

ARTICLE

All meromorphic solutions of Fermat-type functional equations

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Abstract

In this article, by utilizing the properties of elliptic functions, we characterize the meromorphic solutions of Fermat-type functional equations $f(z)^n + f(L(z))^m = 1$ over the complex plane $\mathbb{C}$, where $L(z)$ is a nonconstant entire function, and m and n are two positive integers. As applications, we also investigate the meromorphic solutions of Fermat-type difference and q -difference equations.

1. Introduction and main results

In 1637, Fermat [12] stated the conjecture (which is known as Fermat's last theorem) that the equation $x^m + y^m = 1$ cannot admit positive rational solutions if $m > 2$. Since then, the equation has been the subject of intense and often heated discussions. In 1995, Wiles [41], [43] proved the profound conjecture.

In 1927, Montel in [38] initially considered the functional equation

$$f^m(z) + g^m(z) = 1, \quad (1)$$

which can be regarded as an analog of Fermat diophantine equation $x^m + y^m = 1$ over function fields. He showed that all the entire solutions $f(z)$ and $g(z)$ of (1) must be constant if $m \geq 3$ (see also Jategaonka [25]). Subsequent investigations were carried out by Baker in [1] and Gross in [16], respectively, they generalized Montel's result by proving that (1) does not have nonconstant meromorphic solutions when $m > 4$ and described nonconstant meromorphic solutions for $m = 2, 3$. In 1970, Yang [46] considered more general functional equation

$$f^n(z) + g^m(z) = 1 \quad (2)$$

and derived that this equation does not admit nonconstant meromorphic solutions if $\frac{1}{n} + \frac{2}{m} < 1$. From then on, (2) has been studied in various settings (see [14], [29], [30], [46]). For convenience, some results can be stated as follows (see, e.g., [8, Proposition 1], [1], [15], [45]).

Theorem A. *Suppose that $f(z)$ and $g(z)$ are nonconstant meromorphic solutions of the functional equation (2).*

(i) If $n = m = 3$, then $f(z) = \frac{\frac{1}{2}\left\{1 + \frac{\wp'(h(z))}{\sqrt{3}}\right\}}{\wp(h(z))}$ and $g(z) = \frac{\frac{\eta}{2}\left\{1 - \frac{\wp'(h(z))}{\sqrt{3}}\right\}}{\wp(h(z))}$ for any nonconstant entire function h , where $\eta^3 = 1$ and $\wp$ is the Weierstrass $\wp$ -function satisfying $(\wp')^2 \equiv 4\wp^3 - 1$ after appropriately choosing its periods.

(ii) If $n = 2$ and $m = 3$, then $f(z) = i\wp'(h(z))$ and $g(z) = \eta\sqrt[3]{4}\wp(h(z))$, where $h(z)$ is any nonconstant entire function.

(iii) If $m = 2$ and $n = 4$, then $f(z) = sn'(h(z))$ and $g(z) = sn(h(z))$, where $h(z)$ is any nonconstant entire function, and sn is the Jacobi elliptic function satisfying $sn'^2 = 1 - sn^4$.

In 1989, Yanagihara [45] established the existence of meromorphic solutions to the Fermat-type functional equation in another direction. In fact, Yanagihara proved that $f^3(z) + f^3(z+c) = 1$ does not admit nonconstant meromorphic functions of finite order. Some related results were obtained by Nakamura and Yanagihara in [39]. The above result due to Yanagihara was also independently derived by Lü–Han in [22] by employing the difference analog of the logarithmic derivative lemma of finite-order meromorphic functions, which was established by Chiang and Feng [9] and Halburd and Korhonen [19], independently. In 2014, this difference analog was further improved by Halburd et al. [21] to meromorphic functions of hyper-order strictly less than 1. Here, the order and hyper-order of a meromorphic function f are defined as

$$\rho(f) := \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \rightarrow +\infty} \frac{\log \log T(r, f)}{\log r},$$

where $T(r, f)$ denotes the Nevanlinna characteristic function of f .

By means of these results, Korhonen and Zhang in [27] generalized Yanagihara's theorem to meromorphic solutions of hyper-order strictly less than 1 as follows.

Theorem B. *The functional equation*

$$f^3(z) + f^3(z+c) = 1$$

does not admit nonconstant meromorphic solutions of hyper-order strictly less than 1.

Recently, Lü and his co-workers [7], [22], [36] also considered this type of problem, and their results can be stated as follows.

Theorem C. *Let $P(z)$ be a polynomial and $f(z)$ be a nonconstant meromorphic function of hyper-order strictly less than 1. If $f(z)$ is a solution of*

$$f^3(z) + f^3(z+c) = e^{P(z)}, \quad (3)$$

then, $P(z) = \alpha z + \beta$ and $f(z) = de^{\frac{\alpha z + \beta}{3}}$, where $d(\neq 0)$ is constant and $d^3(1 + e^{\alpha c}) = 1$.

In recent years, Fermat-type difference equations, together with differential–difference equations, have been the subject of extensive research. Consequently, numerous subsequent investigations have been conducted by scholars in this direction (see, e.g., [33], [35]). We note that most of the above results, including Theorems B and C, were derived under the condition that the solutions are of hyper-order strictly less than 1, since the difference analog of the logarithmic derivative lemma is required in the proofs of these results. So, it is natural to ask what will happen if the hyper-order condition is omitted. However, it is found that the conclusions of some previous theorems may not hold, as shown by the following example offered by Han in [22].

Example 1. Let $c = \pi i$ and α, β be fixed constants satisfying $e^{\alpha c} = 1$. Consider $f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} e^{\frac{\alpha z + \beta}{3}}$

with $h(z) = e^z$. Then, a routine computation yields $f(z+c) = \frac{\eta}{2} \frac{1 - \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} e^{\frac{\alpha z + \beta}{3}}$, where $\eta = e^{\frac{\alpha c}{3}}$. Furthermore,

$$f^3(z) + f^3(z+c) = e^{\alpha z + \beta},$$

which implies that the above equation admits the meromorphic solution $f(z)$. Obviously, $f(z)$ is not the form $de^{\frac{\alpha z + \beta}{3}}$ and the hyper-order of f is $\rho_2(f) = 1$.

Therefore, if the hyper-order condition is omitted, the Fermat-type difference equation may admit other types of meromorphic solutions. In this article, we focus on characterizing these solutions. By utilizing the properties of elliptic functions, we describe the forms of all meromorphic solutions to some Fermat-type difference equations.

Before giving the main results, we introduce the Weierstrass $\wp$ -function.

The Weierstrass $\wp$ -function is an elliptic (also doubly periodic) function with periods ω_1 and ω_2 ($\text{Im} \frac{\omega_1}{\omega_2} \neq 0$), defined as follows:

$$\wp(z) = \wp(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2} \right\}$$

and satisfies, after appropriately choosing ω_1 and ω_2 ,

$$(\wp')^2 = 4\wp^3 - 1.$$

The periods of $\wp$ span the lattice $\mathbb{L} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Denote $\mathbb{L}$ as

$$\mathbb{L} = \{\mu\omega_1 + \nu\omega_2 : \mu, \nu = 0, \pm 1, \pm 2, \dots\}.$$

Obviously, all the points in $\mathbb{L}$ are the poles and periods of $\wp$. Suppose that D is the parallelogram with vertices at $0, \omega_1, \omega_2, \omega_1 + \omega_2$. Note that the order of $\wp$ is 2. Here, the order is the number of poles of $\wp$ or the number of zeros of $\wp - a$ ($a \in \mathbb{C}$) in the parallelogram. Together with $(\wp')^2 = 4\wp^3 - 1$, we derive that $\wp$ has two distinct zeros in D , say θ_1 and θ_2 . In view of $\wp(z) = 0 \Leftrightarrow \wp'(z) = \pm i$, without loss of generality, throughout the article, we assume that $\wp'(\theta_1) = -i$ and $\wp'(\theta_2) = i$.

Here, for two meromorphic functions f, g and two points $a, b \in \mathbb{C} \cup \{\infty\}$, the notation $f(z) - a = 0 \Rightarrow g(z) - b = 0$ means that all the zeros of $f - a$ are zeros of $g(z) - b$. And the notation $f(z) - a = 0 \Leftrightarrow g(z) - b = 0$ means that $f - a$ and $g - b$ have the same zeros.

More generally, in the present article, we characterize the meromorphic solutions to Fermat-type functional equations as follows.

