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# SEQUENTIAL COMPLETENESS OF QUOTIENT GROUPS

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We discuss various generalisations of countable compactness for topological groups that are related to completeness. The sequentially complete groups form a class closed with respect to taking direct products and closed subgroups. Surprisingly, the stronger version of sequential completeness called *sequential h-completeness* (all continuous homomorphic images are sequentially complete) implies pseudocompactness in the presence of good algebraic properties such as nilpotency. We also study quotients of sequentially complete groups and find several classes of *sequentially q-complete* groups (all quotients are sequentially complete). Finally, we show that the pseudocompact sequentially complete groups are far from being sequentially $q$-complete in the following sense: every pseudocompact Abelian group is a quotient of a pseudocompact Abelian sequentially complete group.

## 1. INTRODUCTION

The topological groups that are sequentially closed in any other topological group are precisely the *sequentially complete* groups [**17**, **18**], that is, those that are sequentially closed in their Raĭkov completion (equivalently, every Cauchy sequence converges). Obviously, the class of sequentially complete groups contains all complete groups and all countably compact groups. By [**18**, Corollary 4.10], the free topological group $F(X)$ and the free Abelian topological group $A(X)$ are sequentially complete for every countably compact space $X$. The groups $F(X)$ and $A(X)$ are never precompact; however, there are lots of precompact sequentially complete groups that are not pseudocompact, for example, the free precompact Abelian group $F(X, \mathbf{PA})$ on any countably compact space $X$ [**18**, Corollary 5.5]. On the other hand, every minimal, sequentially complete, connected Abelian group of non-measurable cardinality is compact by [**17**, Corollary 3.3].

Since the sequentially complete topological groups are obviously the categorical counterpart of the sequentially closed Tychonoff spaces in the category of topological groups, these facts suggest the natural idea to study categorical properties of the sequentially

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complete groups and compare this class of groups with the narrower class of countably compact groups. Clearly, countable compactness is invariant under continuous mappings, so it is also natural to consider the classes of *sequentially h-complete* (*sequentially q-complete*) groups defined to be the groups all continuous (open) homomorphic images of which are sequentially complete.

We briefly discuss the relations between countably compact, pseudocompact, precompact and sequentially ($h$-)complete groups in Section 2 and present a diagram that visualises (a part of) our knowledge.

In Section 3 we prove that Abelian (more generally, nilpotent) sequentially $h$-complete groups are precompact (Theorem 3.6) and all precompact sequentially $h$-complete groups are pseudocompact (Theorem 3.9). In particular, Abelian sequentially $h$-complete groups are hereditarily pseudocompact (that is, all closed subgroups are pseudocompact). We also show that there are many precompact sequentially $q$-complete groups that fail to be pseudocompact: by Theorem 3.10, every infinite Abelian group $G$ endowed with the maximal totally bounded group topology is sequentially $q$-complete and non-pseudocompact.

Sequential completeness of quotients of free topological groups is considered in Section 4. It is shown that if $X$ is a closed subspace of a $\Sigma(\omega)$-product of compact metric spaces, then the groups $F(X)$ and $A(X)$ are sequentially $q$-complete (Corollary 4.4). We also show that for every infinite cardinal $\tau$, there exists a $\tau$-bounded space $X$ such that the free topological groups $F(X)$, $A(X)$, $F(X, \mathbf{P})$ and $F(X, \mathbf{PA})$ are not sequentially $q$-complete (Corollary 4.10), where $\mathbf{P}$ and $\mathbf{PA}$ are the varieties of precompact and precompact Abelian groups, respectively.

In Section 5 we prove that every pseudocompact Abelian group $H$ can be represented as a quotient $G/N$, where $G$ is a pseudocompact Abelian sequentially complete group and $N$ is a closed pseudocompact subgroup of $G$. This shows that the Abelian pseudocompact sequentially complete groups are far from being $q$-complete.

Finally, in Section 6 we present a list of unsolved problems supplied with short comments.

1.1. Preliminaries We recall here some compactness-like conditions on a topological group $G$. A group $G$ is *precompact* if its completion $\widetilde{G}$ is compact (or, equivalently, if for any open nonempty subset $U$ of $G$ there is a finite subset $F \subseteq G$ such that $F \cdot U = G$), *pseudocompact* if every continuous real-valued function on $G$ is bounded, *$\omega$-bounded* if every countable subset is contained in a compact subgroup (a group $G$ is $\omega$-bounded precisely when all closed separable subgroups of $G$ are compact). Clearly, $\omega$-boundedness implies countable compactness, countable compactness implies pseudocompactness, and pseudocompact groups are precompact [**9**] (see the diagram in Section 2). More generally, for an infinite cardinal $\tau$, a space $X$ is called *$\tau$-bounded* if the closure of every subset of $X$ of cardinality $\leqslant \tau$ is compact. We consider only Tykhonoff spaces here.

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We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive natural numbers and primes, respectively, by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, by $\mathbb{T}$ the unit circle subgroup of the complex plain $\mathbb{C}$, by $\mathbb{Z}_p$ the $p$-adic integers ($p \in \mathbb{P}$), by $\mathbb{Z}(n)$ the cyclic group of order $n$. The cardinality of the continuum $2^\omega$ will be denoted also by $\mathfrak{c}$.

All topological groups we consider are assumed to be Hausdorff. Completeness of topological groups is intended with respect to the two-sided uniformity, so that every topological group $G$ has the (Raĭkov) completion which we denote by $\widetilde{G}$, while $c(G)$ denotes the connected component of a group $G$. A topological group $G$ is called *minimal* if $G$ does not admit a coarser Hausdorff group topology, and *totally minimal* if every quotient of $G$ is minimal. The group $G$ is totally minimal if and only if every continuous epimorphism $\varphi\colon G \to H$ is open [**15**].

The centre of a group $G$ is denoted by $Z(G)$. Recall that the *upper central series* $\{Z_n(G)\}$ of the group $G$ is defined by: $Z_0(G) = \{1\}$ and $Z_{n+1}(G)/Z_n(G)$ is the centre of $G/Z_n(G)$. A group $G$ is *nilpotent* if $Z_n(G) = G$ for some integer $n$. For a topological group $G$, the subgroups $Z_n(G)$ are closed.

## 2. Various degrees of weak completeness

We recall here several closure operators related in a natural way to countable compactness, pseudocompactness and $\omega$-boundedness. For a categorical treatment of closure operators in full generality the reader can consult [**16**]. The topology $P\tau$ on $G$ associated to a topology $\tau$ on $G$, is generated by the family of all $G_\delta$-sets in $(G,\tau)$ taken as a base of open sets for $P\tau$. When referring to this topology we shall simply speak of *$G_\delta$-closed* and *$G_\delta$-dense* sets. The importance of this notion of density is revealed by the following theorem of Comfort and Ross [**9**].

**Theorem 2.1.** *A group $G$ is pseudocompact if and only if it is precompact and $G_\delta$-dense in $\widetilde{G}$.*

It can also be useful to note that if $(G,\tau)$ is a topological group, then so is $(G, P\tau)$ and the latter group is always zero-dimensional.

2.1. $\omega$-complete groups For a subset $A$ of a topological space $X$, define $cl_\omega(A) = \bigcup\{\overline{B} : B \subseteq A,\ |B| \leqslant \omega\}$. We say that $A$ is *$\omega$-closed* (respectively, *$\omega$-dense*) in $X$ if $cl_\omega(A) = A$ (respectively, $cl_\omega(A) = X$). This is an idempotent additive closure operator (in the sense of [**16**]) that induces a finer topology $\tau_\omega$ on $X$ of countable tightness (in fact, $\tau_\omega$ is the coarsest topology on $X$ with these two properties). This closure operator is useful for describing topological properties, for example, a space $X$ has countable tightness if and only if $\tau_\omega = \tau$ [**4**]. Further, by analogy with Theorem 2.1 one can prove that a topological group is $\omega$-bounded if and only if it is precompact and $\omega$-closed in its completion.

Call a group $G$ *$\omega$-complete* if it is $\omega$-closed in every topological group that contains $G$

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as a subgroup, or equivalently, if it is $\omega$-closed in $\widetilde{G}$. Now we can say that $\omega$-boundedness splits up into precompactness and $\omega$-completeness.

Every topological group $G$ admits an *$\omega$-complete hull* $G_\omega$ that has the universal property with respect to $\omega$-complete groups. More precisely, there exists an $\omega$-dense embedding $G \to G_\omega$ into an $\omega$-complete group $G_\omega$ such that every continuous homomorphism $f: G \to K$ to an $\omega$-complete group $K$ extends to a continuous homomorphism $f_\omega: G_\omega \to K$. One can take as $G_\omega$ the $\omega$-closure of $G$ in $\widetilde{G}$; it obviously has the desired universal property.

