1
Introduction

The theory of strong interactions,
now that is quite something.

Elementary particles we know are *leptons* e and μ and their corresponding *neutrinos*, ν_e, ν_μ, a photon (γ) and a graviton, and then, hundreds of strongly interacting particles – *hadrons*: proton p and neutron n, pions $\pi^\pm$ and π^0, kaons $K^\pm$, K^0, $\bar{K}^0$, etc., etc.

1.1 Interaction radius and interaction strength

Electromagnetic interaction has two characteristic features. Firstly, it is characterized by a small coupling,

$$\frac{e^2}{4\pi\hbar c} \simeq \frac{1}{137}.$$

Secondly, it is a long range force,

$$V = \frac{e_1 e_2}{r},$$

so that there is no typical distance, no characteristic interaction radius.

Gravitation behaves (at large distances!) similarly to the electromagnetic interaction,

$$V \; = \; G_{\rm gr} \frac{m_1 m_2}{r};$$

thus it has no radius either. To characterize the magnitude of the interaction one needs to construct a dimensionless parameter. Contrary to the case of the electromagnetic charge, mass is not quantized, so that there is

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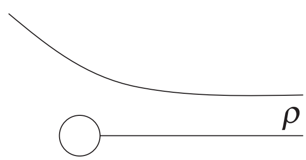
no 'unit mass' to choose. Hence one usually takes the mass of the proton, m_p, to quantify the typical interaction strength:

$$G_{\mathrm{gr}} \frac{m_p^2}{\hbar c} \simeq 7 \cdot 10^{-39}.$$

An important difference with electromagnetism is that here all the 'charges' have the same sign (mass is positive). Therefore the gravitation prevails over the electromagnetic interactions in the macro-world. Moreover, the gravitational interaction grows with energy, making the gravity essential at extremely small distances. This happens solely owing to the existence of the Planck constant $\hbar$, since by confining a system to small distances, Δr, we supply it with a large energy $\Delta E \sim \hbar c (\Delta r)^{-1}$.

Leptons, photons, graviton do not participate in strong interactions.



Rutherford was the first to observe (electromagnetic) scattering of strongly interacting particles. By comparing the scattering pattern of α-particles at large angles with classical formulae he concluded that the size of the gold nucleus was about 10^{-13} cm. By the way, to be able to describe the process as classical particle scattering, all the way down to $\rho \sim 10^{-13}$ cm, one has to have

$$kr_0 \gg 1.$$

However, the energies of α-particles in the Rutherford experiment, $E/m_\alpha = \mathcal{O}(\mathrm{keV}/\mathrm{GeV}) = 10^{-6}$, correspond to momenta k such that

$$k \cdot r_0 \;=\; \sqrt{2m_\alpha \cdot E} \cdot \frac{1}{\mu} \;\ll\; 1, \quad \text{with} \quad (r_0)^{-1} \sim \mu = 140 \text{ MeV},$$

so that the scattering becomes *quantum*, rather than classical, already for the impact parameters much larger than r_0. It was fortunate for Rutherford that the scattering cross section in the Coulomb field happened to be identical to that in the classical theory!

The proton–proton cross section is very small, $\sigma_{pp} \sim 4 \cdot 10^{-26}$ cm^2. Why then do we refer to the 'strong interaction' as *strong*? To really evaluate the strength of interaction, one has to take into consideration the existence of the finite *interaction radius* since the interaction cross section is composed of the actual interaction strength and of the probability to hit the target, measured by the transverse area of the hadron $\sim \pi r_0^2$. This being said, if the interaction cross section turns out to be of the order of the geometric cross section,

$$\sigma \sim r_0^2\,,$$

we call the interaction *strong*; otherwise, if

$$\sigma \ll r_0^2\,,$$

such an interaction we consider as *weak*.

How can one determine experimentally the interaction radius r_0?

In the *classical theory* it is straightforward: from the angular dependence of the scattering cross section $d\sigma(\mathbf{q})$ one can reconstruct the potential, and, subsequently, extract the characteristic radius r_0.