Proposition 1. *Suppose that $L(z)$ is a nonconstant entire function. Then, the nonconstant meromorphic function $f(z)$ is a solution of*

$$f^3(z) + f^3(L(z)) = 1 \tag{4}$$

if and only if $L(z) = qz + c$ is a linear function with $|q| = 1$, and $f(z) = \frac{1}{2} \left\{ 1 + \frac{\wp'(h(z))}{\sqrt{3}} \right\} / \wp(h(z))$, where $h(z)$ is a nonconstant entire function satisfying one of the following equations:

- (1) $h(L(z)) = Ah(z) + \theta_1 + \tau_1$ with $\tau_1 \in \mathbb{L}$;
- (2) $h(L(z)) = Ah(z) + \theta_2 + \tau_2$ with $\tau_2 \in \mathbb{L}$;
- (3) $h(L(z)) = Ah(z) + \tau_3$ with $\tau_3 \in \mathbb{L}$, where A is a constant with $A^3 = -1$ and $A\tau \in \mathbb{L}$ for any period τ of $\wp(z)$.

Proposition 2. *Suppose that $L(z)$ is a nonconstant entire function, suppose that $g(z)$ is an entire function and $f(z)$ is a nonconstant meromorphic function, and suppose that $n(\geq 2)$, $m(\geq 3)$ are two integers such that $(n, m) \neq (3, 3)$. Then, $f(z)$ is a solution of*

$$f^n(z) + f^m(L(z)) = e^{g(z)} \tag{5}$$

if and only if $L(z) = qz + c$ is a linear function, $f(z) = Ae^{\frac{g(z)}{n}}$ and

$$g(L(z)) = \frac{n}{m}g(z) + nLn\frac{B}{A}, \quad (6)$$

where A, B are two nonzero constants with $A^n + B^m = 1$ and $Ln\frac{B}{A} = \log \frac{B}{A} + 2k_0\pi i$ with any fixed integer k_0 . In particular, if $m = n$, then $|q| = 1$. As usual, $\log \frac{B}{A}$ denotes the principal value of $Ln\frac{B}{A}$.

It is pointed out that if $m \neq n$ in Proposition 2, the conclusion $|q| = 1$ may be invalid, as shown by the following example. Suppose $m \neq n$ and $q^2 = \frac{m}{n}$. Consider $g(z) = z^2 + a$ with a constant a satisfying $(1 - \frac{m}{n})a = nLn\frac{B}{A}$. Then, a calculation yields that $g(qz) = \frac{m}{n}g(z) + nLn\frac{B}{A}$ and $f(z) = Ae^{\frac{g(z)}{n}}$ is a solution of (5). However, $|q| \neq 1$.

Let's turn back to the Fermat-type difference equations. Suppose $L(z) = z + c$ with $c \neq 0$. Then, Theorem 1 follows immediately from Proposition 1.

Theorem 1. *The nonconstant meromorphic function $f(z)$ is a solution of*

$$f^3(z) + f^3(z + c) = 1 \quad (7)$$

if and only if $f(z) = \frac{1}{2} \left\{ 1 + \frac{\wp'(h(z))}{\sqrt{3}} \right\} / \wp(h(z))$, where $h(z)$ is a nonconstant entire function satisfying one of the following equations:

- (1) $h(z + c) = Ah(z) + \theta_1 + \tau_1$ with $\tau_1 \in \mathbb{L}$;
- (2) $h(z + c) = Ah(z) + \theta_2 + \tau_2$ with $\tau_2 \in \mathbb{L}$;
- (3) $h(z + c) = Ah(z) + \tau_3$ with $\tau_3 \in \mathbb{L}$, where A is a constant with $A^3 = -1$ and $A\tau \in \mathbb{L}$ for any period τ of $\wp(z)$.

Remark 1. It is known that if $h(z)$ satisfies one of (1)–(3) in Theorem 1, then the order $\rho(h(z)) \geq 1$ and hyper-order $\rho_2(f) \geq 1$. (This fact can be found in [14].) So, if $\rho_2(f) < 1$, then (4) does not admit nonconstant meromorphic solutions. This is the conclusion of Theorem B. Below, we present an example to show that there exists an entire function $h(z)$ with arbitrary order $\rho(h(z)) (\geq 1)$ satisfying one of (1)–(3).

Example 2. In [40, Theorem 3], Ozawa derived that there exists a periodic entire function of arbitrary order $\sigma (\geq 1)$. So, there exists an entire function $g(z)$ such that $\rho(g(z)) = \sigma$ and $g(z + c) = g(z)$. Set $h(z) = e^{az}g(z) + b$, where a, b are constants with $e^{ac} = A$ and $(1 - A)b = \theta_1 + \tau_1$ or $\theta_2 + \tau_2$ or τ_3 with $A^3 = -1$. Then, $\rho(h(z)) = \rho(g(z)) = \sigma$ and a calculation yields that

$$h(z + c) = Ah(z) + (1 - A)b.$$

Thus, $h(z)$ satisfies one of (1)–(3) in Theorem 1.

By Theorems 1 and C, we can get the following result.

Theorem 2. *Suppose that $f(z)$ is a nonconstant meromorphic function and $P(z)$ is a polynomial. Then $f(z)$ is a solution of the functional equation*

$$f^3(z) + f^3(z + c) = e^{P(z)} \quad (8)$$

if and only if $P(z) = \alpha z + \beta$ with two constants α, β , and $f(z)$ satisfies one of the following cases:

- (a) $f(z) = de^{\frac{\alpha z + \beta}{3}}$, where d is a nonzero constant and $d^3(1 + e^{\alpha c}) = 1$;
- (b) $f(z) = e^{\frac{\alpha z + \beta}{3}} \frac{1}{2} \left\{ 1 + \frac{\wp'(h(z))}{\sqrt{3}} \right\} / \wp(h(z))$, where $e^{\alpha c} = 1$ and $h(z)$ is a nonconstant entire function satisfying one of (1)–(3) in Theorem 1.

With Proposition 2, we derive a theorem as follows.

Theorem 3. Suppose that $g(z)$ is an entire function and $f(z)$ is a nonconstant meromorphic function. Suppose that $n(\geq 2)$ and $m(\geq 3)$ are two integers such that $(n, m) \neq (3, 3)$. Then, $f(z)$ is a solution of

$$f^n(z) + f^m(z+c) = e^{g(z)} \quad (9)$$

if and only if $f(z) = Ae^{\frac{g(z)}{n}}$ and $g(z+c) = \frac{n}{m}g(z) + nLn\frac{B}{A}$, where A, B are two nonzero constants with $A^n + B^m = 1$. In particular, if the order of $g(z)$ is less than 1, then (9) admits nonconstant meromorphic solutions if and only if $n = m$ and $g(z) = \alpha z + \beta$, $f(z) = Ae^{\frac{\alpha z + \beta}{n}}$ with a constant $\alpha (\neq 0)$ such that $A^n(1 + e^{\alpha c}) = 1$.

Remark 2. We point out that (9) may admit nonconstant meromorphic functions if $\rho(g(z)) \geq 1$, as shown by the following examples.

Example 3. Suppose that $m \neq n$, $g(z) = e^{\alpha z + \beta} + a$ with constants $\alpha (\neq 0)$, β , a . Consider $f(z) = Ae^{\frac{g(z)}{n}}$ with $e^{\alpha c} = \frac{n}{m}$ and $(1 - \frac{n}{m})a = nLn\frac{B}{A}$, where A, B are nonzero constants with $A^n + B^m = 1$. By a calculation, one gets that $g(z+c) = \frac{ng(z)}{m} + nLn\frac{B}{A}$ and

$$f^n(z) + f^m(z+c) = e^{g(z)}.$$

Obviously, $\rho(g(z)) = 1$.

Example 4. Suppose that $m = n$, and h is any nonconstant entire periodic function with period c . Let $g(z) = h(z) + az$ with $ac = nLn\frac{B}{A}$, where A, B are nonzero constants with $A^n + B^n = 1$. Then, $g(z+c) = g(z) + nLn\frac{B}{A}$, and $f(z) = Ae^{\frac{g(z)}{n}}$ is a nonconstant meromorphic solution of

$$f^n(z) + f^n(z+c) = e^{g(z)}.$$

Obviously, $\rho(g(z)) \geq 1$.

Remark 3. From Theorem 3, we see that (9) does not admit meromorphic solutions with poles. It is pointed out that the same argument as in Theorem 3 can deal with the Fermat-type difference equation $f^m(z) + f^n(z+c) = e^{g(z)}$ with $n(\geq 2)$, $m(\geq 3)$, and $(m, n) \neq (3, 3)$, we omit the details here.

The following corollary follows immediately from Theorem 3.

