THE MINIMAL NUMBER OF PERIODIC ORBITS
OF PERIODS GUARANTEED IN SHARKOVSKII'S THEOREM

Bau-Sen Du

Let $f(x)$ be a continuous function from a compact real interval
into itself with a periodic orbit of minimal period $m$, where $m$
is not an integral power of $2$. Then, by Sharkovskii's theorem,
for every positive integer $n$ with $m < n$ in the Sharkovskii
ordering defined below, a lower bound on the number of periodic
orbits of $f(x)$ with minimal period $n$ is 1. Could we
improve this lower bound from 1 to some larger number? In this
paper, we give a complete answer to this question.

1. Introduction

Let $I$ be a compact real interval and let $f \in C^0(I, I)$. For any
$x_0$ in $I$ and any positive integer $k$, we let $f^k(x_0)$ denote the $k$th
iterate of $x_0$ under $f$ and call $\{f^k(x_0) \mid k \geq 0\}$ the orbit of $x_0$
(under $f$). If $f^m(x_0) = x_0$ for some positive integer $m$, we call $x_0$
a periodic point of $f$ and call the cardinality of the orbit of $x_0$
(under $f$) the minimal period of $x_0$ and of the orbit (under $f$). If $f$
has a periodic orbit of a period $m$, must $f$ also have periodic orbits
of periods $n \neq m$? In 1964, Sharkovskii [11] (see [1], [3], [7], [9],

Received 23 August 1984.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/84
$A2.00 + 0.00.$

89
[12], [13], also) had given a complete answer to this question. Arrange the positive integers according as the following new order (called Sharkovskii ordering):

\[
\begin{align*}
3 & \Delta 5 \Delta 7 \Delta \ldots \\
\Delta & 2 \cdot 3 \Delta 2 \cdot 5 \Delta 2 \cdot 7 \Delta \ldots \\
\vdots & \quad \vdots \\
\Delta & 2^n \cdot 3 \Delta 2^n \cdot 5 \Delta 2^n \cdot 7 \Delta \ldots \\
\vdots & \quad \vdots \\
\Delta & \ldots \Delta 2^3 \Delta 2^2 \Delta 2 \Delta 1.
\end{align*}
\]

Sharkovskii's theorem says that any function \( f \in C^0(I, I) \) with a periodic orbit of minimal period \( m \), must also have at least one periodic orbit of minimal period \( n \) precisely when \( m \Delta n \) in the above Sharkovskii ordering. Therefore, for every positive integer \( n \) with \( m \Delta n \), the number 1 is a lower bound on the number of distinct periodic orbits of \( f \) with minimal period \( n \). One question arises naturally: could we improve this lower bound from 1 to some larger number?

In 1976 Bowen and Franks [2] showed, among other things, that if \( f \in C^0(I, I) \) has a periodic orbit of minimal period \( n = 2^d m \), where \( m > 1 \) is odd, then there is a number \( M_n \) (independent of \( f \)) such that, for all integers \( k \geq M_n \), \( f \) has at least \( \left( \frac{2^{k/m}}{2^d} \right)^2 \) distinct periodic orbits of minimal period \( 2^d k \).

In 1979, Jonker [8] also obtained a similar result on a class of unimodal maps. If \( a \) is an interior point of \( I \), let \( S_a \) denote the collection of all \( f \in C^0(I, I) \) which has either one maximum or one minimum point at \( a \), and is strictly monotone on each component of \( I \setminus \{a\} \) with \( f(3I) \subset 3I \). Jonker showed, among other things, that if \( m, n \) are any two odd integers with \( 1 < m < n \), and if \( f \in S_a \) has a periodic orbit of minimal period \( 2^k m \), where \( k \geq 0 \) is any integer, then \( f \) must also have at least \( 2^{(n-m)/2} \) distinct periodic orbits of minimal period \( 2^k n \).
In [6], a result along this line is also obtained. However, that result is only a partial one. In this paper we give a complete answer to that question.

In Section 2 we state our main results (Theorems 1, 2, and 3). In Section 3 we describe the method used to prove them. This method is the same as that used in [5] and [6]. The proofs of Theorems 1 and 2 will appear in Sections 4 and 5. Theorem 3 then follows easily from Theorems 1 and 2.