In *quantum mechanics* we operate with the partial wave expansion of the scattering amplitude,

$$f(k,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta).$$

Guided by quasi-classical considerations, we can define the interaction radius by comparing the magnitudes of the partial wave amplitudes f_ℓ with different orbital momenta ℓ:

$$f_\ell \sim \begin{cases} 1 & \text{for} \quad \ell \lesssim kr_0, \\ 0 & \qquad\;\; \ell \gg kr_0. \end{cases}$$

Assume that the interaction radius is small, $kr_0 \ll 1$. Then, due to the fact that the partial waves with large orbital momenta are suppressed, $f_\ell \propto (kr_0)^{2\ell+1}$ (centrifugal barrier), the S-wave dominates,

$$f(k,\theta) \simeq f_0(k),$$

and the scattering pattern is spherically symmetric. Increasing the incident momentum we reach $kr_0 \sim 1$ where a few partial waves will start contributing and the corresponding Legendre polynomials with $\ell \neq 0$ will introduce angular dependence into the scattering distribution. Thus we can determine r_0 by studying at what energies the scattering ceases to be spherically symmetric. Alternatively, at large k, we can extract the interaction radius by measuring the characteristic scattering angle, $\theta_{\rm char} \sim (kr_0)^{-1} \ll 1$.

Now that we know how to measure r_0 and may compare σ with r_0^2, let us ask ourselves another question: whether the situation when $\sigma \ll r_0^2$ really means that we are dealing with a *weak* interaction.

The answer is, yes and no!

Consider the scattering of a point-like neutrino off a proton, for example, the process $\nu_\mu + p \to \mu + X$. By examining the momentum dependence of the cross section we will extract that very same proton radius

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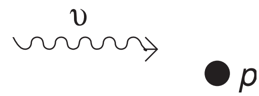

$r_0 \sim 10^{-13}$ cm. At the same time, the interaction cross section is of the order of $\sigma_\nu \sim 10^{-40}$ cm^2. Then, according to our logic we must proclaim the neutrino a weakly interacting particle.

However, imagine that the proton has a tiny core, of the size 10^{-20}, which is smeared over the area of the radius $r_0 = 10^{-13}$. If so, the interaction of the neutrino with the proton actually turns out to be strong: $(10^{-20})^2 \sim \sigma_\nu$. We can only state that ν interacts weakly at the distances *larger* than 10^{-20} cm.

The most important property of the *weak interaction* is its *universality* with respect to hadrons and leptons. They get engaged in the weak interaction in a similar manner and with the same universal *Fermi constant*

$$G_F \simeq \frac{10^{-5}}{m_p^2}, \qquad {m_p}^{-1} \sim 10^{-14}\,\text{cm}.$$

Weak interaction increases with energy. At distances $10^{-3}/m_p \sim 10^{-17}$ cm, corresponding to collision energies of the order of $10^3 m_p \simeq 1000$ GeV, the weak interactions may become *strong*.

The main features of *strong interactions* of hadrons are the following:

(1) probability to interact is $\mathcal{O}(1)$ at the distances $r \lesssim r_0 = 10^{-13}$ cm;

(2) hadrons are intrinsically relativistic objects.

Indeed, to investigate the distances $r_0 = 1/\mu$, momenta $k \sim \mu$ are necessary, which correspond to the proton velocity $v \simeq \mu/m_p \sim 1/6$. (By the way, it is this $1/6$, treated as a small parameter, to which the nuclear physics owes its existence.) At the same time, if we substitute for the proton a π-meson (whose mass is $m_\pi = \mu$) we get $v \simeq 1$ and the very possibility of a non-relativistic approach disappears.

1.2 Symmetries of strong interactions

Imagine that we have an unstable particle whose decay time τ is much larger than $r_0/c \sim 10^{-23}$ s. Does it decay due to the strong or weak interaction? The answer lies in the symmetry of the decay process: the *degeneracy* is much larger in the strong interaction; degeneracy means symmetry, and symmetries, as you know, give rise to conservation laws.

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Electric charge Q. The hadrons have to know themselves about the electromagnetic interaction. Each hadron has a definite electric charge, and the strong interactions must respect its conservation, otherwise quantum electrodynamics would be broken.