Corollary 1. Suppose that $P(z)$ is a polynomial and $f(z)$ is a nonconstant meromorphic function. Suppose that $n(\geq 2)$ and $m(\geq 2)$ are two integers with $m+n \geq 5$ and $(n, m) \neq (3, 3)$. Then, $f(z)$ is a solution of

$$f^n(z) + f^m(z+c) = e^{P(z)} \quad (10)$$

if and only if $n = m$, $f(z) = Ae^{\frac{P(z)}{n}}$ and $P(z) = \alpha z + \beta$ with two constants $\alpha (\neq 0)$, β .

From Theorem 3 and Corollary 1, one can easily get the following result.

Corollary 2. Suppose that $f(z)$ is a nonconstant meromorphic function, and $n(\geq 2)$ and $m(\geq 2)$ are two integers with $m+n \geq 5$. Then, $f(z)$ is a solution of

$$f^n(z) + f^m(z+c) = 1 \quad (11)$$

if and only if $n = m = 3$.

Finally, as applications of Propositions 1 and 2, we consider the meromorphic solutions of Fermat-type q -difference functional equations. As early as 1952, Valiron in [42] showed that the non-autonomous Schröder q -difference equation

$$f(qz) = R(z, f(z)), \quad (12)$$

where $R(z, f(z))$ is rational in both arguments, admits one parameter family of meromorphic solutions, provided that $q \in \mathbb{C}$ is chosen appropriately. Later, Gundersen et al. [18] proved that if $|q| > 1$ and the q -difference equation (12) admits a meromorphic solution of order zero, then (12) reduces to a q -difference Riccati equation, that is, $\deg_f R = 1$. Some scholars, such as Bergweiler–Hayman [5], Bergweiler–Ishizaki–Yanagihara [6], Eremenko–Sodin [13], and Ishizaki–Yanagihara [24], also made contributions to meromorphic solutions of q -difference functional equations. In 2007, Barneet–Halburd–Korhonen–Morgan in [3] derived the q -difference analog of some well-known results in Nevanlinna theory, including the lemma of the logarithmic derivative, the Clunie’s lemma, and the second main theorem. By means of these results, meromorphic solutions to the q -difference functional equations have been further studied (see [26], [32]).

Observe that in Propositions 1 and 2, $L(z)$ reduces to a linear function, say $qz + c$ with $q \neq 0$. Therefore, we have described the meromorphic solutions of Fermat-type q -difference functional equations $f^3(z) + f^3(qz + c) = 1$ and $f^n(z) + f^m(qz + c) = e^{g(z)}$, where n and m satisfy some certain conditions, and $g(z)$ is an entire function.

Remark 4. From the above theorems, we see that the cases $(1, n)$, $(m, 1)$, and $(2, 2)$ are left. Unfortunately, we cannot deal with these cases and thus leave them for further study.

For the proofs, we will assume that the reader is familiar with basic elements in Nevanlinna theory of meromorphic functions in $\mathbb{C}$ (see, e.g., [21], [23], [28], [47]), such as the *first* and *second* main theorems, the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$, the *counting function* $N(r, f)$, and the *reduced counting function* $\bar{N}(r, f)$. We also need the following notation.

The lower order of a meromorphic function f is defined as

$$\mu(f) := \liminf_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}.$$

2. Proofs of main results

In this section, we firstly give the proof of Proposition 1.

Proof of Proposition 1. Firstly, we will prove the necessity. Assume that $h(z)$ satisfies one of (1)–(3), we will prove that $f(z)$ is a meromorphic solution of (4). Suppose that $h(z)$ satisfies (1). Then $h(L(z)) = Ah(z) + \theta_1 + \tau_1$ with $\tau_1 \in \mathbb{L}$. Observe that $A^3 = -1$, $A\tau \in \mathbb{L}$ for any period τ of $\wp(z)$ and the definition of $\wp(z)$ as follows:

$$\wp(z) = \wp(z; \omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2} \right\}. \quad \square$$

A calculation yields

$$\begin{aligned} \wp(Az) &= \frac{1}{(Az)^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{(Az + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2} \right\} \\ &= \frac{1}{A^2} \left\{ \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{\left(z + \frac{1}{A}(\mu\omega_1 + \nu\omega_2)\right)^2} - \frac{1}{\left(\frac{1}{A}(\mu\omega_1 + \nu\omega_2)\right)^2} \right\} \right\} \\ &= -A \left\{ \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{\{z - A^2(\mu\omega_1 + \nu\omega_2)\}^2} - \frac{1}{\{-A^2(\mu\omega_1 + \nu\omega_2)\}^2} \right\} \right\} \quad (13) \\ &= -A \left\{ \frac{1}{z^2} + \sum_{\mu, \nu \in \mathbb{Z}; \mu^2 + \nu^2 \neq 0} \left\{ \frac{1}{(z + \mu\omega_1 + \nu\omega_2)^2} - \frac{1}{(\mu\omega_1 + \nu\omega_2)^2} \right\} \right\} \\ &= -A\wp(z), \end{aligned}$$

which implies that $\wp'(Az) = -\wp'(z)$. It is well known that

$$\wp(w+c) = \frac{1}{4} \left[\frac{\wp'(w) - \wp'(c)}{\wp(w) - \wp(c)} \right]^2 - \wp(w) - \wp(c)$$

for any w and c . Thus,

$$\begin{aligned} \wp(Aw + \theta_1) &= \frac{1}{4} \left[\frac{\wp'(Aw) - \wp'(\theta_1)}{\wp(Aw) - \wp(\theta_1)} \right]^2 - \wp(Aw) - \wp(\theta_1) = \frac{1}{4} \left[\frac{\wp'(Aw) + i}{\wp(Aw)} \right]^2 - \wp(Aw) \\ &= \frac{1}{4} \frac{(\wp'(Aw) + i)^2 \wp(Aw)}{\wp(Aw)^3} - \wp(Aw) = \frac{(\wp'(Aw) + i)^2 \wp(Aw)}{\wp'(Aw)^2 + 1} - \wp(Aw) \\ &= \frac{(\wp'(Aw) + i)^2 \wp(Aw)}{(\wp'(Aw) + i)(\wp'(Aw) - i)} - \wp(Aw) = \frac{2i\wp(Aw)}{\wp'(Aw) - i} \\ &= \frac{-A2i\wp(w)}{-\wp'(w) - i} = \frac{A2i\wp(w)}{\wp'(w) + i}. \end{aligned} \quad (14)$$

$$\begin{aligned} \wp'(Aw + \theta_1) &= \left(\frac{2i\wp(w)}{\wp'(w) + i} \right)' = 2i \frac{\wp'(w)(\wp'(w) + i) - \wp(w)\wp''(w)}{(\wp'(w) + i)^2} \\ &= 2i \frac{\wp'(w)^2 + i\wp'(w) - \wp(w)\wp''(w)}{(\wp'(w) + i)^2} = 2i \frac{\wp'(w)^2 + i\wp'(w) - \frac{3}{2}(\wp'(w)^2 + 1)}{(\wp'(w) + i)^2} \\ &= -i \frac{\wp'(w) - 3i}{\wp'(w) + i}. \end{aligned} \quad (15)$$

Set $h(z) = \omega$. Then $h(L(z)) = Ah(z) + \theta_1 + \tau_1 = Aw + \theta_1 + \tau_1$. Note that both $\wp$ and $\wp'$ are elliptic functions with periods ω_1 and ω_2 , we have

$$\begin{aligned} \wp(h(L(z))) &= \wp(Aw + \theta_1 + \tau_1) = \wp(Aw + \theta_1), \\ \wp'(h(L(z))) &= \wp'(Aw + \theta_1 + \tau_1) = \wp'(Aw + \theta_1). \end{aligned}$$

Observe that $f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))}$. So

$$\begin{aligned} f(L(z)) &= \frac{1}{2} \frac{1 + \frac{\wp'(h(L(z)))}{\sqrt{3}}}{\wp(h(L(z)))} = \frac{1}{2} \frac{1 + \frac{\wp'(Aw + \theta_1)}{\sqrt{3}}}{\wp(Aw + \theta_1)} = \frac{1}{2} \frac{1 - \frac{1}{\sqrt{3}} i \frac{\wp'(w) - 3i}{\wp'(w) + i}}{\frac{A2i\wp(w)}{\wp'(w) + i}} \\ &= \frac{1}{2} \frac{(\wp'(w) + i) - \frac{1}{\sqrt{3}} i (\wp'(w) - 3i)}{A2i\wp(w)} = \frac{1}{2} \frac{-1 - \sqrt{3}i}{2} \frac{1}{-A} \frac{1 - \frac{\wp'(w)}{\sqrt{3}}}{\wp(w)} = \frac{1}{-A} \frac{\eta}{2} \frac{1 - \frac{\wp'(w)}{\sqrt{3}}}{\wp(w)}, \end{aligned} \quad (16)$$

where $\eta = \frac{-1 - \sqrt{3}i}{2}$ and $\eta^3 = 1$. Thus, by (i) of Theorem A, one can easily derive that $f(z)^3 + f(L(z))^3 = 1$ and $f(z)$ is a solution of (4). If $h(z)$ satisfies (2) or (3), then the same reasoning yields that $f(z)$ is also a solution of (4).