2. Statement of main results

Let \( \phi(m) \) be an integer-valued function defined on the set of all positive integers. If \( m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} \), where the \( p_i \)'s are distinct prime numbers, \( r \) and \( k_i \)'s are positive integers, we define

\[
\phi(1, \phi) = \phi(1)
\]

and

\[
\phi(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi(m/p_i) + \sum_{i_1 < i_2} \phi(m/(p_i p_{i_2})) - \sum_{i_1 < i_2 < i_3} \phi(m/(p_i p_{i_2} p_{i_3})) + \ldots + (-1)^r \phi(m/(p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}))
\]

where the summation \( \sum_{i_1 < i_2 < \ldots < i_j} \) is taken over all \( i_1, i_2, \ldots, i_j \) with \( 1 \leq i_1 < i_2 < \ldots < i_j \leq r \). If, when considered as a sequence, \( \langle \phi(m) \rangle \) is the Lucas sequence, that is if \( \phi(1) = 1, \phi(2) = 3 \), and \( \phi(m+2) = \phi(m+1) + \phi(m) \) for all positive integers \( m \), then, for simplicity, we denote \( \phi(m, \phi) \) as \( \phi_1(m) \). Note that if \( f \in C^0(I, I) \) and if, for every positive integer \( m \), \( \phi(m) \) is the number of distinct solutions of the equation \( f^m(x) = x \), then \( \phi(m, \phi) \) is, by the standard inclusion-exclusion argument, the number of periodic points of \( f \) with minimal period \( m \). Now we can state the following theorem.

**Theorem 1.** Let \( f : [1, 3] \to [1, 3] \) be defined by \( f(x) = -2x + 5 \)
if $1 \leq x \leq 2$ and $f(x) = x - 1$ if $2 \leq x \leq 3$. Then the following hold:

(a) for every positive integer $m$, if $a_m$ is the number of distinct solutions of the equation $f^m(x) = x$, then the sequence $(a_m)$ is the Lucas sequence;

(b) for every positive integer $m$, $f$ has exactly $\phi_1(m)/m$ distinct periodic orbits of minimal period $m$;

(c) the sequence $(\phi_1(m)/m)$ is strictly increasing for $m \geq 6$ and $\lim_{n \to \infty} [\phi_1(m+1)/(m+1)]/\phi_1(m)/m = (1+\sqrt{5})/2$.

Fix any integer $n > 1$ and let

$$Q_n = \{(1, n+1)\} \cup \{(m, 2n+2-m) \mid 2 \leq m \leq n\} \cup \{(m, 2n+1-m) \mid n+1 \leq m \leq 2n\}.$$ 

For all integers $i$, $j$, and $k$, with $1 \leq i, j \leq 2n$ and $k \geq 1$, we define $b_{k,i,j,n}$ recursively as follows:

$$b_{1,i,j,n} = \begin{cases} 1 & \text{if } (i, j) \in Q_n, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$b_{k+1,i,j,n} = \begin{cases} b_{k,i,2n+1-j,n} + b_{k,i,n+1,n}, & \text{if } 1 \leq j \leq n-1, \\ b_{k,i,n,n} + b_{k,i,n+1,n}, & \text{if } j = n, \\ b_{k,i,1,n}, & \text{if } j = n+1, \\ b_{k,i,2n+2-j,n}, & \text{if } n+2 \leq j \leq 2n. \end{cases}$$

We also define $c_{k,n}$ by letting

$$c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n}.$$ 

Note that these sequences $(b_{k,i,j,n})$ and $(c_{k,n})$ have the following six properties. Some of these will be used later in the proofs of our main results. (Recall that $n > 1$ is fixed.)
(i) The sequence \( \{b_{k,l,n,n}\} \) is increasing, and for all integers \( k \geq 2 \), we have \( b_{k,l,n,n} \geq b_{k,n+1,l,n,n} \) and \( b_{k,l,i+1,n,n} \geq b_{k,l,i,n,n} \) for all \( 1 \leq i \leq n-1 \).

