_A R_{A^{-1}}) = \{\lambda\mu^{-1} : \lambda, \mu \in \sigma(A)\} \subset \mathbb{C} \setminus \mathbb{R}_-.$$

Applying now Lemma 3.1 to the operator $L_A R_{A^{-1}}$ on the space $B(X)$, we can write

$$\begin{aligned} \|AT - TA\| &= \|(ATA^{-1} - T)A\| \\ &\le \|A\| \|(L_A R_{A^{-1}})T - T\| \\ &\le 2C_T \|A\| \tan\frac{\tau_A}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|A^2T - TA^2\| &= \|A(ATA^{-1} - A^{-1}TA)A\| \\ &\le \|A\|^2 \|(L_A R_{A^{-1}})T - (L_{A^{-1}} R_A)T\| \\ &\le 4C_T \|A\|^2 \sin\frac{\tau_A}{2}. \end{aligned}$$

□

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