Next, we will prove the sufficiency. For convenience, we denote $L(z)$ by $\bar{z}$, which means that $\bar{z} = L(z)$ and a point $\bar{a} = L(a)$.

Suppose that $f(z)$ is a nonconstant meromorphic solution of (4). Then, Via (i) of Theorem A, one has

$$f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} \quad \text{and} \quad f(\bar{z}) = \frac{\eta}{2} \frac{1 - \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))}, \quad (17)$$

where $h(z)$ is a nonconstant entire function. Below, we prove that $h(z)$ satisfies one of (1)–(3).

Obviously, $T(r, f(z)) = T(r, f(\bar{z})) + S(r, f)$. Rewrite the form of $f(z)$ as

$$2f(z)\wp(h(z)) - 1 = \frac{\wp'(h(z))}{\sqrt{3}}.$$

Then,

$$[2f(z)\wp(h(z)) - 1]^2 = \left(\frac{\wp'(h(z))}{\sqrt{3}}\right)^2 = \frac{1}{3}[4\wp(h(z))^3 - 1], \quad (18)$$

We rewrite (18) as

$$f(z)[f(z)\wp(h(z)) - 1] = \frac{1}{3} \frac{[\wp(h(z))^3 - 1]}{\wp(h(z))}, \quad (19)$$

which implies that

$$\begin{aligned} 3T(r, \wp(h(z))) &= T(r, \frac{1}{3} \frac{[\wp(h(z))^3 - 1]}{\wp(h(z))}) + O(1) = T(r, f(z)[f(z)\wp(h(z)) - 1]) + O(1) \\ &\leq T(r, f(z)) + T(r, f(z)\wp(h(z))) + O(1) \\ &\leq 2T(r, f(z)) + T(r, \wp(h(z))) + O(1). \end{aligned}$$

So,

$$T(r, \wp(h(z))) \leq T(r, f(z)) + O(1). \quad (20)$$

The form of $f(z)$ yields $f(\bar{z}) = \frac{1}{2} \frac{1 + \frac{\wp'(h(\bar{z}))}{\sqrt{3}}}{\wp(h(\bar{z}))}$. Then, the same argument leads to

$$T(r, \wp(h(\bar{z}))) \leq T(r, f(\bar{z})) + O(1). \quad (21)$$

Rewrite (4) as $f(\bar{z})^3 = -[f(z)^3 - 1] = -(f(z) - 1)(f(z) - \varsigma)(f(z) - \varsigma^2)$, ($\varsigma \neq 1$, $\varsigma^3 = 1$), which implies that the zeros of $f(z) - 1$, $f(z) - \varsigma$, and $f(z) - \varsigma^2$ are of multiplicities at least 3. Applying Nevanlinna's first and second theorems to $f(z)$ yields that

$$\begin{aligned} 2T(r, f(z)) &\leq \sum_{m=0}^2 \bar{N}(r, \frac{1}{f(z) - \varsigma^m}) + \bar{N}(r, f(z)) + S(r, f(z)), \\ &\leq \frac{1}{3} \sum_{m=0}^2 N(r, \frac{1}{f(z) - \varsigma^m}) + N(r, f(z)) + S(r, f(z)) \\ &\leq 2T(r, f(z)) + S(r, f(z)). \end{aligned}$$

Therefore,

$$T(r, f(z)) = \bar{N}(r, f(z)) + S(r, f(z)) = N(r, f(z)) + S(r, f(z)). \quad (22)$$

Further, in view of the fact that $f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))}$ and $\wp$ has only multiple poles, one derives that

$$\begin{aligned} T(r, f(z)) &= \overline{N}(r, f(z)) + S(r, f(z)) = \overline{N}\left(r, \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))}\right) + S(r, f(z)) \\ &\leq \overline{N}\left(r, \frac{1}{\wp(h(z))}\right) + \overline{N}(r, \wp(h(z))) + S(r, f(z)) \\ &\leq \overline{N}\left(r, \frac{1}{\wp(h(z))}\right) + \frac{1}{2} N(r, \wp(h(z))) + S(r, f(z)) \\ &\leq \frac{3}{2} T(r, \wp(h(z))) + S(r, f(z)). \end{aligned} \quad (23)$$

The equation $f(z)^3 + f(\bar{z})^3 = 1$ implies that $\overline{N}(r, f(\bar{z})) = \overline{N}(r, f(z))$. By (22), one has

$$\begin{aligned} T(r, f(\bar{z})) &= T(r, f(z)) + S(r, f(\bar{z})) = \overline{N}(r, f(z)) + S(r, f(z)) \\ &= \overline{N}(r, f(\bar{z})) + S(r, f(z)). \end{aligned} \quad (24)$$

The same argument as in (23) yields

$$T(r, f(\bar{z})) \leq \frac{3}{2} T(r, \wp(h(\bar{z}))) + S(r, f(z)). \quad (25)$$

All the above discussions yield

$$S(r, f(z)) = S(r, f(\bar{z})) = S(r, \wp(h(z))) = S(r, \wp(h(\bar{z}))).$$

For simplicity, we write

$$S(r) = S(r, f(z)) = S(r, f(\bar{z})) = S(r, \wp(h(z))) = S(r, \wp(h(\bar{z}))).$$

By (17) and a routine computation, we get

$$\eta\left(1 - \frac{\wp'(h(z))}{\sqrt{3}}\right)\wp(h(\bar{z})) = \left(1 + \frac{\wp'(h(\bar{z}))}{\sqrt{3}}\right)\wp(h(z)). \quad (26)$$

We employ the method in [22], [37], [44] to prove this theorem. For a set G , define the counting function $N(r, G)$ as

$$N(r, G) = \int_0^r \frac{n(t, G) - n(0, G)}{t} dt + n(0, G) \log r,$$

where the notation $n(r, G)$ is the number of points in $G \cap \{|z| < r\}$, ignoring multiplicities. We define a set S_1 as

$$S_1 = \{z | \wp(h(z)) = 0 \text{ and } \wp(h(\bar{z})) = 0\}.$$

Arrange S_1 as $S_1 = \{a_s\}_{s=1}^\infty$ and $a_s \rightarrow \infty$ as $s \rightarrow \infty$. Notice, when $\wp(h(a_s)) = 0$, then $[\wp'(h(a_s))]^2 = -1$. Further, when $\wp(h(\bar{a}_s)) = 0$, then $[\wp'(h(\bar{a}_s))]^2 = -1$. Differentiate (26) and apply substitution to observe that

$$\begin{aligned} & \eta \left(1 - \frac{\wp'(h(a_s))}{\sqrt{3}}\right) \wp'(h(\overline{a_s})) [h(\overline{z})]'(a_s) \\ &= \left(1 + \frac{\wp'(h(\overline{a_s}))}{\sqrt{3}}\right) \wp'(h(a_s)) h'(a_s). \end{aligned}$$

From which we have that one and the only one of the following situations occurs:

$$\begin{cases} g_1(a_s) = \eta \left(1 - i \frac{\sqrt{3}}{3}\right) [h(\overline{z})]'(a_s) - \left(1 + i \frac{\sqrt{3}}{3}\right) h'(a_s) = 0, \\ g_2(a_s) = \eta [h(\overline{z})]'(a_s) + h'(a_s) = 0, \\ g_3(a_s) = \eta \left(1 + i \frac{\sqrt{3}}{3}\right) [h(\overline{z})]'(a_s) - \left(1 - i \frac{\sqrt{3}}{3}\right) h'(a_s) = 0. \end{cases}$$

Below, we consider two cases.

Case 1: $g_i(z) \not\equiv 0$ for any $i = 1, 2, 3$.

Here, we employ the result of Clunie [11], which can be stated as follows.