(ii) The sequences \( \{b_{k,l,j,n}\} \), \( 1 \leq j \leq n \), and \( \{b_{k,n+l,n,n}\} \) can also be obtained by the following recursive formulas:

\[
\begin{align*}
b_{1,l,j,n} &= 0, \quad 1 \leq j \leq n, \\
b_{2,l,j,n} &= 1, \quad 1 \leq j \leq n, \\
b_{1,n+l,n,n} &= b_{2,n+l,n,n} = 1, \\
b_{1,n+l,j,n} &= b_{2,n+l,j,n} = 0, \quad 1 \leq j \leq n-1.
\end{align*}
\]

For \( i = 1 \) or \( n+1 \), and \( k \geq 1 \),

\[
\begin{align*}
b_{k+2,i,n,n} &= b_{k,i,n,n} + b_{k+1,i,n,n}, \\
b_{k+2,i,j,n} &= b_{k,i,n,n} + b_{k,i,j+1,n}, \quad 1 \leq j \leq n-1.
\end{align*}
\]

(iii) For every positive integer \( k \), \( a_{k+2n-2,n} \) can also be obtained by the following formulas:

\[
a_{k+2n-2,n} = b_{k+2n-2,n+1,n,n} + 2 \sum_{j=1}^{n} b_{k+2n-2,1,j,n,n}
\]

\[
= b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} (2^{i-2}) b_{k,1,n+1-i,n,n}.
\]

The first identity also holds for all integers \( k \) with \(-2n+3 \leq k \leq 0\) provided we define \( b_{k,l,j,n} = 0 \) for all \(-2n+3 \leq k \leq 0\) and \( 1 \leq j \leq n \).

(iv) For all integers \( k \) with \( 1 \leq k \leq 2n \), \( c_{2k,n} = 2^{k+1} - 1 \).

(v) For all integers \( k \) with \( n+1 \leq k \leq 3n \),

\[
a_{2k+1,n} = 2a_{2k+1,n+1} - 1.
\]

(vi) Since, for every positive integer \( k \geq 2n+1 \),

\[
b_{k,l,n,n} = b_{k-1,l,1,n,n} + \sum_{i=2}^{2n} (-1)^{i} b_{k-i,l,1,n,n},
\]
there exist $2n + 1$ nonzero constants $\alpha_j$'s such that
\[ b_{k, l, n, n} = \sum_{j=1}^{2n+1} \alpha_j x_j^k \] for all positive integers $k$, where \[ x_j \mid 1 \leq j \leq 2n+1 \] is the set of all zeros (including complex zeros) of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$.

For all positive integers $k, m, n$, with $n > 1$, we let $\phi_n(k) = c_{k,n}$ and let $\phi_n(m) = \phi(m, \phi_n)$, where $\phi$ is defined as above.

Now we can state the following theorem.

**THEOREM 2.** For every integer $n > 1$, let
\[ f_n : [1, 2n+1] \rightarrow [1, 2n+1] \]
be the continuous function with the following six properties:

1. $f_n(1) = n + 1$,
2. $f_n(2) = 2n + 1$,
3. $f_n(n+1) = n + 2$,
4. $f_n(n+2) = n$,
5. $f_n(2n+1) = 1$, and
6. $f_n$ is linear on each component of the complement of the set \{2, n+1, n+2\} in $[1, 2n+1]$.

Then the following hold:

(a) for every positive integer $k$, the equation $f_n^k(x) = x$ has exactly $c_{k,n}$ distinct solutions;

(b) for every positive integer $m$, $f_n(x)$ has exactly $\phi_n(m)/m$ distinct periodic orbits of minimal period $m$;

(c) $\lim_{m \to \infty} \left( \log[\phi_n(m)/m] \right)/m = \lambda_n$, where $\lambda_n$ is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. 
From Theorems 1 and 2 above and Theorem 2 of [12, p. 243], we easily obtain the following result.

**THEOREM 3.** Assume that \( f \in C^0(I, I) \) has a periodic orbit of minimal period \( s = 2^k(2n+1) \), where \( n \geq 1 \) and \( k \geq 0 \), and no periodic orbits of minimal period \( r \) with \( r \neq s \) in the Sharkovskii ordering. Then for every positive integer \( t \) with \( s \neq t \) in the Sharkovskii ordering, \( f \) has at least \( \phi_n(t/2^k)/(t/2^k) \) (sharp) distinct periodic orbits of minimal period \( t \).

**REMARK 1.** We call attention to the fact that there exist continuous functions from \( I \) into \( I \) with exactly one periodic orbit of minimal period \( 2^k \) for every positive integer \( k \) (and two fixed points), but no other periods (see [10]).