The class of $\omega$-bounded groups is closed under taking $\omega$-closed subgroups, quotients and products (the third property is easy to check; it also follows from [**46**, Theorems 4.7 and 4.9]). Hence every group admits also an *$\omega$-bounded hull*, $\omega G$ and a continuous homomorphism $G \to \omega G$ which is the restriction of the Bohr compactification $b_G: G \to bG$. Actually, $\omega G$ is the least $\omega$-closed subgroup of $bG$ which contains $b_G(G)$. Obviously, the $\omega$-bounded hull has the usual universal properties with respect to continuous homomorphisms with dense image $G \to K$, where $K$ is an $\omega$-bounded group. Note that the reflector $G \mapsto \omega G$ is the composition of the reflectors $G \mapsto b_G(G)$ and $G \mapsto G_\omega$ considered above. The hull $\omega G$ is an extension of the group $G$ if and only if the group $G$ is precompact (that is, the Bohr pre-compactification $b_G(G)$ coincides with $G$).

Let us note in connection with the $\omega$-bounded hull $\omega G$ that no "countably compact hull" can be defined even for precompact groups. This is due to the failure of productivity of countable compactness in topological groups (see Example 3.3).

2.2. Sequential completeness and its stronger versions As far as countably compact groups are concerned another closure operator seems to be relevant, namely the *sequential closure*. We say that a group $G$ is *sequentially complete* if $G$ is sequentially closed in any other Hausdorff group, that is, all Cauchy sequences in $G$ converge [**17**], [**18**]. Clearly, a group $G$ is sequentially complete if and only if it is sequentially closed in its completion. Therefore, sequential completeness is preserved by arbitrary direct products and inherited by sequentially closed subgroups. Hence, as above, every group $G$ admits a *sequentially complete hull* $G_{seq}$ (namely, the sequential closure of $G$ in $\widetilde{G}$, see [**17**, Proposition 2.1]).

While $\omega$-complete precompact groups are $\omega$-bounded, sequentially complete precompact groups need not be even pseudocompact (see Theorem 3.10 below).

Sequential completeness is not inherited by continuous homomorphic images. Indeed, the group $\mathbb{R}$ is complete while its image $f(\mathbb{R})$ under the continuous homomorphism $f: \mathbb{R} \to \mathbb{T}^2$ defined by $f(r) = (\pi(r), \pi(r\sqrt{2}))$ is a proper dense subgroup of the metric group $\mathbb{T}^2$, where $\pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is the quotient homomorphism.

Call a group $G$ *sequentially h-complete* if all continuous homomorphic images of $G$ are sequentially complete. Countably compact groups are obviously sequentially $h$-complete. This relation is studied more closely in Section 3. We give examples of sequentially

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$h$-complete groups that are neither countably compact nor precompact (Example 3.8), and show that sequential $h$-completeness is not finitely productive under Martin's Axiom (Example 3.3).

In the following diagram we describe several classes of groups with relation between them that frequently appear in the sequel. The only non-trivial relation, namely,

*precompact & sequentially h-complete* $\Rightarrow$ *pseudocompact*

will be proved in Theorem 3.9 where we show that the assumption of precompactness can be omitted in the case of nilpotent groups. We point out several examples showing that all implications are proper with the eventual exception of one. Namely, we have no examples to distinguish the precompact sequentially $h$-complete groups from the countably compact ones, see Question 6.1. The diagram also involves *$h$-$\omega$-complete* groups, that is, the groups whose continuous homomorphic images are $\omega$-complete (these groups appear after Theorem 3.6).

$$
\begin{tikzcd}
 & \boxed{\text{countably compact}} \arrow[d] & & & \boxed{\omega\text{-bounded}} \arrow[lll] \arrow[d] \\
 & \boxed{\text{precompact \& sequentially } h\text{-complete}} \arrow[dl] \arrow[dr] & & & \boxed{\text{precompact \& } h\text{-}\omega\text{-complete}} \arrow[lll] \arrow[d] \\
\boxed{\text{pseudocompact}} \arrow[dr] & & \boxed{\text{precompact \& sequentially complete}} \arrow[dl] \arrow[dr] & & \boxed{\text{precompact \& } \omega\text{-complete}} \arrow[ll] \arrow[d] \\
 & \boxed{\text{precompact}} & & \boxed{\text{sequentially complete}} & \boxed{\omega\text{-complete}} \arrow[l]
\end{tikzcd}
$$

As noted in [**17**, Proposition 2.2] the topological groups that have no convergent sequences other than the trivial ones are sequentially complete. Such precompact groups of arbitrary cardinality can be found in ZFC (see Theorem 3.10 below). There exist also infinite countably compact groups that have no convergent sequences other than the trivial one (see [**21**] for a zero-dimensional example of such a group under Martin's Axiom, or [**45**] for a connected and locally connected one under the Continuum Hypothesis).

## 3. Precompactness of sequentially $h$-complete groups

The class of *$h$-complete* groups was introduced in [**19**] under a different name and studied in [**20**]: these are the groups whose continuous homomorphic images are always complete. Here we establish several properties of the wider class of sequentially $h$-complete groups.

One can prove the following proposition by arguments similar to those in [**20**]:

**Proposition 3.1.** *Sequential $h$-completeness is inherited by closed central subgroups.*

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PROOF: Suppose that $G$ is a sequentially $h$-complete group and $H$ is a closed central subgroup of $G$. To check that $H$ is sequentially $h$-complete consider a continuous surjective homomorphism $f\colon H \to H_1$. Apply Lemma 2.17 of [**20**] to find a continuous homomorphic image $G_1$ of $G$ such that $H_1$ is isomorphic to a closed subgroup of $G_1$. Since $G_1$ is sequentially complete, $H_1$ is sequentially complete as well. ∎

EXAMPLE 3.2. Sequential $h$-completeness is not inherited by closed normal subgroups. Indeed, there exists a locally compact metrisable connected group, namely the semidirect product $G = \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$, where this fails (see [**20**, Example 5.6]). Here the closed normal subgroup $N \cong \mathbb{R}^2$ of $G$ is not sequentially $h$-complete, while $G$ is $h$-complete, and hence sequentially $h$-complete.

Now we show that sequential $h$-completeness is not finitely productive under Martin's Axiom.

EXAMPLE 3.3. Assuming that there exists an infinite countably compact Abelian group $E$ of exponent 2 without non-trivial convergent sequences, van Douwen produced (by an easy "tearing apart" ZFC construction [**21**, 6.2]) two countably compact subgroups $E_1$ and $E_2$ of $E$ such that $E_1 \times E_2$ has a closed countably infinite subgroup $D$. (To get such a countably compact group $E$ he needed $MA$.) Clearly, $D$ is precompact. By Theorem 3.9 below, $D$ cannot be sequentially $h$-complete, and by Proposition 3.1, $E_1 \times E_2$ is not sequentially $h$-complete. The existence of such a pair $E_1, E_2$ in ZFC is still an open problem (see [**5**, Question 1A.2]).

Let us prove that every Abelian (more generally, nilpotent) sequentially $h$-complete group is precompact. The proof of this result follows along the lines of [**20**]. It is based on a couple of lemmas.

**LEMMA 3.4.** *Separable metrisable sequentially $h$-complete Abelian groups are compact.*

PROOF: Let $G$ be a separable metrisable sequentially $h$-complete Abelian group. Clearly, $G$ is complete. We prove first that $G$ is totally minimal. Consider a continuous surjective homomorphism $f\colon G \to H$. Then $H$ has a countable network, and by a theorem of Arhangel'skiĭ [**1**], the group $H$ admits a coarser Hausdorff group topology of countable weight. Denote by $H'$ the group $H$ equipped with that topology. Then the metrisable group $H'$ is a continuous homomorphic image of $G$, and hence $H'$ is complete. The homomorphism $f\colon G \to H'$ of complete separable metrisable groups is open by the Banach Open Mapping Theorem. This implies that $f\colon G \to H$ is open as well, so that $G$ is totally minimal. It remains to note that every totally minimal complete Abelian group is compact [**37**]. ∎

**LEMMA 3.5.** *If all countable subgroups of a topological group $G$ are precompact, then $G$ is precompact.*

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PROOF: If $G$ is not precompact, there exists an open neighbourhood $U$ of the identity $e$ in $G$ such that $K \cdot U \neq G$ for every finite subset $K$ of $G$. Define by induction a sequence $X = \{x_n : n \in \omega\} \subseteq G$ such that $x_n \notin x_k U$ whenever $k < n$ and denote by $S$ the subgroup of $G$ generated by the set $X$. Choose an open symmetric neighbourhood $V$ of $e$ in $G$ with $V^2 \subseteq U$. An easy verification shows that for every $x \in G$, the set $xV$ contains at most one point of $X$. Therefore, the subgroup $S$ of $G$ is not precompact. ∎

In the proof of our precompactness theorem we shall use the following weak form of Guran's embedding theorem [**29**]: a separable topological group is topologically isomorphic to a subgroup of a Cartesian product of separable metrisable groups.