Lemma 1. *Let f be a nonconstant entire function, and let g be a transcendental meromorphic function in the complex plane, then $T(r, f) = S(r, g(f))$ as $r \rightarrow \infty$.*

Note that $\wp$ is transcendental. By Lemma 1, we have $T(r, h(z)) = S(r, \wp(h(z))) = S(r)$ and $T(r, h(\overline{z})) = S(r, \wp(h(\overline{z}))) = S(r)$. We also have

$$T(r, h'(z)) = O(T(r, h(z))) = S(r), \quad T(r, [h(\overline{z})]') = O(T(r, h(\overline{z}))) = S(r).$$

So, $T(r, g_i) = S(r)$. Furthermore,

$$\begin{aligned} N(r, S_1) &\leq \sum_i N(r, \frac{1}{g_i}) \leq \sum_i T(r, \frac{1}{g_i}) \\ &= \sum_i T(r, g_i) + O(1) = S(r). \end{aligned} \tag{27}$$

Assume that E is the set of all zeros of $\wp(h)$, that is, $E = \{z | \wp(h(z)) = 0\}$. Obviously, $S_1 \subseteq E$. Put $S_2 = E \setminus S_1$. For any $b \in S_2$, we see that $\wp(h(b)) = 0$ and $\wp(h(\overline{b})) \neq 0$. In view of $[\wp'(h(b))]^2 = -1$ and (26), one has $\wp(h(\overline{b})) = \infty$. All the above discussions yield

$$\begin{aligned} \overline{N}(r, \frac{1}{\wp(h(z))}) &= N(r, E) = N(r, S_1) + N(r, S_2) \\ &\leq \overline{N}(r, \wp(h(\overline{z}))) + S(r). \end{aligned} \tag{28}$$

Suppose that z_0 is a zero of $\wp(h(z))$ with multiplicity p . Then z_0 is a zero of $(\wp(h(z)))' = \wp'(h(z))h'(z)$ with multiplicity $p-1$. The fact that $\wp'(h(z_0)) = \pm i$ yields that z_0 is a zero of h' with multiplicity $p-1$. Thus,

$$N(r, \frac{1}{\wp(h(z))}) = \overline{N}(r, \frac{1}{\wp(h(z))}) + N(r, \frac{1}{h'(z)}) = \overline{N}(r, \frac{1}{\wp(h(z))}) + S(r). \tag{29}$$

Equation (22) implies that $m(r, f(z)) = S(r)$, since $T(r, f(z)) = m(r, f(z)) + N(r, f(z))$. Rewrite the form of $f(z)$ as

$$\frac{1}{\wp(h(z))} = 2f(z) - \frac{\frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} = 2f(z) - \frac{\frac{\wp'(h(z))h'(z)}{\sqrt{3}}}{\wp(h(z))} \frac{1}{h'(z)}.$$

Then, applying the logarithmic derivative lemma, we have

$$\begin{aligned} m(r, \frac{1}{\wp(h(z))}) &\leq m(r, f(z)) + m(r, \frac{(\wp(h(z)))'}{\wp(h(z))}) + m(r, \frac{1}{h'(z)}) + O(1) \\ &\leq S(r) + S(r, \wp(h(z))) + S(r) = S(r). \end{aligned} \quad (30)$$

Combining (29) and (30) yields

$$\begin{aligned} T(r, \wp(h(z))) &= T(r, \frac{1}{\wp(h(z))}) + O(1) = N(r, \frac{1}{\wp(h(z))}) + m(r, \frac{1}{\wp(h(z))}) + O(1) \\ &= \bar{N}(r, \frac{1}{\wp(h(z))}) + m(r, \frac{1}{\wp(h(z))}) + O(1) \\ &= \bar{N}(r, \frac{1}{\wp(h(z))}) + S(r). \end{aligned} \quad (31)$$

We also know that $[\wp'(h(\bar{z}))]^2 = 4\wp(h(\bar{z}))^3 - 1$. Then,

$$[\wp(h(\bar{z}))]^2 = [h(\bar{z})]'^2 \wp'(h(\bar{z}))^2 = 4[h(\bar{z})]^2 \wp(h(\bar{z}))^3 - [h(\bar{z})]^2.$$

Rewrite it as

$$\wp(h(\bar{z}))^2 \wp(h(\bar{z})) = \wp(h(\bar{z}))^3 = \frac{1}{4[h(\bar{z})]^2} [\wp(h(\bar{z}))]^2 - \frac{1}{4}.$$

By $T(r, \frac{1}{4[h(\bar{z})]^2}) = S(r)$ and Clunie's lemma (see [10]), we get that $m(r, \wp(h(\bar{z}))) = S(r)$ and

$$T(r, \wp(h(\bar{z}))) = m(r, \wp(h(\bar{z}))) + N(r, \wp(h(\bar{z}))) = N(r, \wp(h(\bar{z}))) + S(r). \quad (32)$$

Note that $\wp(h(\bar{z}))$ has only multiple poles. So, $\bar{N}(r, \wp(h(\bar{z}))) \leq \frac{1}{2}N(r, \wp(h(\bar{z})))$. Further, combining (21), (23), (25), (28), (31), and (32), we have that

$$\begin{aligned} \frac{2}{3}T(r, f(z)) &\leq T(r, \wp(h(z))) + S(r) = \bar{N}(r, \frac{1}{\wp(h(z))}) + S(r) \leq \bar{N}(r, \wp(h(\bar{z}))) + S(r) \\ &\leq \frac{1}{2}N(r, \wp(h(\bar{z}))) + S(r) \leq \frac{1}{2}T(r, \wp(h(\bar{z}))) + S(r) \\ &\leq \frac{1}{2}T(r, f(\bar{z})) + S(r) = \frac{1}{2}T(r, f(z)) + S(r), \end{aligned} \quad (33)$$

which is a contradiction. Thus, the case cannot occur.

Case 2: $g_i(z) \equiv 0$, for some $i \in \{1, 2, 3\}$.

Firstly, we assume that $g_1(z) \equiv 0$. Then, $[h(\bar{z})]' = Ah'(z)$, where $A = \frac{(1+i\frac{\sqrt{3}}{3})}{\eta(1-i\frac{\sqrt{3}}{3})}$. Integrating this equation yields $h(\bar{z}) = Ah(z) + B$, where B is a fixed constant.

We know that $\wp(h(z))$ has infinitely many poles. Suppose that $\wp(h(b_0)) = \infty$. Equation (26) yields that $\wp(h(\bar{b}_0)) = 0$ or $\wp(h(\bar{b}_0)) = \infty$. Assume $\wp(h(\bar{b}_0)) = \infty$. Rewrite (26) as

$$\frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} [h(\bar{z})]' = \frac{\eta(1 - \frac{\wp'(h(z))}{\sqrt{3}})}{\wp(h(z))} [h(\bar{z})]' = \frac{\eta(1 - \frac{\wp'(h(z))}{\sqrt{3}})}{\wp(h(z))} Ah'(z). \quad (34)$$

Further, we rewrite (34) as

$$\frac{[h(\bar{z})]'}{\wp(h(\bar{z}))} + \frac{1}{\sqrt{3}} \frac{[\wp(h(\bar{z}))]'}{\wp(h(\bar{z}))} = A\eta \left[\frac{h'(z)}{\wp(h(z))} - \frac{1}{\sqrt{3}} \frac{[\wp(h(z))]'}{\wp(h(z))} \right]. \quad (35)$$

Note that $\wp(h(z))$ has only multiple poles. Assume that b_0 is a pole of $\wp(h(z))$ and $\wp(h(\bar{z}))$ with multiplicities $m(\geq 2)$ and $k(\geq 2)$, respectively. Taking the residue of both sides of (35) at b_0 , we have

$$-k \frac{1}{\sqrt{3}} = A\eta \frac{m}{\sqrt{3}},$$

which implies that $A\eta = -\frac{k}{m}$. It contradicts $A\eta = \frac{1+i\frac{\sqrt{3}}{3}}{(1-i\frac{\sqrt{3}}{3})} = \frac{\sqrt{3}+i}{(\sqrt{3}-i)}$.

