THE MINIMAL NUMBER OF PERIODIC ORBITS
OF PERIODS GUARANTEED IN SHARKOVSKII’S THEOREM

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Let \( f(x) \) be a continuous function from a compact real interval into itself with a periodic orbit of minimal period \( m \), where \( m \) is not an integral power of 2. Then, by Sharkovskii’s theorem, for every positive integer \( n \) with \( m < n \) in the Sharkovskii ordering defined below, a lower bound on the number of periodic orbits of \( f(x) \) with minimal period \( n \) is 1. Could we improve this lower bound from 1 to some larger number? In this paper, we give a complete answer to this question.

1. Introduction

Let \( I \) be a compact real interval and let \( f \in C^0(I, I) \). For any \( x_0 \) in \( I \) and any positive integer \( k \), we let \( f^k(x_0) \) denote the \( k \)th iterate of \( x_0 \) under \( f \) and call \( \{ f^k(x_0) \mid k \geq 0 \} \) the orbit of \( x_0 \) (under \( f \)). If \( f^m(x_0) = x_0 \) for some positive integer \( m \), we call \( x_0 \) a periodic point of \( f \) and call the cardinality of the orbit of \( x_0 \) (under \( f \)) the minimal period of \( x_0 \) and of the orbit (under \( f \)). If \( f \) has a periodic orbit of a period \( m \), must \( f \) also have periodic orbits of periods \( n \neq m \)? In 1964, Sharkovskii [11] (see [1], [3], [7], [9],
[12], [13], also) had given a complete answer to this question. Arrange the positive integers according as the following new order (called Sharkovskii ordering):

$$\begin{align*}
3 & \Delta 5 \Delta 7 \Delta \ldots \\
\Delta & 2 \cdot 3 \Delta 2 \cdot 5 \Delta 2 \cdot 7 \Delta \ldots \\
\vdots & \vdots \vdots \\
\Delta & 2^n \cdot 3 \Delta 2^n \cdot 5 \Delta 2^n \cdot 7 \Delta \ldots \\
\vdots & \vdots \vdots \\
\Delta \ldots & \Delta 2^3 \Delta 2^2 \Delta 2 \Delta 1.
\end{align*}$$

Sharkovskii's theorem says that any function $f \in C^0(I, I)$ with a periodic orbit of minimal period $m$, must also have at least one periodic orbit of minimal period $n$ precisely when $m \Delta n$ in the above Sharkovskii ordering. Therefore, for every positive integer $n$ with $m \Delta n$, the number 1 is a lower bound on the number of distinct periodic orbits of $f$ with minimal period $n$. One question arises naturally: could we improve this lower bound from 1 to some larger number?

In 1976 Bowen and Franks [2] showed, among other things, that if $f \in C^0(I, I)$ has a periodic orbit of minimal period $n = 2^d m$, where $m > 1$ is odd, then there is a number $M_n$ (independent of $f$) such that, for all integers $k \geq M_n$, $f$ has at least $(2^{k/m})/(2^{d_k})$ distinct periodic orbits of minimal period $2^d k$.

In 1979, Jonker [8] also obtained a similar result on a class of unimodal maps. If $c$ is an interior point of $I$, let $S_c$ denote the collection of all $f \in C^0(I, I)$ which has either one maximum or one minimum point at $c$, and is strictly monotone on each component of $I - \{c\}$ with $f(\partial I) \subset \partial I$. Jonker showed, among other things, that if $m, n$ are any two odd integers with $1 < m < n$, and if $f \in S_c$ has a periodic orbit of minimal period $2^k m$, where $k \geq 0$ is any integer, then $f$ must also have at least $2^{(n-m)/2}$ distinct periodic orbits of minimal period $2^k n$. 

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In [6], a result along this line is also obtained. However, that result is only a partial one. In this paper we give a complete answer to that question.

In Section 2 we state our main results (Theorems 1, 2, and 3). In Section 3 we describe the method used to prove them. This method is the same as that used in [5] and [6]. The proofs of Theorems 1 and 2 will appear in Sections 4 and 5. Theorem 3 then follows easily from Theorems 1 and 2.