**THEOREM 3.6.** *Nilpotent sequentially $h$-complete groups are precompact.*

PROOF: From Guran's embedding theorem it follows that every separable sequentially $h$-complete Abelian group $G$ is a subgroup of a product of separable metrisable sequentially $h$-complete Abelian groups, which are compact by Lemma 3.4. Hence $G$ is precompact. Since sequential $h$-completeness in Abelian groups is inherited by closed subgroups (Proposition 3.1), and since precompactness is determined by separable subgroups (Lemma 3.5), we conclude that all sequentially $h$-complete Abelian groups are precompact. In the general case, $Z(G)$ is sequentially $h$-complete by Proposition 3.1 and hence precompact. Now $G/Z(G)$ is again sequentially $h$-complete and the proof goes on by induction on the nilpotency class with the use of the fact that the class of precompact groups is closed under extensions. (See Theorem 6.3 (a) of [**8**].) ∎

Let us call a group $G$ *$h$-$\omega$-complete* if all continuous homomorphic images of $G$ are $\omega$-complete. Note that every $h$-$\omega$-complete group is sequentially $h$-complete and $\omega$-complete.

**COROLLARY 3.7.** *For nilpotent groups, $h$-$\omega$-completeness coincides with $\omega$-boundedness.*

PROOF: Clearly, every $\omega$-bounded group is $h$-$\omega$-complete. Conversely, a nilpotent $h$-$\omega$-complete group is sequentially $h$-complete, and hence precompact by Theorem 3.6. It remains to note that precompact $\omega$-complete groups are $\omega$-bounded. ∎

Corollary 3.7 shows that countable compactness does not imply either $h$-$\omega$-completeness or $\omega$-completeness (take any countably compact Abelian group which is not $\omega$-bounded). The next example shows that both nilpotency and sequential $h$-completeness are essential in Theorem 3.6 (more precisely, neither "Abelian and sequentially $q$-complete" nor "$h$-complete" alone implies "precompact"; however, Abelian sequentially $h$-complete groups are pseudocompact and hence precompact by Theorem 3.9 below). We recall that a topological group $G$ is (sequentially) $q$-complete if all quotients of $G$ are (sequentially) complete. Clearly, $h$-completeness implies $q$-completeness as well as sequential $h$-completeness implies sequential $q$-completeness.

EXAMPLE 3.8.

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(a) The group $SL_2(\mathbb{R})$ is $h$-complete. Indeed, by a theorem of Remus and Stoyanov [**40**] every continuous surjective homomorphism $\phi\colon SL_2(\mathbb{R}) \to G$ is open, hence the group $G$ is locally compact (as a quotient of $SL_2(\mathbb{R})$) and consequently complete. On the other hand, $SL_2(\mathbb{R})$ is not precompact. This shows that a locally compact separable sequentially $h$-complete group need not be precompact.

(b) Let $X$ be a compact space. Then the free Abelian topological group $A(X)$ is sequentially $q$-complete by Proposition 4.2 below, but not precompact.

3.1. SOME SEQUENTIALLY $h$-COMPLETE GROUPS ARE PSEUDOCOMPACT Call a group $G$ *hereditarily pseudocompact* if every closed subgroup of $G$ is pseudocompact. It is known that there exist hereditarily pseudocompact Abelian groups that are not countably compact. In fact, the subgroup $G = M(\mathbb{T}^{\mathfrak{c}})$ of all *metrisable* elements of $\mathbb{T}^{\mathfrak{c}}$ is hereditarily pseudocompact and sequentially dense in $\mathbb{T}^{\mathfrak{c}}$, so that $G$ cannot be countably compact (see [**17**]).

**THEOREM 3.9.** *Any precompact sequentially $h$-complete group is pseudocompact. Nilpotent sequentially $h$-complete groups are pseudocompact. In particular, Abelian sequentially $h$-complete groups are hereditarily pseudocompact.*

PROOF: Let $G$ be a precompact sequentially $h$-complete group. Then for every closed normal $G_\delta$-subgroup $N$ of the completion $\widetilde{G}$ of $G$, the quotient $\widetilde{G}/N$ is metrisable, and hence the image $f(G)$ of $G$ under the canonical homomorphism $f\colon \widetilde{G} \to \widetilde{G}/N$ is sequentially closed and dense in $\widetilde{G}/N$. Therefore, $f(G) = \widetilde{G}/N$. Thus, every coset $gN$ in $\widetilde{G}$ meets $G$, so $G$ is $G_\delta$-dense in $\widetilde{G}$. By Theorem 2.1, $G$ is pseudocompact. This proves the first part of the theorem.

To prove the second part we note that by Theorem 3.6, nilpotent sequentially $h$-complete groups are precompact. Then they are also pseudocompact by the first claim of the theorem. Now assume that $G$ is an Abelian sequentially $h$-complete group. Every closed subgroup of $G$ is sequentially $h$-complete by Proposition 3.1, and hence pseudocompact. Therefore, $G$ is hereditarily pseudocompact. ∎

"Nilpotent" is essential in Theorem 3.9: there exists a non-precompact sequentially $h$-complete separable metrisable locally compact group (see Example 3.8).

The next theorem provides a large class of ZFC examples of precompact sequentially $q$-complete Abelian groups that are not pseudocompact. Following van Douwen [**22**], we denote by $G^{\#}$ the Abelian group $G$ equipped with the maximal precompact group topology.

**THEOREM 3.10.** *The group $G^{\#}$ is sequentially $q$-complete and non-pseudocompact for any infinite Abelian group $G$.*

PROOF: It was proved by Flor [**23**] that $G$ is always sequentially closed in its Bohr compactification, that is, $G^{\#}$ is sequentially closed in its completion (see also [**11**]). To

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finish the proof note that every subgroup $N$ of $G^{\#}$ is closed and $G^{\#}/N \cong (G/N)^{\#}$, so that $G^{\#}$ is sequentially $q$-complete. By a result of Comfort and Saks [**10**, Theorem 2.2], $G^{\#}$ is never pseudocompact. $\square$

Theorems 3.9 and 3.10 also show that many precompact sequentially $q$-complete groups are not sequentially $h$-complete: every infinite Abelian group equipped with its maximal precompact group topology suits.

## 4. Quotients of free topological groups

It is an interesting problem to characterise the class $\mathcal{SQ}$ of spaces $X$ for which the free topological group $F(X)$ is sequentially $q$-complete. Clearly, all discrete spaces have this property, so we have to restrict ourselves to considering spaces satisfying some compact type restrictions. Corollary 4.10 of [**18**] suggests the hypothesis that every countably compact space is in $\mathcal{SQ}$, but we shall see in Theorem 4.9 below that it is not so. Let us show that *$k_\omega$-spaces* are considerably better in this respect (see Proposition 4.2 below). We recall here some related notions.

Let $\gamma$ be a family of subsets of a space $X$ such that $\bigcup \gamma = X$. We say that $\gamma$ *generates* the topology of $X$ if a subset $U \subseteq X$ is open in $X$ if and only if $U \cap K$ is open in $K$ for every $K \in \gamma$. If there exists an increasing sequence $\{K_n : n \in \omega\}$ of compact subsets of $X$ that generates the topology of $X = \bigcup_{n\in\omega} K_n$, then $X$ is called a *$k_\omega$-space* [**24**]. The representation $X = \bigcup_{n\in\omega} K_n$ is said to be a *$k_\omega$-decomposition* of $X$ [**32**].

In the sequel we shall use the following result the proof of which goes almost exactly as in [**26**, Theorem 6] (see also [**39**]). Briefly, a topological group that admits a $k_\omega$-decomposition is complete:

**Theorem 4.1.** *Let $\gamma = \{B_n : n \in \omega\}$ be a sequence of compact symmetric subsets of a topological group $G$ such that $G = \bigcup_{n\in\omega} B_n$ and $B_n \cdot B_k \subseteq B_{n+k}$ for all $n, k \in \omega$. If $\gamma$ generates the topology of $G$, then $G$ is complete.*

Note that every topological group $G$ with a $k_\omega$-decomposition $G = \bigcup_{n\in\omega} K_n$ admits another $k_\omega$-decomposition $G = \bigcup_{n\in\omega} B_n$ satisfying the conditions of the above theorem. To see this, simply put $B_0 = \{1\}$ and $B_n = L_n \cdot \ldots \cdot L_n$ ($n$ times), where $L_n = K_0 \cup K_0^{-1} \cup \cdots \cup K_n \cup K_n^{-1}$ for each $n = 1, 2, \ldots$ .

Here we introduce some notation concerning free (Abelian) groups. Let $X$ be a nonempty set and $F(X)$ be the (abstract) free group on $X$. Every element $g$ of $F(X)$ is a *word* in the alphabet $X$ and has the form $g = x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. The word $g$ may be *reducible,* that is, $g$ can contain two adjacent letters $x_i^{\varepsilon_i}$ and $x_{i+1}^{\varepsilon_{i+1}}$ such that $x_i = x_{i+1}$ and $\varepsilon_i \cdot \varepsilon_{i+1} = -1$. Otherwise the element $g$ is called *irreducible* and we say that the length of $g$ equals to $n$: $l(g) = n$. This permits us

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to define the sets

$$F_n(X) = \{g \in F(X) : l(g) \leqslant n\},$$

where $n \in \mathbb{N}$. We also put $F_0(X) = \{e\}$, where $e$ is the identity of $F(X)$ (the empty word). One defines the subsets $A_n(X)$ of the free Abelian group $A(X)$ in a similar way.