So, $\wp(h(\bar{b}_0)) = 0$ and $\wp'(h(\bar{b}_0)) = \pm i$. Suppose that $\wp'(h(\bar{b}_0)) = i$. Assume that b_0 is a zero of $\wp(h(\bar{z}))$ with multiplicity s . We rewrite (35) as

$$\begin{aligned} \frac{1}{\wp'(h(\bar{z}))} \frac{[\wp(h(\bar{z}))]'}{\wp(h(\bar{z}))} + \frac{1}{\sqrt{3}} \frac{[\wp(h(\bar{z}))]'}{\wp(h(\bar{z}))} &= \frac{[h(\bar{z})]'}{\wp(h(\bar{z}))} + \frac{1}{\sqrt{3}} \frac{[\wp(h(\bar{z}))]'}{\wp(h(\bar{z}))} \\ &= A\eta \left[\frac{h'(z)}{\wp(h(z))} - \frac{1}{\sqrt{3}} \frac{[\wp(h(z))]'}{\wp(h(z))} \right]. \end{aligned} \quad (36)$$

Taking the residue of both sides of (36) again at b_0 , we have

$$s \left[\frac{1}{i} + \frac{1}{\sqrt{3}} \right] = A\eta \frac{m}{\sqrt{3}},$$

which implies that $A\eta = \frac{s}{m}(1-i\sqrt{3})$. Combining $A\eta = \frac{\sqrt{3}+i}{(\sqrt{3}-i)}$ yields that

$$\frac{s}{m} = \frac{\sqrt{3}+i}{(\sqrt{3}-i)(1-i\sqrt{3})} = \frac{-1+\sqrt{3}i}{4},$$

which is a contradiction. Therefore, $\wp'(h(\bar{b}_0)) = -i$ and all the poles of $\wp(h(z))$ must be the zero of $\wp'(h(\bar{z}))+i$. Suppose that $\wp(h(z_0)) = 0$ and $\wp'(h(z_0)) = i$. The same argument also yields $\wp(h(\bar{z}_0)) = 0$ and $\wp'(h(\bar{z}_0)) = i$, which means that all the zeros $\wp'(h(z)) - i$ must be the zero of $\wp'(h(\bar{z})) - i$. We denote these facts as follows:

$$\wp(h(z)) = \infty \Rightarrow \wp'(h(\bar{z})) + i = 0, \quad \wp'(h(z)) - i = 0 \Rightarrow \wp'(h(\bar{z})) - i = 0. \quad (37)$$

Suppose that 0 is a Picard exceptional value of $h(z)$. Then, we can assume that $h(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. The equation $h(\bar{z}) = Ah(z) + B$ yields $e^{\alpha(\bar{z})} = Ae^{\alpha(z)} + B = A[e^{\alpha(z)} - (-\frac{B}{A})]$. So, $-\frac{B}{A}$ is also a Picard exceptional value of $h(z) = e^{\alpha(z)}$, and Picard's little theorem tells us $B = 0$. Thus, $h(\bar{z}) = Ah(z)$.

Suppose that $\tau (\neq 0) \in \mathbb{L}$, which means that τ is any fixed period of $\wp$. Observe that τ is not Picard exceptional value of $h(z)$. Assume that $h(u_0) = \tau$. Then, $\wp(h(u_0)) = \wp(\tau) = \infty$. From (37), one has $\wp(h(\bar{u}_0)) = 0$ and $\wp'(h(\bar{u}_0)) = -i$. Without loss of generality, we assume that

$$h(\bar{u}_0) = Ah(u_0) = A\tau = \theta_1 + \xi_1, \quad (38)$$

where $\xi_1 \in \mathbb{L}$. Note that θ_2 is not a Picard exceptional value of $h(z)$. Assume $h(e_0) = \theta_2$. Thus, $\wp(h(e_0)) = \wp(\theta_2) = 0$ and $\wp'(h(e_0)) = \wp'(\theta_2) = i$. Then, (37) yields $\wp(h(\bar{e}_0)) = 0$ and $\wp'(h(\bar{e}_0)) = i$. Without loss of generality, assume that

$$h(\overline{e_0}) = Ah(e_0) = A\theta_2 = \theta_2 + \xi_2, \quad (39)$$

where $\xi_2 \in \mathbb{L}$. Clearly, $\theta_2 + \tau$ is also not a Picard exceptional value of $h(z)$. Assume $h(t_0) = \theta_2 + \tau$. Then, $\wp(h(t_0)) = \wp(\theta_2 + \tau) = 0$ and $\wp'(h(t_0)) = \wp'(\theta_2 + \tau) = i$. The same argument as above yields

$$h(\overline{t_0}) = Ah(t_0) = A(\theta_2 + \tau) = \theta_2 + \xi_3, \quad (40)$$

where $\xi_3 \in \mathbb{L}$. Combining (39) and (40) leads to $A\tau = \xi_3 - \xi_2 \in \mathbb{L}$. Together with (38), we derive that $\theta_1 \in \mathbb{L}$ is a period of $\wp$, a contradiction. Thus, 0 is not a Picard exceptional value of $h(z)$.

Below, for any period τ of $\wp$, we prove that $A\tau \in \mathbb{L}$, which means that $A\tau$ is also a period of $\wp$.

Observe that 0 is not a Picard exceptional value of $h(z)$. Assume $h(d_0) = 0$. Thus, $\wp(h(d_0)) = \wp(0) = \infty$. Then, (37) yields $\wp(h(\overline{d_0})) = 0$ and $\wp'(h(\overline{d_0})) = -i$. Without loss of generality, we assume that

$$h(\overline{d_0}) = Ah(d_0) + B = B = \theta_1 + \xi_5, \quad (41)$$

where $\xi_5 \in \mathbb{L}$. In view of the fact that $h(z)$ has at most one finite Picard exceptional value, so one of ω_1 and ω_2 is not a Picard exceptional value of $h(z)$. Without loss of generality, assume that ω_1 is not a Picard exceptional value of $h(z)$ and $h(c_0) = \omega_1$. Then, $\wp(h(c_0)) = \wp(\omega_1) = \wp(0) = \infty$, which plus (37) implies that $\wp(h(\overline{c_0})) = 0$ and $\wp'(h(\overline{c_0})) = -i$. So, we can set $h(\overline{c_0}) = \theta_1 + \xi_6$ with $\xi_6 \in \mathbb{L}$. Further,

$$h(\overline{c_0}) = Ah(c_0) + B = Ah(c_0) + \theta_1 + \xi_5 = A\omega_1 + \theta_1 + \xi_5 = \theta_1 + \xi_6, \quad (42)$$

which implies that $A\omega_1 = \xi_6 - \xi_5 \in \mathbb{L}$.

Clearly, there exists an integer N such that $N\omega_1 + \tau$ is not a Picard exceptional value of $h(z)$. Suppose that $h(t_1) = N\omega_1 + \tau$. Then, $\wp(h(t_1)) = \wp(N\omega_1 + \tau) = \wp(0) = \infty$, which plus (37) implies that $\wp(h(\overline{t_1})) = 0$ and $\wp'(h(\overline{t_1})) = -i$. And we can set $h(\overline{t_1}) = \theta_1 + \xi_7$ with $\xi_7 \in \mathbb{L}$. Further,

$$h(\overline{t_1}) = Ah(t_1) + \theta_1 + \xi_5 = A(N\omega_1 + \tau) + \theta_1 + \xi_5 = \theta_1 + \xi_7, \quad (43)$$

which leads to

$$A\tau = \xi_7 - \xi_5 - NA\omega_1 \in \mathbb{L}.$$

Thus, we prove that $A\tau$ is also a period of $\wp$.

Observe that $A = \frac{(1+i\frac{\sqrt{3}}{3})}{\eta(1-i\frac{\sqrt{3}}{3})}$ and $\eta^3 = 1$. A calculation yields $A^3 = -1$ and

$$A = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}i}, \quad \text{or} \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{-\frac{\pi}{3}i}, \quad \text{or} \quad -1.$$

Meanwhile, (41) implies that

$$h(\overline{z}) = Ah(z) + B = Ah(z) + \theta_1 + \xi_5,$$

which is (1) with $\tau_1 = \xi_5$.

Suppose that $A = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}i}$. Among all the periods of $\wp(\omega)$, we list the periods which may take the smallest modulus as follows: ω_1 , $\omega_1 + \omega_2$, ω_2 , $\omega_2 - \omega_1$, $-\omega_1$, $-\omega_2 - \omega_1$, $-\omega_2$, and $\omega_1 - \omega_2$. Then, in view of the fact that $A\tau = \tau e^{\frac{\pi}{3}i}$ is also a period of $\wp$, we illuminate two possible subcases (see Figures 1 and 2).

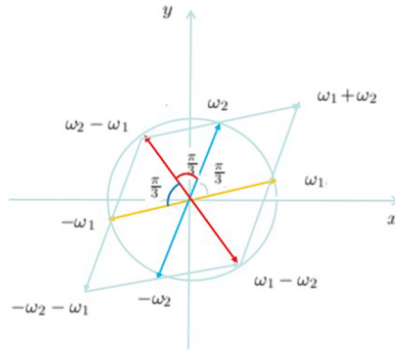


Figure 1. Subcase 2.1.