**REMARK 2.** With the help of Theorem 2 of [12, p. 243] on the distribution along the real line of points in a periodic orbit of odd period \( n > 1 \), when there are no periodic orbits of odd period \( m \) with \( 1 < m < n \), our results give a new proof of Sharkovskii’s theorem.

**REMARK 3.** Table 1 (see p. 96) lists the first 31 values of \( \phi_n(m)/m \) for \( 1 \leq n \leq 5 \). It seems that, for all positive integers \( n \) and \( m \), we have

\[
\phi_n(2m+1)/(2m+1) = 2^{m-n} \quad \text{for} \quad n \leq m \leq 3n+1 ,
\]

and

\[
\phi_n(2m+1)/(2m+1) > 2^{m-n} \quad \text{for} \quad m > 3n + 1 .
\]

**REMARK 4.** For all positive integers \( k \) and \( m \), let \( \psi(k) = 2^k \) and \( \psi(m) = \Phi(m, \psi) \), where \( \Phi \) is defined as in Section 2. It is obvious that \( \psi(m)/m \) is the number of distinct periodic orbits of minimal period \( m \)

for, say, the mapping \( g(x) = 4x(1-x) \) from \([0, 1]\) onto itself. Since, for all positive integers \( k \) and \( n \) with \( 1 \leq k \leq 2n \), \( c_{2k,n} = 2^{k+1} - 1 \)

\( c_{1,n} = 1 \), we obtain that \( \phi_n(2k+2)/(2k+2) = \psi(k+1)/(k+1) \) for all
<table>
<thead>
<tr>
<th>$m$</th>
<th>$\Phi_1(m)/m$</th>
<th>$\Phi_2(m)/m$</th>
<th>$\Phi_3(m)/m$</th>
<th>$\Phi_4(m)/m$</th>
<th>$\Phi_5(m)/m$</th>
<th>$\Psi(m)/m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>56</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>99</td>
</tr>
<tr>
<td>11</td>
<td>18</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>186</td>
</tr>
<tr>
<td>12</td>
<td>25</td>
<td>11</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>335</td>
</tr>
<tr>
<td>13</td>
<td>40</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>630</td>
</tr>
<tr>
<td>14</td>
<td>58</td>
<td>23</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>1161</td>
</tr>
<tr>
<td>15</td>
<td>90</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>4</td>
<td>2182</td>
</tr>
<tr>
<td>16</td>
<td>135</td>
<td>46</td>
<td>32</td>
<td>30</td>
<td>30</td>
<td>4080</td>
</tr>
<tr>
<td>17</td>
<td>210</td>
<td>66</td>
<td>32</td>
<td>16</td>
<td>8</td>
<td>7710</td>
</tr>
<tr>
<td>18</td>
<td>316</td>
<td>94</td>
<td>61</td>
<td>56</td>
<td>56</td>
<td>14560</td>
</tr>
<tr>
<td>19</td>
<td>492</td>
<td>136</td>
<td>64</td>
<td>32</td>
<td>16</td>
<td>27594</td>
</tr>
<tr>
<td>20</td>
<td>750</td>
<td>195</td>
<td>115</td>
<td>101</td>
<td>99</td>
<td>52377</td>
</tr>
<tr>
<td>21</td>
<td>1164</td>
<td>282</td>
<td>128</td>
<td>64</td>
<td>32</td>
<td>99858</td>
</tr>
<tr>
<td>22</td>
<td>1791</td>
<td>408</td>
<td>224</td>
<td>191</td>
<td>186</td>
<td>190557</td>
</tr>
<tr>
<td>23</td>
<td>2786</td>
<td>592</td>
<td>258</td>
<td>128</td>
<td>64</td>
<td>364722</td>
</tr>
<tr>
<td>24</td>
<td>4305</td>
<td>856</td>
<td>431</td>
<td>351</td>
<td>337</td>
<td>696870</td>
</tr>
<tr>
<td>25</td>
<td>6710</td>
<td>1248</td>
<td>520</td>
<td>256</td>
<td>128</td>
<td>1342176</td>
</tr>
<tr>
<td>26</td>
<td>10420</td>
<td>1814</td>
<td>850</td>
<td>668</td>
<td>635</td>
<td>2580795</td>
</tr>
<tr>
<td>27</td>
<td>16264</td>
<td>2646</td>
<td>1050</td>
<td>512</td>
<td>256</td>
<td>4971008</td>
</tr>
<tr>
<td>28</td>
<td>25350</td>
<td>3858</td>
<td>1673</td>
<td>1257</td>
<td>1177</td>
<td>9586395</td>
</tr>
<tr>
<td>29</td>
<td>39650</td>
<td>5644</td>
<td>2128</td>
<td>1026</td>
<td>512</td>
<td>18512790</td>
</tr>
<tr>
<td>30</td>
<td>61967</td>
<td>8246</td>
<td>3328</td>
<td>2402</td>
<td>2220</td>
<td>35790267</td>
</tr>
<tr>
<td>31</td>
<td>97108</td>
<td>12088</td>
<td>4320</td>
<td>2056</td>
<td>1024</td>
<td>69273666</td>
</tr>
</tbody>
</table>
1 \leq k \leq 2n. It seems that \( \frac{\phi_n(2k+2)}{(2k+2)} > \frac{\psi(k+1)}{(k+1)} \) for all \( k > 2n \). But note that

