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PREMIUM RATES UNDER INFLATIONARY CONDITIONS

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1. INTRODUCTION

This short note has as its starting point an interesting article by TAYLOR (1979) in which he considered the effects of inflation on a risk process. Taylor showed that if the premium density increased at the same rate as the cost of individual claims then, under certain conditions, ultimate ruin was certain. This raises a natural question, viz. "If the cost of individual claims is increasing how should the premiums be increased in order to keep the probability of ruin under control?" It is this question that we shall be considering in this note.

In the next section we define the risk process that we shall be studying for the remainder of this note. Our process is essentially a compound Poisson process except that we allow the distribution function of an individual claim to depend on the time at which the claim occurs. We start the third section by deriving, with the help of a general result of GERBER (1973), a formula for the future premium density that will keep the probability of ruin for our process below a predetermined bound. We then derive a simple approximation to this formula that shows more clearly how we require the premium density to change in relation to the change in claims costs. Finally we show that if we consider the same process with annual premiums then the probability of ultimate ruin will be kept below a predetermined bound if the annual premium is calculated according to the principle of zero utility with an exponential utility function or, as a first approximation, according to the variance principle.

2. THE RISK PROCESS

In this section we describe the risk process that we shall be studying in this note.

We assume claims are independent of each other and occur as the points of a Poisson process with a mean rate of p claims each year. The amount of a single claim occurring at time t years has distribution function F_t where

$$(1) \qquad F_t(x) = F_0(x/i(t)) \qquad t \geqslant 0$$

and where $i(t)$ (> 0) is a non-stochastic index of claims inflation at time t and

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$i(0) = 1$. If we denote the moment generating function of F_t by m_t it is clear that

(2) $$m_t(\theta) = m_0(\theta i(t))$$

We shall assume that $m_0(\theta)$ is finite for all values of θ but note the remarks following the proof of the theorem in the next section.

We denote by X_t the accumulated claims in the interval $[0, t]$. The moment generating function of X_t can be shown to be

(3) $$M_t(\theta) = \exp\{pt\,[1/t \int_0^t m_s(\theta)ds - 1]\}$$

For a derivation of this formula see either BÜHLMANN (1970, p. 60) or TAYLOR (1979, p. 153). For $n = 1, 2, 3 \ldots$ we define

(4) $$Y_n = X_n - X_{n-1}$$

so that Y_n is the total claims in the n-th year. Using (3) it is easy to show that

(5) $$E[Y_n] = pm \int_{n-1}^{n} i(s)\,ds$$

(6) $$Var[Y_n] = p\alpha \int_{n-1}^{n} i^2(s)\,ds$$

where m and α are the first and second moments of F_0 about the origin respectively.

We denote by U the insurer's free reserves at $t = 0$. We make no specific allowance for investment income to be added to the insurer's reserves but following the remarks on p. 161 of TAYLOR (1979) we can regard $i(t)$ as $i_1(t)/\,i_2(t)$ where $i_1(t)$ is a true index of claims inflation at time t and $i_2(t)$ is the accumulated amount at time t of a unit sum invested at time 0.

TAYLOR (1979) has shown that if the total premium income in $[0, t]$ is

(7) $$C_t = c \int_0^t i(s)\,ds$$

for some constant c then ultimate ruin is certain for our risk process provided only that there exists some constant k such that

(8) $$F_0(c/k) < 1 \text{ and } \int_0^t i(s)\,ds \leqslant ki(t) \text{ for all } t \geqslant 0.$$

i.e. provided only that sufficiently large claims are possible and that the rate of inflation is large enough. The above conditions on F_0 and $i(s)$ are not necessary for the results of the next section.

3. PREMIUM RATES

We denote by c_t the insurer's instantaneous rate of premium income at time t. We start this section by showing how to determine c_t in such a way that the insurer's probability of ultimate ruin can be kept below a predetermined bound. We do this in the following theorem.

Theorem

The insurer's probability of ultimate ruin will be bounded above by $\exp\{-RU\}$ if c_t is chosen so that

(9) $$c_t = p[m_t(R) - 1]/R \qquad \text{for } t \geqslant 0$$

where R is any positive number.

Proof

We define the process $\{Z_t\}_{t \geqslant 0}$ by

(10) $$Z_t = \int_0^t c_s ds - X_t$$

(where c_s is as defined in (9)) so that the insurer's reserves at time t are $U + Z_t$. This process has independent increments so we can use a result of GERBER (1973) which states that the probability of ultimate ruin for such a process is bounded above by

(11) $$\min_r \exp\{-rU\} \max_{0 \leqslant t} \quad E[\exp\{-rZ_t\}]$$

But, using (3), we have for any r

(12) $$E[\exp\{-rZ_t\}] = \exp\{-r \int_0^t c_s\, ds\} M_t(r)$$

$$= \exp\{-r \int_0^t c_s\, ds + pt\,[1/t \int_0^t m_s(r)\, ds - 1]\}$$

$$= \exp\{- \int_0^t [p + rc_s - pm_s(r)]\, ds\}$$

By putting $r = R$ in (12) and then using (9) we can see that

(13) $$E[\exp\{-RZ_t\}] = 1 \qquad \text{for all } t \geqslant 0.$$

Our theorem is then a simple consequence of Gerber's result.