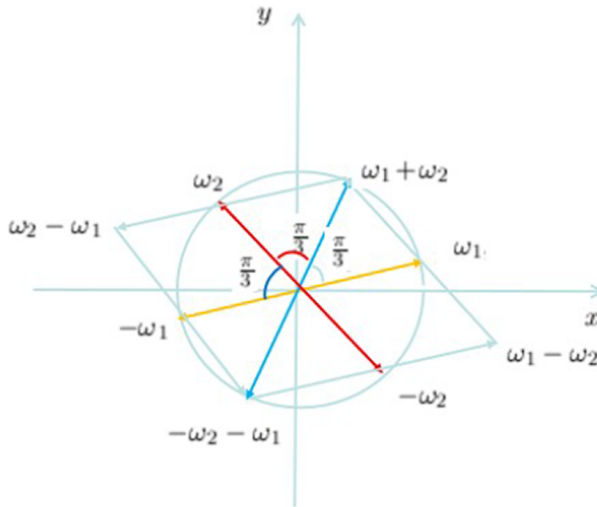


Figure 2. Subcase 2.2.

Subcase 2.1: $A\omega_1 = \omega_2$ and $A\omega_2 = \omega_2 - \omega_1$ and $A(\omega_2 - \omega_1) = -\omega_1$.

Subcase 2.2: $A\omega_1 = \omega_1 + \omega_2$ and $A(\omega_1 + \omega_2) = \omega_2$ and $A\omega_2 = -\omega_1$.

For $A = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ or $A = -1$, analogous subcases can also be obtained as above.

Next, we consider $g_3(z) \equiv 0$ or $g_2(z) \equiv 0$.

If $g_3(z) \equiv 0$, on the basis of the above discussion, we can derive that

$$\wp(h(z)) = \infty \Rightarrow \wp'(h(\bar{z})) - i = 0, \quad \wp'(h(z)) + i = 0 \Rightarrow \wp'(h(\bar{z})) + i = 0.$$

If $g_2(z) \equiv 0$, on the basis of the above discussion, we can derive that

$$\wp(h(z)) = \infty \Rightarrow \wp(h(\bar{z})) = \infty, \quad \wp'(h(z)) - i = 0 \Rightarrow \wp'(h(\bar{z})) + i = 0.$$

Furthermore, with the same argument, we can obtain conclusions (2) and (3) when $g_3(z) \equiv 0$ and $g_2(z) \equiv 0$, respectively. Here, we omit the details.

Next, we will prove that $L(z)$ is a linear function. Without loss of generality, assume that

$$h(L(z)) = Ah(z) + C, \tag{44}$$

where $A^3 = -1$ and C is a constant. Suppose that h is transcendental. We recall the following result (see [11, Theorem 2] or [17, p. 370]).

Lemma 2. *If f (meromorphic) and g (entire) are transcendental, then*

$$\limsup_{r \notin E, r \rightarrow \infty} \frac{T(r, f(g))}{T(r, f)} = \infty,$$

where E is a set of finite Lebesgue measure.

If L is transcendental, then Lemma 2 implies that

$$\infty = \limsup_{r \notin E, r \rightarrow \infty} \frac{T(r, h(L(z)))}{T(r, h(z))} = \limsup_{r \notin E, r \rightarrow \infty} \frac{T(r, Ah(z) + C)}{T(r, h(z))} = 1,$$

a contradiction. Thus, L is a polynomial. Without loss of generality, assume $L(z) = a_m z^m + \dots + a_1 z + a_0$ with $m \geq 1$ and $a_m \neq 0$. We employ the following result, which can be seen in [17, Inequality (19)] and [31, Inequality (2.7)], respectively.

Lemma 3. *Suppose that $g(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$ ($a_m \neq 0$) is a nonconstant polynomial, then for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$,*

$$T(r, f(g)) \geq (1 - \epsilon_2) T\left(\frac{a_m}{2} r^m, f\right),$$

$$T(r, f) \leq \frac{1}{m} (1 + \epsilon_1) T\left(\frac{a_m}{2} r^m, f\right),$$

for large r outside possibly a set of finite Lebesgue measure.

Applying Lemma 3 to the function $h(L(z))$, we obtain

$$\begin{aligned} T(r, h(z)) &\leq \frac{1}{m} (1 + \epsilon_1) T\left(\frac{a_m}{2} r^m, h(z)\right) \leq \frac{1}{m} \frac{1 + \epsilon_1}{1 - \epsilon_2} T(r, h(L(z))) \\ &= \frac{1}{m} \frac{1 + \epsilon_1}{1 - \epsilon_2} T(r, Ah(z) + C) = \frac{1}{m} \frac{1 + \epsilon_1}{1 - \epsilon_2} T(r, h(z)) + O(1), \end{aligned} \quad (45)$$

which implies that $m = 1$, since ϵ_1 and ϵ_2 can be chosen small enough. Thus, $L(z)$ is a linear function.

Suppose that h is a polynomial of degree n . Then, (44) implies that $h(L(z))$ is also a polynomial, which implies that $L(z)$ is also a polynomial. Furthermore, (44) leads to the conclusion that $L(z)$ is a linear function.

So, the above discussions yield that $L(z) = qz + c$ with $q \neq 0$ and c is a constant. Then, $h(qz + c) = Ah(z) + C$, where C is a constant.

At the end, we will prove $|q| = 1$ by an elementary method. Differentiating the above equation leads to

$$qh'(qz + c) = Ah'(z). \quad (46)$$

If $h'(z)$ is a constant, then, $q = A$ and $|q| = |A| = 1$. Next, assume that $h'(z)$ is nonconstant. Suppose that $|q| < 1$. For each $t^* \in \mathbb{C}$, by (46), we get that

$$h'(t^*) = \frac{q}{A} h'(qt^* + c) = \left(\frac{q}{A}\right)^2 h'[q(qt^* + c) + c] = \dots = \left(\frac{q}{A}\right)^n h'[q^n t^* + c] = \sum_{k=0}^{n-1} \left(\frac{q}{A}\right)^k q^k.$$

Note that $|q| < 1$ and $q^n t^* + c \sum_{k=0}^{n-1} q^k \rightarrow c \frac{1}{1-q}$ as $n \rightarrow \infty$. Then, let $n \rightarrow \infty$, one has that

$$h'(t^*) = h'(c \frac{1}{1-q}) \lim_{n \rightarrow \infty} (\frac{q}{A})^n = 0,$$

which means that $h'(z) \equiv 0$, a contradiction. If $|q| > 1$, then we rewrite (46) as $h'(z) = Aph'(pz + c')$ with $p = \frac{1}{q}$ and $c' = -\frac{c}{q}$. Note that $|p| = |\frac{1}{q}| < 1$. The same argument as above yields a contradiction. So, we obtain $|q| = 1$.

Therefore, we finish the proof of Proposition 1.

Proof of Proposition 2. The necessity is obvious. Below, we prove the sufficiency. Rewrite (5) as

$$[\frac{f(z)}{e^{\frac{g(z)}{n}}}]^n + [\frac{f(L(z))}{e^{\frac{g(z)}{m}}}]^m = 1.$$

We will prove that both $\frac{f(z)}{e^{\frac{g(z)}{n}}}$ and $\frac{f(L(z))}{e^{\frac{g(z)}{m}}}$ are constants. □

Yang's theorem yields that both $\frac{f(z)}{e^{\frac{g(z)}{n}}}$ and $\frac{f(L(z))}{e^{\frac{g(z)}{m}}}$ are constants if $\frac{1}{n} + \frac{2}{m} < 1$. Therefore, it suffices to consider the cases $(n, m) = (2, 3)$ and $(2, 4)$.

Case 1: $(n, m) = (2, 3)$.

Suppose that one of $\frac{f(z)}{e^{\frac{g(z)}{2}}}$ and $\frac{f(L(z))}{e^{\frac{g(z)}{3}}}$ is not constant. Then neither of them is constant. Via (ii) of Theorem A, we have that $\frac{f(z)}{e^{\frac{g(z)}{2}}} = i\varphi'(h(z))$ and $\frac{f(L(z))}{e^{\frac{g(z)}{3}}} = \eta\sqrt[3]{4}\varphi(h(z))$, where $h(z)$ is any nonconstant entire function. So,

$$f(L(z)) = i\varphi'(h(L(z)))e^{\frac{g(L(z))}{2}} = \eta\sqrt[3]{4}\varphi(h(z))e^{\frac{g(z)}{3}}. \quad (47)$$

Note that $w_n = n\omega_1$ ($n = 0, 1, 2, \dots$) is a pole of $\varphi(z)$ with multiplicity 2. It is known that a nonconstant meromorphic function has four complete multiple values at most. (Here, the constant a is a complete multiple value of f if $f - a$ has only multiple zeros.) So, there exists a point w_N such that $h(z) - w_N$ has a simple zero, say z_0 . Then, $h(z_0) = w_N$ and z_0 is a pole of $\eta\sqrt[3]{4}\varphi(h(z))e^{\frac{g(z)}{3}}$ with multiplicity 2. On the other hand, since that the multiplicity of any pole of $\varphi'(w)$ is 3, we derive that z_0 is a pole of $i\varphi'(h(L(z)))e^{\frac{g(L(z))}{2}}$ with multiplicity at least 3, or z_0 is not a pole of this function. Comparing the multiplicities of both sides of (47) at pole-point z_0 , we thereby arrive at a contradiction.