If $Y$ is a subset of $X$, $F(Y, X)$ will denote the subgroup of $F(X)$ generated by the set $Y$. The same applies to the subgroup $A(Y, X)$ of $A(X)$. In addition, for every $n \in \omega$ we use the abbreviation $F_n(Y, X)$ for the subspace $F_n(X) \cap F(Y, X)$ of $F(X)$. Analogously, we put $A_n(Y, X) = A_n(X) \cap A(Y, X)$ in the case of the group $A(X)$.

The following proposition refines the result of Hunt and Morris [**31**] about completeness of free topological groups on $k_\omega$-spaces.

**PROPOSITION 4.2.** *If $X$ is a $k_\omega$-space, then the groups $F(X)$ and $A(X)$ are $q$-complete.*

PROOF: Suppose that $K$ is a closed normal subgroup of $F(X)$ and put $G = F(X)/K$. Let $p\colon F(X) \to G$ be the quotient homomorphism. By assumption, there exists a $k_\omega$-decomposition $X = \bigcup_{n\in\omega} C_n$. For every $n \in \omega$, the sets $A_n = F_n(C_n, X)$ and $B_n = p(A_n)$ are symmetric and compact. Clearly, we have $F(X) = \bigcup_{n\in\omega} A_n$ and $G = \bigcup_{n\in\omega} B_n$. In addition, $A_n \cdot A_k \subseteq A_{n+k}$ for all $n, k \in \omega$. By a theorem of [**32**], the sets $A_n$ generate the topology of $F(X)$. We claim that the sets $B_n$ generate the topology of $G$. Indeed, let $U$ be a subset of $G$ such that $U \cap B_n$ is open in $B_n$ for each $n \in \omega$. Then $p^{-1}(U) \cap A_n = p_n^{-1}(U \cap B_n)$, where $p_n = p|_{A_n}$, $n \in \omega$. Since $p_n$ is a continuous mapping, the set $p^{-1}(U) \cap A_n$ is open in $A_n$ for each $n \in \omega$, and hence $p^{-1}(U)$ is open in $F(X)$. So, $U = p\big(p^{-1}(U)\big)$ is open in $G$. This proves our claim.

It is clear that $B_n^{-1} = B_n$, $B_n \cdot B_k = p(A_n \cdot A_k) \subseteq p(A_{n+k}) = B_{n+k}$ for all $n, k \in \omega$, and $G = \bigcup_{n\in\omega} B_n$. Since the compact sets $B_n$ generate the topology of $G$, Theorem 4.1 implies that the group $G$ is complete. The same argument works for the quotients of the group $A(X)$. ∎

The above result implies that every $k_\omega$-space is in the class $\mathcal{SQ}$. By [**18**, Corollary 4.10], the groups $F(X)$ and $A(X)$ are sequentially complete for any countably compact space $X$, but not every countably compact space is in $\mathcal{SQ}$ by Corollary 4.10 below. Let us show that some special countably compact spaces are in $\mathcal{SQ}$.

**THEOREM 4.3.** *Let $X$ be a space such that $X^n$ is countably compact and normal for each $n \in \omega$. Then the groups $F(X)$ and $A(X)$ are sequentially $q$-complete.*

PROOF: Let $K$ be a closed normal subgroup of $F(X)$. Since $X$ is countably compact, we can identify $F(X)$ with the subgroup $F(X, \beta X)$ of $F(\beta X)$ (see [**36**] or [**35**]). Denote by $L$ the closure of $K$ in $F(\beta X)$ and put $G = F(\beta X)/L$. Let $p\colon F(\beta X) \to G$ be the quotient homomorphism. Since $K$ is dense in $L$, [**27**, Lemma 1.3] implies that $H = F(X)/K$ is topologically isomorphic to the dense subgroup $p\big(F(X, \beta X)\big)$ of $G$. For every $n \in \omega$, put

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$A_n = F_n(\beta X)$ and $B_n = p(A_n)$. Then the compact sets $B_n$ cover the group $G$ and, by the argument in the proof of Proposition 4.2, generate its topology. Therefore, following assertion is immediate:

(∗) If $S \subseteq G$ and $|S \cap B_n| < \aleph_0$ for each $n \in \omega$, then $S$ is closed and discrete in $G$.

The group $G$ is complete by Proposition 4.2, so it suffices to show that $p\big(F(X, \beta X)\big) \cong H$ is sequentially closed in $G$. Suppose that $S \subseteq H$ is a non-trivial sequence converging to a point $x \in G \setminus H$. Then (∗) implies that $S \subseteq B_k$ for some $k \in \omega$. Since all finite powers of $X$ are countably compact, we conclude that $F_k(X)$ and $C_k = p\big(F_k(X)\big)$ are also countably compact. We claim that $C_k$ is closed in $H$. Indeed, let $q$ be the restriction of $p$ to $F(X)$. Then $q^{-1}(C_k) = K \cdot F_k(X)$, and since all finite powers of $X$ are countably compact and normal, the latter set is closed in $F(X)$ by [**43**, Assertion 4]. Since $q\colon F(X) \to H$ is an open epimorphism, we conclude that $C_k$ is closed in $H$. Note that $C_k$ is dense in $B_k \cap H$, so that $C_k = B_k \cap H$. Being countably compact, $B_k \cap H$ is sequentially closed in $B_k$ and in $G$. Clearly, $S \subseteq B_k \cap H$, so $S$ cannot converge to $x \in G \setminus H$. This contradiction shows that $H$ is sequentially complete. The same argument applies to the group $A(X)$. ∎

We do not know whether Theorem 4.3 remains valid for spaces $X$ whose finite powers are countably compact (see Question 6.6).

In the sequel we make the use of $\Sigma(\tau)$-*products* defined as follows. Let $\Pi = \prod_{\alpha \in A} X_\alpha$ be a product space and $p \in \Pi$ be an arbitrary point. For every $x \in \Pi$, put $\operatorname{supp}(x) = \{\alpha \in A : \pi_\alpha(x) \neq \pi_\alpha(p)\}$, where $\pi_\alpha\colon \Pi \to X_\alpha$ is the projection. Given a cardinal $\tau \geqslant \omega$, we put

$$\Sigma(p, \tau) = \big\{x \in \Pi : |\operatorname{supp}(x)| \leqslant \tau\big\}.$$

Clearly, $\Sigma(p, \tau)$ is a dense subspace of $\Pi$. In addition, if all $X_\alpha$ are topological groups and $p$ is the identity of $\Pi$, then $\Sigma(p, \tau)$ is a subgroup of $\Pi$. We shall usually abbreviate $\Sigma(p, \tau)$ to $\Sigma(\tau)$. Topological properties of $\Sigma(\omega)$-products of (separable) metrisable spaces and compact spaces were studied in [**12, 28, 34**].

**Corollary 4.4.** *Let $X$ be a closed subspace of a $\Sigma(\omega)$-product of compact metric spaces. Then the groups $F(X)$ and $A(X)$ are sequentially $q$-complete.*

Proof: Any $\Sigma(\omega)$-product $Y$ of compact metric spaces is $\omega$-bounded [**34**] and normal [**12**]. Since a finite power of $Y$ is homeomorphic to a closed subspace of a larger $\Sigma(\omega)$-product, every closed subspace $X$ of $Y$ satisfies the conditions of Theorem 4.3. ∎

For an ordinal $\alpha$, let $T(\alpha)$ be the space with the underlying set $\alpha$ endowed with the well-order topology. We show now that even if the space $T(\alpha)$ is not countably compact, it belongs to the class $\mathcal{SQ}$. Hence $\mathcal{SQ}$ does not consist only of countably compact spaces.

**Corollary 4.5.** *The groups $F\big(T(\alpha)\big)$ and $A\big(T(\alpha)\big)$ are sequentially $q$-complete for each ordinal $\alpha$.*

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PROOF: If $cf(\alpha) \leqslant \omega$, Proposition 4.2 implies that the groups $F(X)$ and $A(X)$ are $q$-complete and hence sequentially $q$-complete. If $cf(\alpha) > \omega$, then the space $T(\alpha)^n$ is normal and countably compact for each $n \in \omega$, so Theorem 4.3 applies. ∎

We shall show now that the class $\mathcal{SQ}$ fails to contain very nice countably compact spaces. Our construction requires three auxiliary results. To avoid a repetition of similar arguments for the free groups $F(X)$, $F(X, \mathbf{P})$, et cetera, we shall use the notion of a *variety* of topological groups understood as a class of groups which is closed under taking Cartesian products, quotient groups and closed subgroups [**33**]. A variety $\mathcal{V}$ closed under taking arbitrary subgroups will be called an *S-variety*. Finally, if an $S$-variety $\mathcal{V}$ contains all homomorphic images of the groups in $\mathcal{V}$, then it has the name of a *J-variety* [**38**].