2. Statement of main results

Let \( \phi(m) \) be an integer-valued function defined on the set of all positive integers. If \( m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \), where the \( p_i \)'s are distinct prime numbers, \( r \) and \( k_i \)'s are positive integers, we define

\[
\phi(1, \phi) = \phi(1)
\]

and

\[
\phi(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi(m/p_i) + \sum_{i_1 < i_2} \phi(m/(p_{i_1} p_{i_2}))
\]

\[
- \sum_{i_1 < i_2 < i_3} \phi(m/(p_{i_1} p_{i_2} p_{i_3})) + \cdots + (-1)^r \phi(m/(p_1 p_2 \cdots p_r))
\]

where the summation \( \sum_{i_1 < i_2 < \cdots < i_j} \) is taken over all \( i_1, i_2, \ldots, i_j \) with \( 1 \leq i_1 < i_2 < \ldots < i_j \leq r \). If, when considered as a sequence, \( \langle \phi(m) \rangle \) is the Lucas sequence, that is if \( \phi(1) = 1 \), \( \phi(2) = 3 \), and \( \phi(m+2) = \phi(m+1) + \phi(m) \) for all positive integers \( m \), then, for simplicity, we denote \( \phi(m, \phi) \) as \( \Phi_1(m) \). Note that if \( f \in C^0(I, I) \) and if, for every positive integer \( m \), \( \phi(m) \) is the number of distinct solutions of the equation \( f^m(x) = x \), then \( \Phi(m, \phi) \) is, by the standard inclusion-exclusion argument, the number of periodic points of \( f \) with minimal period \( m \). Now we can state the following theorem.

THEOREM 1. Let \( f : [1, 3] \to [1, 3] \) be defined by \( f(x) = -2x + 5 \).
if \( 1 \leq x \leq 2 \) and \( f(x) = x - 1 \) if \( 2 \leq x \leq 3 \). Then, the following hold:

(a) for every positive integer \( m \), if \( a_m \) is the number of distinct solutions of the equation \( f^m(x) = x \), then the sequence \( \{a_m\} \) is the Lucas sequence;

(b) for every positive integer \( m \), \( f \) has exactly \( \phi_1(m)/m \) distinct periodic orbits of minimal period \( m \);

(c) the sequence \( \{\phi_1(m)/m\} \) is strictly increasing for \( m \geq 6 \) and \( \lim_{n \to \infty} \left[ \frac{\phi_1(m+1)/(m+1)}{\phi_1(m)/m} \right] = \frac{1+\sqrt{5}}{2} \)

Fix any integer \( n > 1 \) and let
\[ Q_n = \{(1, n+1)\} \cup \{(m, 2n+2-m) \mid 2 \leq m \leq n\} \cup \{(m, 2n+1-m) \mid n+1 \leq m \leq 2n\} . \]

For all integers \( i, j, \) and \( k \), with \( 1 \leq i, j \leq 2n \) and \( k \geq 1 \), we define \( b_{k,i,j,n} \) recursively as follows:
\[
b_{1,i,j,n} = \begin{cases} 1 & \text{if } (i, j) \in Q_n, \\ 0 & \text{otherwise,} \end{cases}
\]
and
\[
b_{k+1,i,j,n} = \begin{cases} b_{k,i,2n+1-j,n} + b_{k,i,n+1,n}, & \text{if } 1 \leq j \leq n-1, \\ b_{k,i,n,n} + b_{k,i,n+1,n}, & \text{if } j = n, \\ b_{k,i,j,n}, & \text{if } j = n+1, \\ b_{k,i,2n+2-j,n}, & \text{if } n+2 \leq j \leq 2n. \end{cases}
\]

We also define \( c_{k,n} \) by letting
\[
c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n}.
\]

Note that these sequences \( \{b_{k,i,j,n}\} \) and \( \{c_{k,n}\} \) have the following six properties. Some of these will be used later in the proofs of our main results. (Recall that \( n > 1 \) is fixed.)
(i) The sequence \( \{b_{k,1,n,n}\} \) is increasing, and for all integers \( k \geq 2 \), we have \( b_{k,1,n,n} \geq b_{k,n+1,n,n} \) and \( b_{k,1,i+1,n} \geq b_{k,1,i,n} \) for all \( 1 \leq i \leq n-1 \).