Case 2: $(n, m) = (2, 4)$.

Assume that one of $\frac{f(z)}{e^{\frac{g(z)}{2}}}$ and $\frac{f(L(z))}{e^{\frac{g(z)}{4}}}$ is not a constant. Via (iii) of Theorem A, we have that $\frac{f(z)}{e^{\frac{g(z)}{2}}} = sn'(h(z))$ and $\frac{f(L(z))}{e^{\frac{g(z)}{4}}} = sn(h(z))$, where $h(z)$ is any nonconstant entire function and sn is the Jacobi elliptic function satisfying $sn'^2 = 1 - sn^4$. So,

$$f(L(z)) = sn'(h(L(z)))e^{\frac{g(L(z))}{2}} = sn(h(z))e^{\frac{g(z)}{4}}. \quad (48)$$

Suppose that w_n ($n = 0, 1, 2, \dots$) is a simple pole of $sn(z)$. As in Case 1, there exists a point w_N such that $h(z) - w_N$ has a simple zero, say z_1 . Then, $h(z_1) = w_N$ and z_1 is a pole of $sn(h(z))e^{\frac{g(z)}{4}}$ with multiplicity 1. On the other hand, in view of the fact that the multiplicity of any pole of $sn'(w)$ is 2, we get that z_1 is a pole of $sn'(h(z))e^{\frac{g(L(z))}{2}}$ with multiplicity at least 2, or z_1 is not a pole of this function. Comparing the multiplicities of both sides of (48) at pole-point z_1 , we have a contradiction.

Therefore, we derive that both $\frac{f(z)}{e^{\frac{g(z)}{n}}}$ and $\frac{f(L(z))}{e^{\frac{g(z)}{m}}}$ are nonzero constants, say A and B with $A^n + B^m = 1$. Thus, $f(z) = Ae^{\frac{g(z)}{n}}$ and

$$f(L(z)) = Be^{\frac{g(z)}{m}} = Ae^{\frac{g(L(z))}{n}},$$

which implies that

$$g(L(z)) = \frac{n}{m}g(z) + nLn\frac{B}{A}. \quad (49)$$

With the same argument as in Proposition 1, we can derive that $L(z)$ is a linear function, say $L(z) = qz + c$ with constants $q(\neq 0)$, c .

If $m = n$, then (49) reduces to $g(qz + c) = g(z) + nLn\frac{B}{A}$. The same discussion as in Proposition 1 yields $|q| = 1$.

Therefore, we finish the proof of Proposition 2.

The proof of Theorem 2 is now given as follows.

Proof of Theorem 2. The necessity is obvious. Below, we prove the sufficiency. Suppose that the hyper-order of $f(z)$ is less than 1. Then the conclusion (a) follows immediately from Theorem B. Next, we assume that the hyper-order $\rho_2(f) \geq 1$. Rewrite (8) as

$$\left(\frac{f(z+c)}{e^{\frac{P(z)}{3}}}\right)^3 + \left(\frac{f(z)}{e^{\frac{P(z)}{3}}}\right)^3 = 1.$$

Via (i) of Theorem A, we have

$$f(z) = \frac{1}{2} \frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))} e^{P(z)/3}, \quad (50)$$

where $h(z)$ is an entire function. The fact $\rho_2(f) \geq 1$ and (50) yields that h is transcendental. Here, we employ the result of Bergweiler in [4, Lemma 1]. $\square$

Lemma 4. *Let f be meromorphic and let g be entire and transcendental. If the lower order $\mu(f \circ g) < \infty$, then $\mu(f) = 0$.*

Note that $\rho(\wp) = \mu(\wp) = 2$ (see [2]). It follows from Lemma 3 that $\mu\left[\frac{1 + \frac{\wp'(h(z))}{\sqrt{3}}}{\wp(h(z))}\right] = \infty$, so is $\mu(f)$. Thus, e^P is a small function of $f(z)$ and $T(r, e^P) = S(r, f)$. Based on the idea of Liu and Ma in [34], we will obtain the desired result.

Set $a(z) = e^{P(z)}$ and rewrite (8) as $f^3(z+c) = -(f^3(z) - a(z))$, which implies that the multiplicities of the zeros of $f^3(z) - a(z)$ are at least 3. Rewrite (8) as $f^3(z) = -(f^3(z+c) - a(z))$, which implies that the multiplicities of the zeros of $f^3(z+c) - a(z)$ are at least 3. So, the zeros of $f^3(z) - a(z-c)$ are of multiplicities at least 3. Set $G(z) = f^3(z)$. Assume that the functions $a(z-c)$ and $a(z)$ are distinct from each other. Then, applying the second main theorem of Nevanlinna to G , one gets that

$$\begin{aligned} 2T(r, G) &\leq \overline{N}\left(r, \frac{1}{G(z) - a(z-c)}\right) + \overline{N}\left(r, \frac{1}{G(z) - a(z)}\right) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + S(r, G) \leq \frac{1}{3} \left[N\left(r, \frac{1}{G(z) - a(z-c)}\right) + N\left(r, \frac{1}{G(z) - a(z)}\right) + N(r, G) \right. \\ &\quad \left. + N\left(r, \frac{1}{G}\right) \right] + S(r, G) \leq \frac{4}{3} T(r, G) + S(r, G), \end{aligned}$$

which is a contradiction. Thus, $a(z) = a(z - c)$, which implies that $e^{P(z+c)} = e^{P(z)}$ and $e^{P(z+c)-P(z)} = 1$. In view of the fact that $P(z)$ is a polynomial, we derive that $P(z) = \alpha z + \beta$ with two constants α, β and $e^{\alpha c} = 1$. Therefore, we can rewrite (8) as

$$1 = \left[\frac{f(z)}{e^{\frac{P(z)}{3}}} \right]^3 + \left[\frac{f(z+c)}{e^{\frac{P(z)}{3}}} \right]^3 = \left[\frac{f(z)}{e^{\frac{P(z)}{3}}} \right]^3 + \left[\frac{f(z)}{e^{\frac{P(z+c)}{3}}} \right]^3.$$

Set $F(z) = \frac{f(z)}{e^{\frac{P(z)}{3}}}$. Then, $F(z)^3 + F(z+c)^3 = 1$. By Theorem 1, we arrive at the desired conclusion (b).

Therefore, we finish the proof of Theorem 2.

At the end of this section, we give the proof of Theorem 3.

Proof of Theorem 3. From Proposition 2, it suffices to consider the case $\rho(g(z)) < 1$. Suppose that f is a nonconstant meromorphic solution of (9), then g is nonconstant, since $f(z) = Ae^{\frac{g(z)}{n}}$. Differentiating the equation $g(z+c) = \frac{n}{m}g(z) + nLn\frac{B}{A}$ yields that

$$g'(z+c) = \frac{n}{m}g'(z). \quad (51)$$

Assume that $g'(z)$ is not a constant. If 0 is a Picard exceptional value of $g'(z)$, then $g'(z) = e^{h(z)}$, where $h(z)$ is a nonconstant entire function. So, the order $\rho(g(z)) = \rho(g'(z)) = \rho(e^{h(z)}) \geq 1$, a contradiction. Therefore, there exists a point z_2 such that $g'(z_2) = 0$. Equation (51) yields that $z_2 + nc$ is also a zero of g' . So, $n(d|c| + |z_2|, g') \geq d$ for any $d \in \mathbb{N}$, where $n(r, g')$ denotes the number of zeros of g' in $\{z : |z| < r\}$. Based on the method in [20, Lemma 3.2], we will derive a contradiction as follows:

$$\begin{aligned} \rho(g(z)) = \rho(g'(z)) &\geq \limsup_{r \rightarrow \infty} \frac{\log n(r, g')}{\log r} \geq \limsup_{d \rightarrow \infty} \frac{\log n(d|c| + |z_2|, g')}{\log(d|c| + |z_2|)} \\ &\geq \limsup_{d \rightarrow \infty} \frac{\log d}{\log(d|c| + |z_2|)} = 1, \end{aligned}$$

a contradiction. Thus, $g'(z)$ is a nonzero constant and (51) yields $m = n$. Furthermore, $g(z) = \alpha z + \beta$, and $f(z) = Ae^{\frac{\alpha z + \beta}{n}}$ with $A^n(1 + e^{\alpha c}) = 1$. $\square$

Therefore, we finish the proof of Theorem 3.

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