Baryon charge B. This is another quantum number whose conservation is verified with a fantastic accuracy (stability of the Universe). The baryon charge equals $+1$ for *baryons* like p, n, Λ, Σ, Ξ, $\ldots$ (and -1 for their antiparticles), and 0 for *mesons* (π, K, ρ, ω, φ, $\ldots$).

Isotopic spin I. Phenomenologically, hadrons split in groups of particles with close masses, and can be classified as belonging to *isotopic $SU(2)$* multiplets. For example, the doublet of the proton and the neutron, p, n ($I = \frac{1}{2}$); the triplet of pions, $\pi^{\pm}$ and π^0 ($I = 1$), etc. The relative mass difference of hadrons in one multiplet is 10^{-2}–10^{-3}, that is, of the order of the electromagnetic 'fine-structure constant':

$$\frac{m_n - m_p}{m_p} \sim \frac{m_{\pi^0} - m_{\pi^+}}{m_{\pi^+}} \sim \alpha \simeq \frac{1}{137}.$$

It looks that if we switched off the electromagnetic interaction, we would arrive at a complete degeneracy in the mass spectrum of strongly interacting particles. Independently of the hypothesis about the nature of this tiny mass splitting, these states can be treated as degenerate in the first approximation and therefore, there must be a symmetry and the corresponding conservation law.

Are the pn and pp scattering cross sections the same, if electromagnetic interactions are switched off? No – in the second case the particles are identical. In order to distinguish p from n, a new quantum number is introduced: the proton is treated as a *nucleon* with the isospin projection $I_3 = +\frac{1}{2}$, and the neutron with $I_3 = -\frac{1}{2}$. Thus, the nucleon wave function depends on coordinates, spin and *isospin* variables, $\psi(\mathbf{r}, \sigma, \tau)$. In strong interactions isospin is conserved.

For example, the lightest stable nuclei – the deuteron and the helium – consist of equal number of protons and neutrons, $D = (pn)$, $\mathrm{He}^4 = (2p2n)$, and both have $I = 0$ (isotopic singlets). Therefore, the fusion reaction

$$D + D \not\to \mathrm{He}^4 + \pi^0$$

is forbidden, since the pion has isospin $I = 1$.

Strangeness S. Any reaction takes place that is allowed by conservation laws. At the same time, it was observed that long-living hadrons like K-mesons, and Λ- and Σ-baryons, cannot be produced *alone* in the interactions of nucleons and pions. They always go *in pairs*, e.g.

$$\pi^- + p \to \Lambda + \bar{K}^0 ,$$

while the reactions

$$\pi^- + p \to n + K^0, \quad \text{or} \quad \pi^- + p \to \Lambda + \pi^0$$

are forbidden.

By prescribing to these hadrons a new quantum number – *strangeness* S,

$$S(K^-, K^0) = S(\Lambda) = -1, \qquad S(\bar{K}^0, K^+) = S(\bar{\Lambda}) = +1,$$

we get the relation between the conserved quantities:

$$Q = I_3 + \frac{B}{2} + \frac{S}{2}.$$

There is one more approximate symmetry which combines strange and non-strange hadrons into of SU(3) multiplets, like *octets* of pseudoscalar mesons,

$$\begin{array}{lc} S = 1 & (\bar{K}^0, K^+)_{I=\frac{1}{2}}, \\ S = 0 & (\pi^-, \pi^0, \pi^+)_{I=1}, \quad \eta_{I=0}, \\ S = -1 & (K^-, K^0)_{I=\frac{1}{2}}, \end{array}$$

and baryons,

$$\begin{array}{lc} S = 0 & (n, p)_{I=\frac{1}{2}}, \\ S = -1 & (\Sigma^-, \Sigma^0, \Sigma^+)_{I=1}, \quad \Lambda_{I=0}, \\ S = -2 & (\Xi^-, \Xi^0)_{I=\frac{1}{2}}, \end{array}$$

baryon *decuplet*, etc.