Given a variety $\mathcal{V}$, one defines the free $\mathcal{V}$-group $F(X, \mathcal{V}) \in \mathcal{V}$ on a Tychonov space $X$ to satisfy the following conditions (see [**33, 6**]):

(1) there exists a canonical continuous mapping $i_X \colon X \to F(X, \mathcal{V})$;

(2) $i_X(X)$ generates a dense subgroup of $F(X, \mathcal{V})$;

(3) for every group $G \in \mathcal{V}$ and every continuous mapping $f \colon X \to G$, there exists a continuous homomorphism $\widehat{f} \colon F(X, \mathcal{V}) \to G$ such that $\widehat{f} \circ i_X = f$.

If $Y \subseteq X$, we use the symbol $F(Y, X, \mathcal{V})$ for the subgroup of $F(X, \mathcal{V})$ generated by the set $i_X(Y)$. Note that $i_X$ is not necessarily a topological embedding; it may even fail to be injective [**33**]. It is important to note that if $\mathcal{V}$ is an $S$-variety, then for any space $X$, the set $i_X(X)$ algebraically generates the group $F(X, \mathcal{V})$. Indeed, denote by $G$ the subgroup $\langle i_X(X) \rangle$ of $F(X, \mathcal{V})$. By (3), there exists a continuous homomorphism $\varphi \colon F(X, \mathcal{V}) \to G$ such that $\varphi \circ i_X = i_X$. Then $\varphi|_G = id_G$ and since $G$ is dense in $F(X, \mathcal{V})$, we conclude that $F(X, \mathcal{V}) = G$.

Finally, for every variety $\mathcal{V}$ and every $G \in \mathcal{V}$, the mapping $i_G$ is a topological embedding, and hence the canonical continuous homomorphism $p \colon F(G, \mathcal{V}) \to G$ satisfying $p \circ i_G = id_G$ is open. For the same reason, the restriction of $p$ to the subgroup $\langle i_G(G) \rangle$ of $F(G, \mathcal{V})$ is also open.

**LEMMA 4.6.** *Let $\mathcal{V}$ be a variety of topological groups. Suppose that $H$ is a dense subgroup of a group $G \in \mathcal{V}$ and $X$ is a subspace of $G$ such that $H \subseteq X$. If $p \colon F(G, \mathcal{V}) \to G$ is the homomorphism extending the identity mapping $id_G$, then the restriction $p|_{F(X,G,\mathcal{V})} \colon F(X, G, \mathcal{V}) \to \langle X \rangle \subseteq G$ is an open epimorphism.*

PROOF: Consider the free topological group $F(G)$ on $G$ and the continuous isomorphism $\pi \colon F(G) \to F(G, \mathcal{V})$ which fixes the points of $G$. Then the composition $p \circ \pi \colon F(G) \to G$ coincides with the homomorphism $\varphi \colon F(G) \to G$ extending the identity mapping $id_G$. By Assertion C of [**26**, Section 4], the continuous homomorphism $\varphi$ is open, so $p$ is open too. Therefore, by Lemma 1.3 of [**27**], it suffices to verify that $F(X, G, \mathcal{V}) \cap \ker p$ is dense in $\langle i_G(G) \rangle \cap \ker p$. In what follows we identify $G$ with its image $i_G(G) \subseteq F(G, \mathcal{V})$.

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Let $W$ be an open subset of $F(G,\mathcal{V})$ such that $W \cap \ker p \neq \emptyset$. Choose an element $u \in W \cap \ker p$, say $u = g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}$, where $g_1, \dots, g_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$. Then $g_1^{\varepsilon_1} \cdot \ldots \cdot g_n^{\varepsilon_n} = e_G$ in $G$, where $e_G$ is the identity of $G$. Since $W$ is open in $F(G)$, we can find open sets $U_1, \dots, U_n$ and $U$ in $G$ such that $g_1 \in U_1, \dots, g_n \in U_n$, $e_G \in U$ and $U_n^{\varepsilon_1} \cdots U_n^{\varepsilon_n} U^{-1} U \subseteq W$ in $F(G)$. There exist open sets $V_1, \dots, V_n$ in $G$ such that $g_i \in V_i \subseteq U_i$ for each $i \leqslant n$ and $V_1^{\varepsilon_1} \cdot \ldots \cdot V_n^{\varepsilon_n} \subseteq U$ in $G$. For every $i \leqslant n$, pick an element $h_i \in V_i \cap H$. Then $h = h_1^{\varepsilon_1} \cdot \ldots \cdot h_n^{\varepsilon_n} \in H \cap U$. Consider the element $v = h_1^{\varepsilon_1} \cdots h_n^{\varepsilon_n} h^{-1} e_G$ of $F(X,G)$. It is easy to see that $v \in W$ and $p(v) = e_G$, that is, $v \in W \cap F(X,G) \cap \ker p$. Thus, $F(X,G,\mathcal{V}) \cap \ker p$ is dense in $\langle i_G(G) \rangle \cap \ker p$. This proves the lemma. ∎

The proof of the following result is close to that in [**36, 35**].

**Lemma 4.7.** *Let $X$ be a pseudocompact space. Then for any $S$-variety $\mathcal{V}$, the free $\mathcal{V}$-group $F(X,\mathcal{V})$ is topologically isomorphic to the subgroup $F(X,\beta X,\mathcal{V})$ of the free $\mathcal{V}$-group $F(\beta X,\mathcal{V})$.*

Proof: Denote by $i_X \colon X \to F(X,\mathcal{V})$ and $i_{\beta X} \colon \beta X \to F(\beta X,\mathcal{V})$ the canonical continuous mappings of $X$ and $\beta X$ respectively. The restriction $j = i_{\beta X}|_X \colon X \to F(\beta X,\mathcal{V})$ admits an "extension" to a continuous homomorphism $\widehat{\jmath} \colon F(X,\mathcal{V}) \to F(\beta X,\mathcal{V})$ satisfying $\widehat{\jmath} \circ i_X = j$. Note that the image $\widehat{\jmath}\big(F(X,\mathcal{V})\big)$ coincides with $F(X,\beta X,\mathcal{V})$. We claim that $\widehat{\jmath}$ is a topological isomorphism between $F(X,\mathcal{V})$ and $F(X,\beta X,\mathcal{V})$. Consider a continuous epimorphism $\varphi \colon F(X,\mathcal{V}) \to G$, where $G \in \mathcal{V}$. It suffices to show that there exists a continuous homomorphism $\psi \colon F(\beta X,\mathcal{V}) \to \widetilde{G}$ such that $\varphi = \psi \circ \widehat{\jmath}$. We shall do this in two steps.

I. Suppose that the group $G$ admits a coarser metrisable topology. We shall show a bit more, namely: there exists a continuous homomorphism $\psi \colon F(X,\mathcal{V}) \to G$ satisfying $\varphi = \psi \circ \widehat{\jmath}$. Indeed, the set $Y = \varphi\big(i_X(X)\big)$ is pseudocompact and hence compact [**3**, Lemma 5.10]. Therefore, $\varphi \circ i_X$ admits an extension to a continuous mapping $f \colon \beta X \to Y$. Let $\widehat{f} \colon F(\beta X,\mathcal{V}) \to G$ be a continuous homomorphism satisfying $\widehat{f} \circ i_{\beta X} = f$. Let us verify that $\varphi = \widehat{f} \circ \widehat{\jmath}$ (in other words, one can take $\psi = \widehat{f}$). We have:

$$\widehat{f} \circ \widehat{\jmath} \circ i_X = \widehat{f} \circ j = \widehat{f} \circ i_{\beta X}|_X = f|_X = \varphi \circ i_X,$$

whence it follows that $\widehat{f} \circ \widehat{\jmath}|_{i_X(X)} = \varphi|_{i_X(X)}$. Since $i_X(X)$ generates a dense subgroup of $F(X,\mathcal{V})$, the equality $\widehat{f} \circ \widehat{\jmath} = \varphi$ is immediate.

II. In the general case, the set $Y = \varphi\big(i_X(X)\big)$ is pseudocompact and algebraically generates the group $G$. Therefore, the group $G$ can be embedded as a topological subgroup into the direct product $\Pi = \prod_{i \in I} G_i$ of separable metrisable groups $G_i$ by Corollary 2 of [**36**] (see also [**2**, Corollary on page 140]). Let $p_i \colon \Pi \to G_i$ be the projection, $i \in I$. For every $i \in I$, denote by $K_i$ the kernel of the homomorphism $\pi_i = p_i \circ \varphi \colon F(X,\mathcal{V}) \to G_i$ and put $H_i = F(X,\mathcal{V})/K_i$. Since the quotient homomorphism $\phi_i \colon F(X,\mathcal{V}) \to H_i$ is open, the group $H_i$ belongs to $\mathcal{V}$. Clearly, there exists an isomorphism $h_i \colon H_i \to p_i(G)$

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satisfying $h_i \circ \phi_i = \pi_i$. Apply the fact that $\phi_i$ is open to deduce that the isomorphism $h_i$ is continuous. So, the group $H_i$ admits a coarser metrisable topology and by I, we can find a continuous homomorphism $\psi_i \colon F(\beta X, \mathcal{V}) \to H_i$ such that $\phi_i = \psi_i \circ \widehat{\jmath}$. For every $i \in I$, the homomorphism $\widetilde{\psi}_i = h_i \circ \psi_i \colon F(\beta X, \mathcal{V}) \to G_i$ is continuous, so the diagonal product $\psi$ of the family $\{\widetilde{\psi}_i : i \in I\}$ is a continuous homomorphism of $F(\beta X, \mathcal{V})$ to $\Pi$. By the definition of $\psi$, we have $p_i \circ \psi \circ \widehat{\jmath} = \pi_i = p_i \circ \varphi$ for each $i \in I$, whence it follows that $\psi \circ \widehat{\jmath} = \varphi$. It remains to note that $\widehat{\jmath}\big(F(X, \mathcal{V})\big)$ is a dense subgroup of $F(\beta X, \mathcal{V})$, and hence the continuity of $\psi$ implies that $\psi\big(F(\beta X, \mathcal{V})\big) \subseteq \overline{G}$, where $\overline{G}$ is the closure of $G$ in $\Pi$. Since $G \subseteq \overline{G} \subseteq \widetilde{G}$, the homomorphism $\psi \colon F(\beta X, \mathcal{V}) \to \widetilde{G}$ is as required. ∎

The following lemma is well known in topological group folklore, so we give only a brief sketch of its proof here.