\[
\lim_{k \to \infty} \left( \log \left( \frac{\phi_n(2k+2)}{(2k+2)} \right) \right) / (2k+2)
= \log \lambda_n > \frac{1}{2} \log 2 = \frac{1}{2} \lim_{k \to \infty} \left( \log \left( \frac{\psi(k+1)}{(k+1)} \right) \right) / (k+1),
\]

where \( \lambda_n \) is the unique positive zero of the polynomial

\[ x^{2n+1} - 2x^{2n-1} - 1. \]

3. Symbolic representation for continuous piecewise linear functions

In this section we describe a method. This method was first introduced in [4], and then generalized in [5] to construct, for every positive integer \( n \), a continuous piecewise linear function from \([0, 1]\) into itself which has a periodic orbit of minimal period 3, but with the property that almost all (in the sense of Lebesgue) points of \([0, 1]\) are eventually periodic of minimal period \( n \) with the periodic orbit the same as the orbit of a fixed known period \( n \) point. The same method was also used in [6] to give a new proof of a result of Block et al. [1] on the topological entropy of interval maps. In this paper we will use this method to prove our main results.

Throughout this section, let \( g \) be a continuous piecewise linear function from the interval \([a, b]\) into itself. We call the set \( \{ (x_i, y_i) \mid i = 1, 2, \ldots, k \} \) a set of nodes for (the graph of) \( y = g(x) \) if the following three conditions hold:

1. \( k \geq 2 \),
2. \( x_1 = a \), \( x_k = b \), \( x_1 < x_2 < \ldots < x_k \), and
3. \( g \) is linear on \([x_i, x_{i+1}]\) for all \( 1 \leq i \leq k-1 \) and \( y_i = g(x_i) \) for all \( 1 \leq i \leq k \).

For any such set, we will use its \( y \)-coordinates \( y_1, y_2, \ldots, y_k \) to represent the graph of \( y = g(x) \) and call \( y_1 y_2 \ldots y_k \) (in that order) a (symbolic) representation for (the graph of) \( y = g(x) \). For
1 \leq i < j \leq k$, we will call \( y_i y_{i+1} \ldots y_j \) the representation for \( y = g(x) \) on \([x_i, x_j]\) obtained by restricting \( y_1 y_2 \ldots y_k \) to \([x_i, x_j]\). For convenience, we will also call every \( y_i \) in \( y_1 y_2 \ldots y_k \) a node. If \( y_i = y_{i+1} \) for some \( i \) (that is, \( g \) is constant on \([x_i, x_{i+1}]\)), we will simply write \( y_1 \ldots y_i y_{i+1} y_{i+2} \ldots y_k \) instead of \( y_1 \ldots y_i y_{i+1} y_{i+2} \ldots y_k \). Therefore, every two consecutive nodes in a (symbolic) representation are distinct.