The isospin symmetry is broken by electromagnetic interactions. The weak interaction breaks *everything* except B, Q and, maybe, the lepton charge L. (Apparently, the electron and the muon lepton charges conserve separately, since $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, but $\mu^- \not\to e^- + \gamma$.)

The lightest strange particles are stable under strong interactions. However, K-mesons decay into pions, and the Λ-baryon into $\pi^- p$, due to the weak interaction, disrespecting the strangeness conservation. The weak forces violate spatial parity P, charge parity C, and even the time reflection symmetry T (the later equivalent to the 'combined parity' CP).

1.3 Basic properties of the strong interaction

1.3.1 Interaction radius

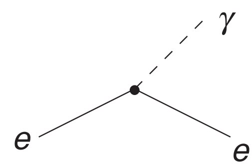
The question arises, what is r_0: is this an *interaction radius* specific for the strong interaction, or rather a real size of an object? This question can be answered using, for example, weak interactions as a short-range probe. It turns out that r_0 is the actual size of the proton that can be extracted, in particular, from the measurement of the spatial distribution of the electric charge inside the proton.

The hadron radius r_0 appears to be equal to the pion Compton wavelength,

$$r_0 \simeq m_\pi{}^{-1} \equiv \mu^{-1} \simeq 10^{-13}\,\text{cm}.$$

Is this coincidence an accident? In the past it was thought to be of fundamental importance; it is not so clear any more that it really is.

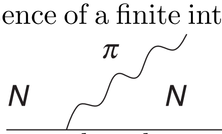
What is the problem with the description of the strong interactions? As we have discussed above, a non-relativistic description does not make sense here. We have just one example which may help us to construct a relativistic theory: electrodynamics. In the quantum electrodynamics, the electron e and the photon γ are point-like, and so is the interaction between them.



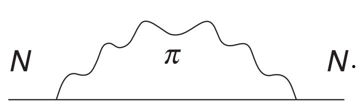
(1.1)

Now we want to describe hadrons: p, n, π. Are these particles point-like? The existence of the finite radius r_0 confirms, apparently, the opposite. There is no way, however, to give a *relativistic* description of a particle of finite radius. So we have to assume that the particles we consider are, in a sense, point-like.

Yukawa suggested that the point-likeness of a hadron does not contradict the existence of a finite interaction radius. Let us draw a pion–nucleon interaction



taking (1.1) for a model. The existence of this vertex means that there are processes of virtual emission and absorption of pions by the nucleon,



(1.2)

Let us imagine now that this happens quite frequently. What will we see as a result of a scattering of an external particle off such a fluctuating

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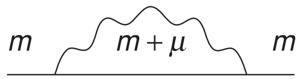
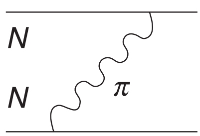
nucleon? Estimating the energy uncertainty as

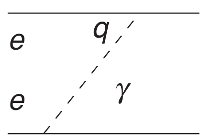


$$\Delta E \simeq (m+\mu) - m = \mu, \tag{1.3}$$

we conclude that the lifetime of the fluctuation is $\Delta t \sim (\Delta E)^{-1} \sim \mu^{-1}$. During this time interval, a pion (with a velocity $v \sim 1$) will cover the distance $\Delta r \sim \mu^{-1}$. Thus, our object, which was point-like in the beginning, is now spread over a distance μ^{-1}, and, in the process of scattering, it will interact with the projectile at impact parameters $\rho \sim \mu^{-1}$. In other words, the scattering of an incident particle with our nucleon can be depicted as a pion exchange between the two nucleons – the process that has a characteristic radius $r_0 \sim \mu^{-1}$!



Without any theory, let us first calculate this amplitude in a naive way, by analogy. What would be the difference between the above process and the scattering of electrons that we have studied in the quantum electrodynamics,



$$= \frac{e^2}{q^2}\big(\bar{u}\gamma^\mu u\big)\big(\bar{u}\gamma_\mu u\big). \tag{1.4}$$

We must replace the photon propagator $1/q^2$ in (1.4) by the Green function of the massive π meson:

$$D_\pi(q) = \frac{1}{\mu^2 - q^2}.$$

The corresponding scattering amplitude will have the form

$$A = \frac{g^2}{\mu^2 - q^2}, \tag{1.5}$$

with g the pion–nucleon interaction constant, replacing the electric charge e in the QED amplitude (1.4).