(ii) The sequences \( \{b_{k,1,j,n}\} \), \( 1 \leq j \leq n \), and \( \{b_{k,n+1,n,n}\} \) can also be obtained by the following recursive formulas:

\[
\begin{align*}
b_{1,1,j,n} & = 0, \quad 1 \leq j \leq n, \\
b_{2,1,j,n} & = 1, \quad 1 \leq j \leq n, \\
b_{1,n+1,n,n} & = b_{2,n+1,n,n} = 1, \\
b_{1,n+1,j,n} & = b_{2,n+1,j,n} = 0, \quad 1 \leq j \leq n-1.
\end{align*}
\]

For \( i = 1 \) or \( n + 1 \), and \( k \geq 1 \),

\[
\begin{align*}
b_{k+2,i,n,n} & = b_{k,i,1,n} + b_{k+1,i,n,n}, \\
b_{k+2,i,j,n} & = b_{k,i,1,n} + b_{k,i,j+1,n}, \quad 1 \leq j \leq n-1.
\end{align*}
\]

(iii) For every positive integer \( k \), \( a_{k+2n-2,n} \) can also be obtained by the following formulas:

\[
\begin{align*}
a_{k+2n-2,n} & = \sum_{j=1}^{n} b_{k+2n-2,2j,1,j,n} + 2 \sum_{j=1}^{n} b_{k+2n-2,1,j,n} \\
& = b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} (2^{i-2})b_{k,1,n+1-i,n}.
\end{align*}
\]

The first identity also holds for all integers \( k \) with \(-2n+3 \leq k \leq 0\) provided we define \( b_{k,1,j,n} = 0 \) for all \(-2n+3 \leq k \leq 0\) and \( 1 \leq j \leq n \).

(iv) For all integers \( k \) with \( 1 \leq k \leq 2n \), \( c_{2k,n} = 2^{k+1} - 1 \).

(v) For all integers \( k \) with \( n+1 \leq k \leq 3n \),

\[
a_{2k+1,n} = 2a_{2k+1,n+1} - 1.
\]

(vi) Since, for every positive integer \( k \geq 2n+1 \),

\[
b_{k,1,n,n} = b_{k-1,1,n,n} + \sum_{i=2}^{2n} (-1)^{i}b_{k-i,1,n,n},
\]
there exist $2n + 1$ nonzero constants $\alpha_i$'s such that
\[ b_{k,1,n,n} = \sum_{j=1}^{2n+1} \alpha_j x_j^k \]
for all positive integers $k$, where
\[ \{ x_j | 1 \leq j \leq 2n+1 \} \]
is the set of all zeros (including complex zeros) of the polynomial
\[ x^{2n+1} - 2x^{2n-1} - 1. \]

For all positive integers $k, m, n$, with $n > 1$, we let
\[ \phi_n(k) = \alpha_{k,n} \]
and let $\Phi(m) = \Phi(m, \phi_n)$, where $\Phi$ is defined as above.

Now we can state the following theorem.

**THEOREM 2.** For every integer $n > 1$, let
\[ f_n : [1, 2n+1] \rightarrow [1, 2n+1] \]
be the continuous function with the following six properties:

1. $f_n(1) = n + 1$,
2. $f_n(2) = 2n + 1$,
3. $f_n(n+1) = n + 2$,
4. $f_n(n+2) = n$,
5. $f_n(2n+1) = 1$, and
6. $f_n$ is linear on each component of the complement of the set
\{2, n+1, n+2\} in $[1, 2n+1]$.

Then the following hold:

(a) for every positive integer $k$, the equation $f_n^k(x) = x$ has
exactly $\alpha_{k,n}$ distinct solutions;

(b) for every positive integer $m$, $f_n(x)$ has exactly $\Phi_n(m)/m$
distinct periodic orbits of minimal period $m$;

(c) $\lim_{m \to \infty} \left( \log \left( \frac{\Phi_n(m)}{m} \right) \right) / m = \lambda_n$, where $\lambda_n$ is the (unique)
positive (and the largest in absolute value) zero of the
polynomial $x^{2n+1} - 2x^{2n-1} - 1$. 
From Theorems 1 and 2 above and Theorem 2 of [12, p. 243], we easily obtain the following result.

**THEOREM 3.** Assume that \( f \in C^0(I, I) \) has a periodic orbit of minimal period \( s = 2^k(2n+1) \), where \( n \geq 1 \) and \( k \geq 0 \), and no periodic orbits of minimal period \( r \) with \( r \neq s \) in the Sharkovskii ordering. Then for every positive integer \( t \) with \( s \leq t \) in the Sharkovskii ordering, \( f \) has at least \( \frac{\phi_n(t/2^k)}{(t/2^k)} \) (sharp) distinct periodic orbits of minimal period \( t \).