Now assume that \( \{(x_i, y_i) \mid i = 1, 2, \ldots, k\} \) is a set of nodes for \( y = g(x) \) and \( a_1 a_2 \ldots a_r \) is a representation for \( y = g(x) \) with \( \{a_1, a_2, \ldots, a_r\} \subset \{y_1, y_2, \ldots, y_k\} \) and \( a_i \neq a_{i+1} \) for all \( 1 \leq i \leq r-1 \). If \( \{y_1, y_2, \ldots, y_k\} \subset \{x_1, x_2, \ldots, x_k\} \), then there is an easy way to obtain a representation for \( y = g^2(x) \) from the one \( a_1 a_2 \ldots a_r \) for \( y = g(x) \). The procedure is as follows. First, for any two distinct real numbers \( u \) and \( v \), let \([u : v]\) denote the closed interval with endpoints \( u \) and \( v \). Then let \( b_1, b_2, \ldots, b_{i-1}, t_i \) be the representation for \( y = g(x) \) on \([a_i : a_{i+1}]\) which is obtained by restricting \( a_1 a_2 \ldots a_r \) to \([a_i : a_{i+1}]\). We use the following notation to indicate this fact: \( a_i a_{i+1} b_{i-1} t_i b_{i+1} \) (under \( g \)) if \( a_i < a_{i+1} \), or \( a_i a_{i+1} t_i b_{i-1} b_{i+1} \) (under \( g \)) if \( a_i > a_{i+1} \).

The above representation on \([a_i : a_{i+1}]\) exists since \( \{a_1, a_2, \ldots, a_r\} \subset \{x_1, x_2, \ldots, x_k\} \). Finally, if \( a_i < a_{i+1} \), let \( z_{i,j} = b_{j-1} \) for all \( 1 \leq j \leq t_i \). If \( a_i > a_{i+1} \), let \( z_{i,j} = b_{j-1} t_{i+1} \) for all \( 1 \leq j \leq t_i \). It is easy to see that \( z_{i,t_i} = z_{i+1,1} \) for all \( 1 \leq i \leq r-1 \). So, if we define

\[
Z = z_{1,1} \ldots z_{1,t_1} z_{2,2} \ldots z_{2,t_2} \ldots z_{r,2} \ldots z_{r,t_r},
\]

then it is obvious that \( Z \) is a representation for \( y = g^2(x) \). It is
also obvious that the above procedure can be applied to the representation $Z$ for $y = g^2(x)$ to obtain one for $y = g^3(x)$, and so on.

4. Proof of Theorem 1

In this section we let $f(x)$ denote the map as defined in Theorem 1, that is $f(x) = -2x + 5$ if $1 \leq x \leq 2$, and $f(x) = x - 1$ if $2 \leq x \leq 3$. The proof of part (a) of Theorem 1 will follow from two easy lemmas.

**LEMMA 4.** Under $f$, we have

\[ 13 \rightarrow 312, \quad 31 \rightarrow 213, \quad 12 \rightarrow 31, \quad 21 \rightarrow 13. \]

In the following when we say the representation for $y = f^k(x)$, we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 4 to the representation $312$ for $y = f(x)$ successively until we get to the one for $y = f^k(x)$.

For every positive integer $k$, let $u_{1,k}$ ($u_{2,k}$ respectively) denote the number of $13$'s and $31$'s in the representation for $y = f^k(x)$ whose corresponding $x$-coordinates are $\leq$ ($\geq$ respectively) $2$. We also let $v_{1,k}$ ($v_{2,k}$ respectively) denote the number of $12$'s and $21$'s in the representation for $y = f^k(x)$ whose corresponding $x$-coordinates are $\leq$ ($\geq$ respectively) $2$. It is clear that $u_{1,1} = v_{2,1} = 1$ and $u_{2,1} = v_{1,1} = 0$. Now from Lemma 4, we have

**LEMMA 5.** For every positive integer $k$ and integers $i = 1, 2$

\[ u_{i,k+1} = u_{i,k} + v_{i,k} \quad \text{and} \quad v_{i,k+1} = u_{i,k}. \]

Furthermore, if

\[ w_k = u_{1,k} + v_{1,k} + u_{2,k}, \]

then $w_1 = 1$, $w_2 = 3$, and $w_{k+2} = w_{k+1} + w_k$. That is, $\{w_k\}$ is the Lucas sequence.