**Lemma 4.8.** *Every non-trivial compact topological group contains a non-trivial closed metrisable Abelian subgroup.*

Proof: Let $G$ be a non-trivial compact group. By a theorem of Stoyanov (see also [**17**, Theorem 2.9]) there exists a prime number $p$ and a non-trivial closed subgroup $N_p$ of $G$ that is either a cyclic $p$-group or isomorphic to the group of $p$-adic integers. (The more precise result is that the subgroup of $G$ generated by all subgroups $N_p$ with this property, when the prime $p$ varies in $\mathbb{P}$, is dense in $G$.) Since $N_p$ is metrisable in both cases, we are done. ∎

We shall say that a variety $\mathcal{V}$ is *good* if it contains a non-trivial compact group. It is known that there exist $J$-varieties that are not good [**38**]. The following result shows that very strong forms of countable compactness of a space $X$ do not imply sequential $q$-completeness of the free group $F(X, \mathcal{V})$ for any good $S$-variety $\mathcal{V}$ of topological groups.

**Theorem 4.9.** *Let $\mathcal{V}$ be a good $S$-variety of topological groups. For every infinite cardinal $\tau$, there exists a $\tau$-bounded space $X$ such that the group $F(X, \mathcal{V})$ is not sequentially $q$-complete.*

Proof: By Lemma 4.8, $\mathcal{V}$ contains a non-trivial compact metrisable Abelian group $J$. Put $K = J^\omega$. Then $K \in \mathcal{V}$, and Theorem 1.12 of [**30**] implies that there exists a non-trivial sequence $S = \{a_n : n \in \omega\} \subseteq K$ converging to the neutral element $e_K$ of $K$ such that $\langle S \rangle$ is dense in $K$ (see also Theorem 5.13 of [**7**]). Then $C = S \cup \{e_K\}$ is a compact subset of $K$ which generates a proper dense subgroup of $K$. Since $K$ is metrisable, we can choose a sequence $\{b_n : n \in \omega\} \subseteq \langle C \rangle$ converging to an element $b \in K \setminus \langle C \rangle$.

Let $G$ be the compact group $K^\lambda$, where $\lambda = \tau^+$ and $\tau \geqslant \aleph_0$. Clearly, $G \in \mathcal{V}$. Denote by $\Sigma(\tau)$ the subgroup of $G$ consisting of all points $g \in G$ which have at most $\tau$ coordinates distinct from $0_K$. Then $\Sigma(\tau)$ is dense in $G$ and $\tau$-bounded. Let $i \colon K \to K^\lambda$ be the topological monomorphism that sends $x \in K$ to the point $i(x) \in K^\lambda$ whose coordinates are equal to $x$. Put $X = \Sigma(\tau) \cup i(C)$. Since $i(C)$ is compact, the subspace $X$ of $G$ is $\tau$-bounded. In addition, from $\Sigma(\tau) \subseteq X \subsetneq K^\lambda$ and $\beta\Sigma(\tau) = K^\lambda$ it follows that

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$\beta X = K^\lambda = G$. It is easy to see that the subgroup $\langle X\rangle$ of $G$ is not sequentially closed in $G$. Indeed, the sequence $\{i(b_n) : n \in \omega\} \subseteq \langle i(C)\rangle \subseteq \langle X\rangle$ converges to the point $i(b)$, so it suffices to verify that $i(b) \notin \langle X\rangle$. Suppose the contrary, then $i(b) \in \langle X\rangle \cap i(K)$ since the subgroup $i(K)$ is closed. Let us see now that this intersection coincides with the subgroup $\langle i(C)\rangle$. Indeed,

$$\langle X\rangle \cap i(K) = i(K) \cap \big(\Sigma(\tau) + \langle i(C)\rangle\big) = \big(i(K) \cap \Sigma(\tau)\big) + \langle i(C)\rangle = \langle i(C)\rangle.$$

Here the second equality is the modular law applied in the lattice of all subgroups of $G$, while the last one is due to the fact that $i(K)$ trivially intersects $\Sigma(\tau)$. This contradicts the choice of $b \in K \setminus \langle C\rangle$ in view of the injectivity of $i$.

Consider the group $F(G,\mathcal{V})$ and its subgroup $H = F(X,G,\mathcal{V})$ generated by $X$. By Lemma 4.7, the group $F(X,\mathcal{V})$ is topologically isomorphic to $H$. Let $p\colon F(G,\mathcal{V}) \to G$ be the homomorphism extending the canonical mapping $j_G\colon G \to F(G,\mathcal{V})$. Since $G \in \mathcal{V}$, the mapping $j_G$ is a homeomorphic embedding. Apply Lemma 4.6 to $G$, $X$ and $H = \Sigma(\tau)$ to conclude that the restriction of $p$ to the subgroup $F(X,G,\mathcal{V})$ of $F(G,\mathcal{V})$ is an open continuous epimorphism of $F(X,G,\mathcal{V}) \cong F(X,\mathcal{V})$ onto $\langle X\rangle$. Since $\langle X\rangle$ is not sequentially closed in $G$, the group $F(X,\mathcal{V})$ fails to be sequentially $q$-complete. ∎

Denote by $\mathbf{P}$ ($\mathbf{PA}$) the $S$-variety of all precompact (Abelian) topological groups. Clearly, the $S$-varieties $\mathbf{P}$ and $\mathbf{PA}$ are good, so the following result is immediate.

**Corollary 4.10.** *For every infinite cardinal $\tau$, there exists a $\tau$-bounded space $X$ such that the groups $F(X)$, $A(X)$, $F(X,\mathbf{P})$ and $F(X,\mathbf{PA})$ are not sequentially $q$-complete.*

Remark 4.11. Theorem 4.9 shows that by adding a convergent sequence $C$ to the space $X = \Sigma(\omega) \subseteq \mathbb{Z}(2)^{\mathfrak{c}}$ we destroy sequential $q$-completeness of the groups $F(X \cup C)$ and $A(X \cup C)$, while the groups $F\big(\Sigma(\omega)\big)$ and $A\big(\Sigma(\omega)\big)$ are sequentially $q$-complete in view of Corollary 4.4.

## 5. Quotients of pseudocompact sequentially complete groups

Corollary 4.10 (combined with [**18**, Corollary 5.5]) gives a series of precompact Abelian sequentially complete groups of the form $F(X,\mathbf{PA})$ that fail to be sequentially $q$-complete. However, these groups are never pseudocompact. In fact, it is impossible to define a "free pseudocompact topological group" on a nonempty space [**6**]. This gives rise to the question whether pseudocompact Abelian sequentially complete groups are sequentially $q$-complete. Theorem 5.5 below gives a strongly negative answer to this question. As usual, we start with auxiliary results.

**Lemma 5.1.** *Let $S$ be a subgroup of the free group $A(X)$ and $\varphi\colon S \to \mathbf{T}$ be a homomorphism. Then there exists a mapping $f\colon X \to \mathbb{T}$ such that $\widehat{f}|_S = \varphi$, where $\widehat{f}\colon A(X) \to \mathbb{T}$ is the homomorphism extending $f$.*

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PROOF: Since the group $\mathbb{T}$ is divisible, $\varphi$ extends to a homomorphism $\widetilde{\varphi}\colon A(X) \to \mathbb{T}$. Put $f = \widetilde{\varphi}|_X$. Clearly, then $\widetilde{\varphi} = \widehat{f}$, and hence $\varphi = \widetilde{\varphi}|_S = \widehat{f}|_S$. $\square$

The next lemma appeared in [**17**], but it actually goes back to [**23**]. We reproduce its proof here for the sake of completeness.