**REMARK 1.** We call attention to the fact that there exist continuous functions from \( I \) into \( I \) with exactly one periodic orbit of minimal period \( 2^i \) for every positive integer \( i \) (and two fixed points), but no other periods (see [10]).

**REMARK 2.** With the help of Theorem 2 of [12, p. 243] on the distribution along the real line of points in a periodic orbit of odd period \( n > 1 \), when there are no periodic orbits of odd period \( m \) with \( 1 < m < n \), our results give a new proof of Sharkovskii's theorem.

**REMARK 3.** Table 1 (see p. 96) lists the first 31 values of \( \frac{\phi_n(m)}{m} \) for \( 1 \leq n \leq 5 \). It seems that, for all positive integers \( n \) and \( m \), we have

\[
\frac{\phi_n(2m+1)}{(2m+1)} = 2^{m-n} \quad \text{for } n \leq m \leq 3n+1,
\]

and

\[
\frac{\phi_n(2m+1)}{(2m+1)} > 2^{m-n} \quad \text{for } m > 3n + 1.
\]

**REMARK 4.** For all positive integers \( k \) and \( m \), let \( \psi(k) = 2^k \) and \( \Psi(m) = \phi(m, \psi) \), where \( \phi \) is defined as in Section 2. It is obvious that \( \Psi(m)/m \) is the number of distinct periodic orbits of minimal period \( m \) for, say, the mapping \( g(x) = 4x(1-x) \) from \([0, 1]\) onto itself. Since, for all positive integers \( k \) and \( n \) with \( 1 \leq k \leq 2n \), \( \phi_{2k,n} = 2^{k+1} - 1 \) \( c_{1,n} = 1 \), we obtain that \( \frac{\phi_n(2k+2)}{(2k+2)} = \Psi(k+1)/(k+1) \) for all
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It seems that \( \frac{(2k+2)}{(2k+2)} > \frac{\Psi(k+1)}{(k+1)} \) for all \( k > 2n \).

But note that
\[
\lim_{k \to \infty} \left( \log \left( \frac{\Phi_n(2k+2)}{2k+2} \right) \right) / (2k+2) = \frac{1}{2} \log 2 = \frac{1}{2} \lim_{k \to \infty} \left( \log \left( \frac{\Psi(k+1)}{(k+1)} \right) \right) / (k+1),
\]
where \( \lambda_n \) is the unique positive zero of the polynomial
\[ x^{2n+1} - 2x^{2n-1} - 1. \]

3. **Symbolic representation for continuous piecewise linear functions**

In this section we describe a method. This method was first introduced in [4], and then generalized in [5] to construct, for every positive integer \( n \), a continuous piecewise linear function from \([0, 1]\) into itself which has a periodic orbit of minimal period \( 3 \), but with the property that almost all (in the sense of Lebesgue) points of \([0, 1]\) are eventually periodic of minimal period \( n \) with the periodic orbit the same as the orbit of a fixed known period \( n \) point. The same method was also used in [6] to give a new proof of a result of Block et al. [7] on the topological entropy of interval maps. In this paper we will use this method to prove our main results.

Throughout this section, let \( g \) be a continuous piecewise linear function from the interval \([a, d]\) into itself. We call the set \( \{(x_i, y_i) \mid i = 1, 2, \ldots, k\} \) a set of nodes for (the graph of) \( y = g(x) \) if the following three conditions hold:

1. \( k \geq 2 \),
2. \( x_1 = a, \ x_k = d, \ x_1 < x_2 < \ldots < x_k \), and
3. \( g \) is linear on \([x_i, x_{i+1}]\) for all \( 1 \leq i \leq k-1 \) and \( y_i = g(x_i) \) for all \( 1 \leq i \leq k \).

For any such set, we will use its \( y \)-coordinates \( y_1, y_2, \ldots, y_k \) to represent the graph of \( y = g(x) \) and call \( y_1y_2 \ldots y_k \) (in that order) a (symbolic) representation for (the graph of) \( y = g(x) \). For
1 \leq i < j \leq k$, we will call $y_i y_{i+1} \ldots y_j$ the representation for $y = g(x)$ on $[x_i, x_j]$ obtained by restricting $y_1 y_2 \ldots y_k$ to $[x_i, x_j]$. For convenience, we will also call every $y_i$ in $y_1 y_2 \ldots y_k$ a node. If $y_i = y_{i+1}$ for some $i$ (that is, $g$ is constant on $[x_i, x_{i+1}]$), we will simply write $y_1 \ldots y_i y_{i+2} \ldots y_k$ instead of $y_1 \ldots y_i y_{i+1} y_{i+2} \ldots y_k$. Therefore, every two consecutive nodes in a (symbolic) representation are distinct.