Since, for every positive integer $k$, the number of distinct solutions of the equation $f^k(x) = x$ equals $w_k$, part (a) of Theorem 1 follows from Lemma 5. Part (b) follows from the standard inclusion-exclusion argument. As for part (c), we note that, for every positive
integer $k$,
\[ a_{k+2} = \sum_{i=1}^{k} a_i + 3. \]

So, for $k \geq 6$,
\[
(k+2)\phi_1(k+3) > (k+2)(a_{k+2} - a_{\lfloor (k+3)/2 \rfloor + 1})
> (k+3)(a_{k+2} - a_{\lfloor (k+3)/2 \rfloor + 1})
> (k+3)\phi_1(k+2),
\]
where \(\lfloor (k+3)/2 \rfloor\) is the largest integer less than or equal to \((k+3)/2\).
The proof of the other statement of part (a) is easy and omitted. This completes the proof of Theorem 1.

5. Proof of Theorem 2

In this section we fix any integer $n > 1$ and let $f_n(x)$ denote the map as defined in Theorem 2. For convenience, we also let $S_n$ denote the set of all these \(\lfloor n \rfloor\) symbolic pairs: \((i+1)i, (i+1)i, 1 \leq i \leq n-1; n(n+2), (n+2)n, (n+1)(2n+1), (2n+1)(n+1), j(j+1), (j+1)j, n+2 \leq j \leq 2n\).

The following lemma is easy.

**Lemma 6.** Under $f_n$, we have
\[
n(n+2) \to (n+3)(n+2)n, (n+2)n \to (n+2)(n+2)(n+3),
(n+1)(2n+1) \to (n+2)n(n-1)(n-2) \ldots 321,
(2n+1)(n+1) \to 123 \ldots (n-2)(n-1)n(n+2),
\]
and $uv \to f_n(u)f_n(v)$ for every $uv$ in
\[ S_n = \{ n(n+2), (n+2)n, (n+1)(2n+1), (2n+1)(n+1) \}. \]

In the following when we say the representation for $y = f_n^k(x)$, we mean the representation obtained, following the procedure as described in Section 3, by applying Lemma 6 to the representation
\[
(n+1)(2n+1)(2n)(2n-1) \ldots (n+2)n(n-1)(n-2) \ldots 321
\]
for $y = f^k_n(x)$ successively until we get to the one for $y = f^k_n(x)$.

For every positive integer $k$ and all integers $i, j$ with $1 \leq i, j \leq 2n$, let $b_{k,i,j,n}$ denote the number of $uv$'s and $vu$'s in the representation for $y = f^k_n(x)$ whose corresponding $x$-coordinates are in $[i, i+1]$, where $uv = j(j+1)$ if $1 \leq j \leq n-1$ or $n+2 \leq j \leq 2n$, $uv = n(n+2)$ if $j = n$, and $uv = (n+1)(2n+1)$ if $j = n+1$. It is obvious that $b_{1,1,n+1,n} = 1$, $b_{1,i,2n+2-i,n} = 1$ if $2 \leq i \leq n$, $b_{1,i,2n+1-i,n} = 1$ if $n+1 \leq i \leq 2n$, and $b_{1,i,j,n} = 0$ elsewhere. From Lemma 6, we see that the sequences $\{b_{k,i,j,n}\}$ are exactly the same as those defined in Section 2.

Since

$$c_{k,n} = \sum_{i=1}^{2n} b_{k,i,i,n} + b_{k,n+1,n,n} + \sum_{i=n+2}^{2n} b_{k,i,n+1,n},$$

it is clear that $c_{k,n}$ is the number of intersection points of the graph of $y = f^k_n(x)$ with the diagonal $y = x$. This proves part (a) of Theorem 2. Part (b) follows from the standard inclusion-exclusion argument. As for part (c), we note that there exist $2n+1$ nonzero constants $a_j$'s such that

$$b_{k,1,n,n} = \sum_{j=1}^{2n+1} a_j x^k_j$$

for all positive integers $k$, where $\{x_j \mid 1 \leq j \leq 2n+1\}$ is the set of all zeros (including complex zeros) of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$. Since $c_{k+2n-2,n}$ can also be expressed as

$$b_{k+2n-2,n+1,n,n} + 2nb_{k,1,n,n} + \sum_{i=2}^{n} (2^i-2)b_{k,1,n+1-i,n},$$

part (c) follows from property (i) of the sequences $\{b_{k,i,j,n}\}$ stated in Section 2. This completes the proof of Theorem 2.
References


Institute of Mathematics,
Academia Sinica,
Nankang,
Taipei,
Taiwan 115,
Republic of China.