**LEMMA 5.2.** *A topological group without non-trivial convergent sequences is sequentially complete.*

PROOF: Suppose that a topological group $G$ contains a non-trivial sequence $\{x_n : n \in \omega\}$ converging to a point of $\widetilde{G} \setminus G$. For every $n \in \omega$, put $y_n = x_n^{-1} \cdot x_{n+1}$. Then $\{y_n : n \in \omega\}$ is a non-trivial sequence in $G$ converging to the identity of $G$. $\square$

Our third lemma is a very special case of Glicksberg's theorem in [**25**] saying that a locally compact group $G$ and the group $G^+$ (the same group $G$ but endowed with the Bohr topology) have the same compact sets. An elementary proof of this theorem that does not require the methods of Functional Analysis can be found in [**15**, Theorem 3.4.3]. Following van Douwen's [**22**], we use $G^\#$ instead of $G^+$ for a discrete group $G$.

**LEMMA 5.3.** *The group $A(X)^\#$ does not contain non-trivial convergent sequences for any set $X$.*

Now we present the main result of this section in a slightly more general form than it is necessary for further applications here.

**THEOREM 5.4.** *Every Abelian topological group $H$ is topologically isomorphic to the quotient group $G/N$, where $G$ is a sequentially complete Abelian group and $N$ is a closed pseudocompact subgroup of $G$.*

PROOF: Our construction resembles the one applied in [**44**, Example 3.6], but the arrangement of the main details here is different. Let $\tau = \left(|H| \cdot \omega\right)^\omega$. Our aim is to define a dense subgroup $G$ of $H \times \mathbb{T}^\tau$ such that $p(G) = H$ and $N = G \cap p^{-1}(0_H)$ is a dense pseudocompact subgroup of $\{0_H\} \times \mathbb{T}^\tau$, where $p\colon H \times \mathbb{T}^\tau \to H$ is the projection. We have to guarantee that $G$ will be sequentially closed in $\widetilde{H} \times \mathbb{T}^\tau$, where $\widetilde{H}$ is the completion of the group $H$. Clearly, the projection $p$ is open and its restriction to $G$ is an open homomorphism of $G$ onto $H$ by [**27**, Lemma 1.3], so that $H \cong G/N$.

In fact, we shall construct the subgroup $G$ of $H \times \mathbb{T}^\tau$ with two additional properties. First, the restriction to $G$ of the projection $\pi\colon H \times \mathbb{T}^\tau \to \mathbb{T}^\tau$ will be injective. Second, the subgroup $\pi(G)$ of $\mathbb{T}^\tau$ won't have non-trivial convergent sequences.

In its turn, to ensure that $G$ will have these two properties, we take care of constructing it in such a way that *every* homomorphism $h\colon S \to \mathbb{T}$ defined on any countable subgroup $S$ of $G$ extends to a *continuous* homomorphism $\widetilde{h}\colon G \to \mathbb{T}$, and this continuous extension is simply the restriction of the projection of $G$ to the factor $\mathbb{T}_\alpha$ for some $\alpha < \tau$.

Let us enumerate the set

$$\Sigma = \bigcup\{H \times \mathbb{T}^K : K \subseteq \tau,\ |K| \leqslant \omega\},$$

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say $\Sigma = \{x_\gamma : \gamma < \tau\}$. By definition, for every $\gamma < \tau$ there exists a countable subset $K_\gamma$ of $\tau$ such that $x_\gamma \in H \times \mathbb{T}^{K_\gamma}$. Denote by $A(\tau)$ the abstract free Abelian group on the set $\tau$. For every $g \in A(\tau)$, denote by $\operatorname{supp}(g)$ the minimal finite subset $F$ of $\tau$ such that $g \in \langle F\rangle$. Enumerate the family $\mathcal{S}$ of countable subgroups of $A(\tau)$, say $\mathcal{S} = \{S_\nu : \nu < \tau\}$. For every $\nu < \tau$, let $\mathcal{H}_\nu$ be the family of all homomorphisms from $S_\nu$ to $\mathbb{T}$. Clearly, $|\mathcal{H}_\nu| \leqslant \mathfrak{c} \leqslant \tau$ for each $\nu < \tau$. Therefore, the family $\mathcal{H} = \bigcup\limits_{\nu<\tau} \mathcal{H}_\nu$ has cardinality $\leqslant \tau$ and we can write $\mathcal{H} = \{h_\alpha : \alpha < \tau\}$. For every $\alpha < \tau$, denote by $\nu(\alpha)$ the ordinal $< \tau$ such that $h_\alpha \in \mathcal{H}_{\nu(\alpha)}$. Finally, let $\pi_K\colon H \times \mathbb{T}^\tau \to H \times \mathbb{T}^K$ and $\pi^L_K\colon H \times \mathbb{T}^L \to H \times \mathbb{T}^K$ be the natural projections, where $K \subseteq L \subseteq \tau$.

Our recursive construction of the group $G$ is quite easy after all preliminary definitions have been given. For every $\alpha < \tau$, we shall define a subset $A_\alpha$ of $\tau$ and the points $y_{\gamma,\alpha} \in H \times \mathbb{T}^{B_{\gamma,\alpha}}$ for all $\gamma < \tau$, where $B_{\gamma,\alpha} = K_\gamma \cup A_\alpha$, satisfying the following conditions:

(1) $A_\beta \subseteq A_\alpha$ if $\beta < \alpha < \tau$;

(2) $\alpha \subseteq A_\alpha$;

(3) $|A_\alpha| \leqslant |\alpha| \cdot \omega$;

(4) $\pi^{B_{\gamma,\alpha}}_{K_\gamma}(y_{\gamma,\alpha}) = x_\gamma$ for each $\gamma < \tau$;

(5) $\pi^{B_{\gamma,\alpha}}_{B_{\gamma,\beta}}(y_{\gamma,\alpha}) = y_{\gamma,\beta}$ whenever $\beta < \alpha$ and $\gamma < \tau$.

From (5) it follows that $y_{\gamma,\alpha}$ extends $y_{\gamma,\beta}$ if $\beta < \alpha < \tau$. Therefore, (2) implies that for every $\gamma < \tau$ there exists the unique point $y_\gamma \in H \times \mathbb{T}^\tau$ such that $\pi_{B_{\gamma,\alpha}}(y_\gamma) = y_{\gamma,\alpha}$ for all $\alpha < \tau$. Clearly, $\pi_{K_\gamma}(y_\gamma) = x_\gamma$ (see (4)). We put $X = \{y_\gamma : \gamma < \tau\}$ and define $G$ to be the subgroup of $H \times \mathbb{T}^\tau$ generated by $X$. However, to guarantee that every homomorphism $h\colon S \to \mathbb{T}$ defined on a countable subgroup $S$ of $G$ extends to a continuous homomorphism $\widetilde{h}\colon G \to \mathbf{T}$, we have to perform the construction more carefully.

Suppose that for some $\alpha < \tau$, we have defined the sets $A_\beta$ and the points $y_{\gamma,\beta}$ that satisfy (1)–(5) for all $\beta < \alpha$ and $\gamma < \tau$. If $\alpha$ is a limit, put $A_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$ and for every $\gamma < \tau$, define a point $y_{\gamma,\alpha} \in H \times \mathbb{T}^{B_{\gamma,\alpha}}$ that satisfies (4) and (5) for all $\beta < \alpha$. This completely defines $y_{\gamma,\alpha}$. Clearly, the set $A_\alpha$ and the points $y_{\gamma,\alpha}$ satisfy (1)–(5).

Suppose now that $\alpha = \beta + 1$. Consider the homomorphism $h_\alpha\colon S_\nu \to \mathbf{T}$, where $\nu = \nu(\alpha)$, and put

$$C_\nu = \bigcup\{\operatorname{supp}(g) : g \in S_\nu\} \text{ and } L_\nu = \bigcup\{K_\gamma : \gamma \in C_\nu\}.$$

Then $|C_\nu| \leqslant \omega$, $|L_\nu| \leqslant \omega$ and $S_\nu \subseteq A(C_\nu, \tau)$. By Lemma 5.1, there exists a mapping $f\colon C_\nu \to \mathbb{T}$ such that $h_\alpha = \widehat{f}|_{S_\nu}$, where $\widehat{f}\colon A(C_\nu,\tau) \to \mathbf{T}$ is the homomorphism extending $f$. Here $A(C_\nu, \tau)$ denotes the subgroup of $A(\tau)$ generated by the set $C_\nu$. Let us define the subset $A_\alpha$ of $\tau$ by $A_\alpha = A_\beta \cup L_\nu \cup \{\beta, \delta\}$, where $\delta = \delta(\alpha) \in \tau \setminus (A_\beta \cup L_\nu \cup \{\beta\})$ is arbitrary. For every $\gamma < \tau$, put $B_{\gamma,\alpha} = K_\gamma \cup A_\alpha$. It remains to define the points $y_{\gamma,\alpha} \in H \times \mathbb{T}^{B_{\gamma,\alpha}}$ for all $\gamma < \tau$. To this end, we put $y_{\gamma,\alpha}(\delta) = f(\gamma)$ for every $\gamma \in C_\nu$, $y_{\gamma,\alpha} = x_\gamma(\delta)$ if $\gamma \in \tau \setminus C_\nu$

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and $\delta \in K_\gamma$, and $y_{\gamma,\alpha}(\delta) = 0$ otherwise. This defines the points $y_{\gamma,\alpha}$ at the $\delta$th coordinate. Then we define $y_{\gamma,\alpha}$ at the rest of coordinates in $B_{\gamma,\alpha}$ to satisfy (4) and (5). It is easy to verify that the set $A_\alpha$ and the points $y_{\gamma,\alpha}$ satisfy (1)–(5). This finishes our construction.