Now assume that \{$(x_i, y_i) \mid i = 1, 2, \ldots, k$\} is a set of nodes for $y = g(x)$ and $a_1 a_2 \ldots a_r$ is a representation for $y = g(x)$ with \{$a_1, a_2, \ldots, a_r$\} $\subseteq \{y_1, y_2, \ldots, y_k\}$ and $a_i \neq a_{i+1}$ for all $1 \leq i \leq r-1$. If $\{y_1, y_2, \ldots, y_k\} \subseteq \{x_1, x_2, \ldots, x_k\}$, then there is an easy way to obtain a representation for $y = g^2(x)$ from the one $a_1 a_2 \ldots a_r$ for $y = g(x)$. The procedure is as follows. First, for any two distinct real numbers $u$ and $v$, let $[u : v]$ denote the closed interval with endpoints $u$ and $v$. Then let $b_1 b_2 \ldots b_k$ be the representation for $y = g(x)$ on $[a_1 : a_{i+1}]$ which is obtained by restricting $a_1 a_2 \ldots a_r$ to $[a_1 : a_{i+1}]$. We use the following notation to indicate this fact: $a_i a_{i+1} \rightarrow b_1 b_2 \ldots b_{i, i}$ (under $g$) if $a_i < a_{i+1}$, or $a_i a_{i+1} \rightarrow b_i, b_{i, i} \ldots b_{i}, b_{i, i}$ (under $g$) if $a_i > a_{i+1}$.

The above representation on $[a_1 : a_{i+1}]$ exists since \{$a_1, a_2, \ldots, a_r$\} $\subseteq \{x_1, x_2, \ldots, x_k\}$. Finally, if $a_i < a_{i+1}$, let $s_{i, j} = b_{i, j}$ for all $1 \leq j \leq t_i$. If $a_i > a_{i+1}$, let $s_{i, j} = b_{i, t_i-j}$ for all $1 \leq j \leq t_i$. It is easy to see that $s_{i, t_i} = s_{i+1, 1}$ for all $1 \leq i \leq r-1$. So, if we define

$$Z = s_{1, 1} \ldots s_{1, t_1} s_{2, 2} \ldots s_{2, t_2} \ldots s_{r, 2} \ldots s_{r, t_r},$$

then it is obvious that $Z$ is a representation for $y = g^2(x)$. It is
also obvious that the above procedure can be applied to the representation \( Z \) for \( y = g^2(x) \) to obtain one for \( y = g^3(x) \), and so on.

4. Proof of Theorem 1

In this section we let \( f(x) \) denote the map as defined in Theorem 1, that is \( f(x) = -2x + 5 \) if \( 1 \leq x \leq 2 \), and \( f(x) = x - 1 \) if \( 2 \leq x \leq 3 \). The proof of part (a) of Theorem 1 will follow from two easy lemmas.

**Lemma 4.** Under \( f \), we have

\[
13 \rightarrow 312, \quad 31 \rightarrow 213,
\]
\[
12 \rightarrow 31, \quad 21 \rightarrow 13.
\]

In the following when we say the representation for \( y = f^k(x) \), we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 4 to the representation 312 for \( y = f(x) \) successively until we get to the one for \( y = f^k(x) \).

For every positive integer \( k \), let \( u_{1,k} \) (\( u_{2,k} \) respectively) denote the number of 13's and 31's in the representation for \( y = f^k(x) \) whose corresponding \( x \)-coordinates are \( \leq \) (\( \geq \) respectively) 2. We also let \( v_{1,k} \) (\( v_{2,k} \) respectively) denote the number of 12's and 21's in the representation for \( y = f^k(x) \) whose corresponding \( x \)-coordinates are \( \leq \) (\( \geq \) respectively) 2. It is clear that \( u_{1,1} = v_{2,1} = 1 \) and \( u_{2,1} = v_{1,1} = 0 \). Now from Lemma 4, we have

**Lemma 5.** For every positive integer \( k \) and integers \( i = 1, 2 \),

\[
u_{i,k+1} = u_{i,k} + v_{i,k} \quad \text{and} \quad v_{i,k+1} = u_{i,k}.
\]

Furthermore, if

\[
w_k = u_{1,k} + v_{1,k} + u_{2,k},
\]
then \( w_1 = 1 \), \( w_2 = 3 \), and \( w_{k+2} = w_{k+1} + w_k \). That is, \( \{w_k\} \) is the Lucas sequence.