Remarks

1. In the special case $i(s) = 1$ for all s (i.e. for a standard compound Poisson risk process) the above theorem is nothing more than Lundberg's inequality

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for the probability of ruin since it is clear from (9) that R is the insurer's insolvency constant. What we have done is to extend this result to the case where the cost of a claim depends on the time at which it occurs and we have achieved this by requiring c_t to be calculated in such a way that the insurer's "instantaneous insolvency constant" at time t is held fixed at some value $R > 0$.

2. An alternative interpretation of our result is that we have chosen c_t in such a way that the process $\{\exp(-RZ_t)\}_{t \geqslant 0}$ is a martingale. See GERBER (1975).
3. The assumption that $m_0(\theta)$ is finite for all θ implies that $m_t(R)$ and hence c_t will be finite for any values of R and t. Suppose that only the weaker condition

(14) $$m_t(\theta) < \infty \qquad \text{for all } 0 \leqslant \theta \leqslant \theta_0 \text{ and } 0 \leqslant t \leqslant t_0$$

holds where $\theta_0, t_0 > 0$. We can then show that the probability of ruin before time t_0 is bounded above by $\exp\{-RU\}$ provided $R \leqslant \theta_0$ and provided c_t is calculated as in (9) for $0 \leqslant t \leqslant t_0$. The proof is as before except that it requires the finite-time version of Geıber's result. See p. 207 of GERBER (1973).

Formula (9) gives little indication of the way in which we require c_t to change relative to $i(t)$. We try to provide this, at least for small values of t, in the following corollary.

Corollary 1

The rate of premium income c_t specified by (9) gives the following approximation for small values of t:

(15) $$c_t \doteqdot c_0 i(t)\,[1+\lambda i(t)]/[1+\lambda]$$

where $\lambda = R\alpha/2m$.

Proof

Formula (9) gives

(16) $$c_t = c_0[m_t(R)-1]/[m_0(R)-1]$$

and we have

(17) $$m_t(R)-1 = Ri(t)m + \alpha(Ri(t))^2/2 + \sum_{j=3}^{\infty} (Ri(t))^j \alpha_j / j!$$

where α_j is the j-th moment of F_0 about the origin. By assumption, (17) is a convergent series. In practice R is likely to be small so that if $i(t)$ is not large the first two terms on the right hand side of (18) should give a reasonable approximation to $[m_t(R)-1]$. Making a similar approximation to $[m_0(R)-1]$ and putting these two approximations into (16) we obtain (15).

Remarks

1. It is interesting to compare (15) with (7).
2. It can be easily checked that if $i(t) \geqslant 1$ and $m, \alpha, \alpha_j \geqslant 0$ for $j \geqslant 3$ then the "$\doteqdot$" sign in (15) can be changed to a "$\geqslant$" sign.
3. The range of values of t for which (15) is likely to be a reasonable approximation is not immediately clear since it depends on the relationships between R, $i(t)$ and F_0. However, in the special case where F_0 is a negative exponential distribution we can get a clearer idea of the accuracy of (15). Let us suppose then that $F_0(x) = 1 - \exp\{-ax\}$ for some $a > 0$ so that $m = 1/a$ and $\alpha = 2/a^2$. Formula (9) gives

(18) $$c_t = pi(t)/[a - Ri(t)] \qquad \text{provided } a > Ri(t).$$

This gives the exact relationship

(19) $$c_t = c_0 i(t)[1 - R/a]/[1 - Ri(t)/a]$$

Formula (15) gives the following approximation

(20) $$c_t \doteqdot c_0 i(t)[1 + Ri(t)/a]/[1 + R/a]$$

So far in this section we have been concerned with c_t, the instantaneous rate of premium income, which we have assumed to be continuously variable. We now suppose that in the time interval $[n-1, n]$, where n is a positive integer, a total premium P_n is payable at a constant rate throughout the year.

Corollary 2

Assuming either that $i(t)$ is a non-decreasing function of t or that "ruin" can only occur after an integral number of years, the probability of ultimate ruin for our risk process will be bounded above by $\exp\{-RU\}$ if P_n is calculated by the formula

(21) $$P_n = p\left[\int_{n-1}^{n} m_t(R)\, dt - 1\right]/R \qquad n = 1, 2, \ldots$$

This formula for P_n gives the following approximation for small values of n

(22) $$P_n \doteqdot E[Y_n] + (R/2)\operatorname{Var}[Y_n]$$

Proof

The first part of the corollary is easily proved since (21) can be written

(23) $$P_n = \int_{n-1}^{n} c_t\, dt$$

where c_t is as in (9). Using similar approximations to those used in the proof of

Corollary 1 we have that

(24) $$P_n \doteqdot p \int_{n-1}^{n} [mi(t) + \alpha R i^2 (t)/2] \, dt$$

and so (22) follows from (5) and (6).

Remarks

1. If $i(t) \geqslant 1$ and $\alpha_j \geqslant 0$ for $j \geqslant 3$ the "$\doteqdot$" sign in (22) can be replaced by a "$\geqslant$" sign.
2. Formula (21) is equivalent to

(25) $$P_n = (1/R) \ln [E[\exp \{RY_n\}]]$$

In other words to keep the probability of ruin below $\exp\{-RU\}$ the annual premium should be calculated using the principle of zero utility with the utility function

(26) $$u(x) = (1/R)(1 - \exp\{-Rx\})$$

See Gerber (1974).

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