Define the homomorphism $\varphi\colon A(\tau) \to H \times \mathbb{T}^\tau$ as follows. Suppose that $g = n_1\gamma_1 + \cdots + n_k\gamma_k \in A(\tau)$, where $n_1, \dots, n_k \in \mathbb{Z} \setminus \{0\}$ and $\gamma_1, \dots, \gamma_k < \tau$. Then put $\varphi(g) = n_1 y_{\gamma_1} + \cdots + n_k y_{\gamma_k}$ (we use additive notation here). It is clear that $\varphi(A(\tau)) = G$. Our definition of the points $y_{\gamma,\alpha}$ at the $\delta$th coordinate and (5) together imply the following:

(6) $h_\alpha(g) = \varphi(g)(\delta)$ for each $g \in S_\nu$ (here $\nu = \nu(\alpha)$ and $\delta = \delta(\alpha)$).

Informally, (6) means that the projection of $\varphi(S_\nu)$ to the $\delta$th coordinate "represents" the homomorphism $h_\alpha$. Let $\pi_\delta\colon H \times \mathbb{T}^\tau \to \mathbf{T}_\delta$ be the projection. Since every countable subgroup $S$ of $G$ is of the form $\varphi(S_\nu)$ for some $\nu < \tau$, we conclude that any homomorphism $h\colon S \to \mathbb{T}$ admits a continuous extension to a homomorphism $\widetilde{h}\colon G \to \mathbb{T}$; this extension $\widetilde{h}$ is simply the restriction of the projection $\pi_\delta$ to $G$ for some $\delta < \tau$. We thus have:

(6*) If $S$ is a countable subgroup of $G$ and $h\colon S \to \mathbb{T}$ is a homomorphism, then there exists $\delta < \tau$ such that $\pi_\delta|_S = h$.

It is important to note that the assertion similar to (6*) remains valid for countable subgroups of the group $\pi(G) \subseteq \mathbb{T}^\tau$. Indeed, let $R$ be a countable subgroup of $\pi(G)$ and $g\colon R \to \mathbb{T}$ be a homomorphism. Then one can find a countable subgroup $S$ of $G$ with $\pi(S) = R$ and then apply (6*) to the homomorphism $g \circ \pi|_S\colon S \to \mathbb{T}$ in order to choose $\delta < \tau$ such that $g \circ \pi|_S = \pi_\delta|_S$. Since $\pi(S) = R$, we conclude that $g$ coincides with the projection of $R$ to the $\delta$th coordinate.

This property of $\pi(G)$ implies that every countable subgroup $R$ of $\pi(G)$ inherits from $\pi(G)$ the maximal totally bounded group topology, that is, $R = R_d^\#$, where $R_d$ is the group $R$ with the discrete topology.

The next step is to show that $\pi(G)$ does not contain non-trivial convergent sequences. Let us first note that the group $\pi(G)$ is algebraically isomorphic to the free Abelian group $A(\tau)$. Indeed, it suffices to verify that $\pi \circ \varphi\colon A(\tau) \to \pi(G)$ is an isomorphism. From the definition of $G$ it readily follows that $\psi = \pi \circ \varphi$ is an epimorphism, so we have to show that the kernel of $\psi$ is trivial. Let $g \in A(\tau)$ be an arbitrary element, $g \neq 0$. Denote by $S$ the cyclic subgroup of $A(\tau)$ generated by $g$ and consider a homomorphism $h\colon S \to \mathbb{T}$ such that $h(g) \neq 0$. Then $h = h_\alpha$ for some $\alpha < \tau$. Put $\nu = \nu(\alpha)$ and $\delta = \delta(\alpha)$. Clearly, $S = S_\nu$ and, by (6),

$$0 \neq h(g) = h_\alpha(g) = \varphi(g)(\delta) = \psi(g)(\delta).$$

Therefore, $g \notin \ker(\psi)$. This proves that $\ker(\psi) = \{0\}$, so that $\psi$ is an isomorphism. In particular, we conclude that the restriction of $\pi$ to $G$ is a monomorphism.

Let $K$ be a sequence in $\pi(G)$. Denote by $R$ the subgroup of $\pi(G)$ generated by $K$. Then $R$ is countable, and hence $R = R_d^\#$. By [**41**, 4.2.3], the subgroup $R$ of $\pi(G)$ is algebraically isomorphic to the free Abelian group $A(Y)$ with $\leqslant \omega$ generators, whence

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$R \cong A(Y)^{\#}$. Apply Lemma 5.3 to conclude that $R$ does not contain non-trivial convergent sequences, and hence the same is true for $\pi(G)$. Since $\pi|_G \colon G \to \pi(G)$ is a continuous isomorphism of $G$ to $\pi(G)$, the group $\pi(G)$ does not contain non-trivial convergent sequences either. Therefore, from Lemma 5.2 it follows that $G$ is sequentially complete.

Denote by $q$ the restriction to $G$ of the projection $p\colon H \times \mathbb{T}^\tau \to H$. We claim that the kernel $N$ of $q$ is a dense pseudocompact subgroup of $\{0_H\} \times \mathbb{T}^\tau$. This is equivalent to saying that $N$ fills in all countable faces of $\mathbb{T}^\tau$. Indeed, let $x$ be an arbitrary point of $\{0_H\} \times \mathbb{T}^K$, where $K$ is a countable subset of $\tau$. Then $x \in \Sigma$, so that $x = x_\gamma$ and $K = K_\gamma$ for some $\gamma < \tau$. The projection of $y_\gamma$ to $H \times \mathbb{T}^{K_\gamma}$ coincides with $x_\gamma$ by (4), and $y_\gamma \in G \cap \ker(p) = N$. This proves our claim. Note that this also implies that $G$ is dense in $H \times \mathbb{T}^\tau$. Therefore, the quotient group $G/N$ is topologically isomorphic to $H$, thus finishing the proof. ∎

**THEOREM 5.5.** *Every pseudocompact Abelian group $H$ is topologically isomorphic to the quotient group $G/N$, where $G$ is a sequentially complete pseudocompact Abelian group and $N$ is a closed pseudocompact subgroup of $G$.*

PROOF: By Theorem 5.4, we can find a sequentially complete Abelian group $G$ and a closed pseudocompact subgroup $N$ of $G$ such that $H \cong G/N$. Since both $N$ and $G/N$ are pseudocompact, [**8**, Theorem 6.6 (c)] (or [**42**, Corollary 2.26]) implies that $G$ is also pseudocompact. ∎

REMARK 5.6. The group $G$ in Theorem 5.5 has several additional properties that follow from the construction given in Theorem 5.4:

(a) $G$ is algebraically the free Abelian group of cardinality $|H|^\omega \cdot \mathfrak{c}$;

(b) $G$ is a pseudocompact group without non-trivial convergent sequences;

(c) every countable subgroup of $G$ is closed in $G$; in particular, every countable subgroup of $G$ is sequentially complete.

## 6. QUESTIONS

The first two questions are taken from [**14**]. Our first question concerns the diagram in Section 2.

QUESTION 6.1.

(1) Must a precompact sequentially $h$-complete group $G$ be countably compact? What if $G$ is Abelian?

(2) Find an example of a pseudocompact sequentially $q$-complete group that is not sequentially $h$-complete.

QUESTION 6.2. Are minimal sequentially complete totally (hereditarily) disconnected groups zero-dimensional?

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In view of Theorem 3.9 and [**13**, Theorem 1.2], the answer to the above question is "yes" for sequentially $h$-complete Abelian groups. By [**17**, Corollary 4.9] this is also true for minimal hereditarily disconnected sequentially complete Abelian groups.

QUESTION 6.3.

(1) Is $h$-$\omega$-completeness finitely productive?

(2) Is sequential $h$-completeness consistently finitely productive (see Example 3.3)?

We do not know if adding minimality (nilpotency) may help in answering Question 6.3. Note that nilpotency gives "yes" in (1) and (2) of the above question.

QUESTION 6.4. Is sequential completeness preserved by perfect continuous epimorphisms? More generally, find classes of continuous homomorphisms that preserve sequential completeness.

As Theorem 5.5 shows, continuous open homomorphisms with pseudocompact kernel do not preserve sequential completeness even in the class of pseudocompact groups. The groups that arise on the basis of our construction in Theorem 5.5 are very far from being hereditarily pseudocompact. It is also known that hereditary pseudocompact groups need not be sequentially complete (see the comment before Theorem 3.9). This gives rise to the following problem.

QUESTION 6.5. Is a hereditarily pseudocompact sequentially complete group sequentially $q$-complete?

QUESTION 6.6. Does Theorem 4.3 remain valid for any space $X$ whose finite powers are countably compact?

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