Since, for every positive integer \( k \), the number of distinct solutions of the equation \( f^k(x) = x \) equals \( w_k \), part (a) of Theorem 1 follows from Lemma 5. Part (b) follows from the standard inclusion-exclusion argument. As for part (c), we note that, for every positive
integer $k$,

$$a_{k+2} = \sum_{i=1}^{k} a_i + 3.$$ 

So, for $k \geq 6$,

$$(k+2) \Phi_1(k+3) > (k+2)[a_{(k+3)/2} + 1] > (k+3)[a_{k+2} + 1] > (k+3) \Phi_1(k+2),$$

where $[(k+3)/2]$ is the largest integer less than or equal to $(k+3)/2$.

The proof of the other statement of part (a) is easy and omitted. This completes the proof of Theorem 1.

5. Proof of Theorem 2

In this section we fix any integer $n > 1$ and let $f_n(x)$ denote the map as defined in Theorem 2. For convenience, we also let $S_n$ denote the set of all these $hn$ symbolic pairs: $i(i+1), (i+1)i$, $1 \leq i \leq n-1$; $n(n+2), (n+2)n, (n+1)(2n+1), (2n+1)(n+1), j(j+1), (j+1)j$, $n+2 \leq j \leq 2n$.

The following lemma is easy.

**LEMMA 6.** Under $f_n$, we have

$$n(n+2) \rightarrow (n+3)(n+2)n, (n+2)n \rightarrow (n+2)(n+3),$$

$$(n+1)(2n+1) \rightarrow (n+2)n(n-1)(n-2) \ldots 321,$$

$$(2n+1)(n+1) \rightarrow 123 \ldots (n-2)(n-1)n(n+2),$$

and $uv \rightarrow f_n(u)f_n(v)$ for every $uv$ in

$$S_n = \{n(n+2), (n+2)n, (n+1)(2n+1), (2n+1)(n+1)\}.$$

In the following when we say the representation for $y = f^n_n(x)$, we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 6 to the representation

$$(n+1)(2n+1)(2n)(2n-1) \ldots (n+2)n(n-1)(n-2) \ldots 321$$
for $y = f_n^k(x)$ successively until we get to the one for $y = f_n^k(x)$.

For every positive integer $k$ and all integers $i, j$ with $1 \leq i, j \leq 2n$, let $b_{k,i,j,n}$ denote the number of $uv$'s and $vu$'s in the representation for $y = f_n^k(x)$ whose corresponding $x$-coordinates are in $[i, i+1]$, where $uv = j(j+1)$ if $1 \leq j \leq n-1$ or $n+2 \leq j \leq 2n$, $uv = n(n+2)$ if $j = n$, and $uv = (n+1)(2n+1)$ if $j = n + 1$. It is obvious that $b_{1,i,n+1,n} = 1$, $b_{1,i,2n+2-i,n} = 1$ if $2 \leq i \leq n$, $b_{1,i,2n+1-i,n} = 1$ if $n+1 \leq i \leq 2n$, and $b_{1,i,j,n} = 0$ elsewhere. From Lemma 6, we see that the sequences $(b_{k,i,j,n})$ are exactly the same as those defined in Section 2.

Since

$$c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n},$$

it is clear that $c_{k,n}$ is the number of intersection points of the graph of $y = f_n^k(x)$ with the diagonal $y = x$. This proves part (a) of Theorem 2. Part (b) follows from the standard inclusion-exclusion argument. As for part (c), we note that there exist $2n + 1$ nonzero constants $\alpha_j$'s such that

$$b_{k,1,n,n} = \sum_{j=1}^{2n+1} \alpha_j x_j^k$$

for all positive integers $k$, where $\{x_j \mid 1 \leq j \leq 2n+1\}$ is the set of all zeros (including complex zeros) of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. Since $c_{k+2n-2,n}$ can also be expressed as

$$b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} (2i-2)b_{k,1,n+1-i,n},$$

part (c) follows from property (i) of the sequences $(b_{k,i,j,n})$ stated in Section 2. This completes the proof of Theorem 2.
References


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