What does this amplitude correspond to in the case of the non-relativistic scattering? The non-relativistic scattering amplitude reads

$$f = -\frac{2m}{4\pi}\int \mathrm{e}^{i\mathbf{k}'\cdot\mathbf{r}} V(r)\,\psi(\mathbf{r})\,d^3r. \tag{1.6}$$

In the Born approximation, replacing the wave function $\psi(\mathbf{r})$ by a plane wave with momentum $\mathbf{k}$, we obtain

$$f_B = -\frac{2m}{4\pi}\int \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} V(r)\,d^3r, \tag{1.7}$$

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where $\mathbf{q}$ is the momentum transfer, $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. For non-relativistic particles, the kinetic energy, $E = \mathbf{k}^2/2m$, is small, and the *energy* transfer component can be neglected:

$$|q_0| \sim \mathbf{q}^2/m \ll |\mathbf{q}|\,, \qquad \text{so that} \quad q^2 = q_0^2 - \mathbf{q}^2 \simeq -\mathbf{q}^2.$$

The scattering amplitude (1.5) becomes

$$A \simeq \frac{g^2}{\mu^2 + \mathbf{q}^2}\,.$$

What is the potential corresponding to this amplitude? Evaluating the inverse Fourier transform of the Born amplitude f_B in (1.7) we obtain the Yukawa potential,

$$V(r) = -\frac{4\pi}{2m}\int \mathrm{e}^{-i\mathbf{qr}}\frac{g^2}{\mu^2+\mathbf{q}^2}\frac{d^3q}{(2\pi)^3} = \frac{g^2}{2m}\cdot\frac{\mathrm{e}^{-\mu r}}{r}. \tag{1.8}$$

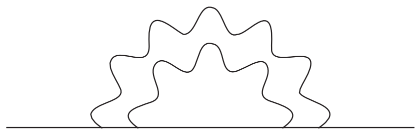
So, indeed, the effective interaction is characterized by a finite radius $r_0 = 1/\mu$.

We conclude that the assumption of the point-like nature of the interaction does not exclude the finiteness of the interaction radius. Moreover, having adopted the point of view that the hadron has no intrinsic size (having no other option), we see that the interaction radius is not an independent quantity but is determined by the *masses* of the particles.

From the point of view of a relativistic theory the π-meson has to exist in nature, otherwise there would be no explanation for such a 'large' value of the proton radius.

1.3.2 *Interaction strength*

The other side of the strong interactions is their *strength*: once the particles approach each other to the distance r_0, the interaction is inevitable. Since a nucleon is always surrounded by a pion cloud, see (1.2), this means that the coupling constant g^2 (if it exists at all) is obliged to be *large*, $g^2 \sim 1$, contrary to the electromagnetic interaction, characterized by the small coupling $\alpha = 1/137 \ll 1$. Now that is bad indeed, because under these circumstances anything will go. For example, a virtual state with two pions will be there, having a typical lifetime $\Delta t \sim \frac{1}{2}\mu$ and, correspondingly, a spatial spread of the order of $\sim \frac{1}{2} r_0$.



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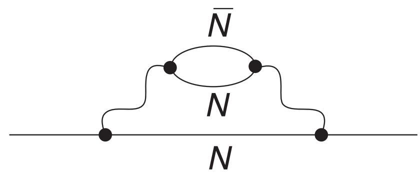
We may even have a three-nucleon state, $N \to N\pi$, $\pi \to N\bar{N}$. Since the nucleon is much heavier than the pion, this fluctuation is short-lived: $\Delta t \sim 1/(2m_N) \ll r_0$. However, we cannot state a priori that such a process does not contribute to the radius of the nucleon since these 'second-order' amplitudes may be actually *larger* than the one-pion emission amplitude (1.2), because the coupling constant is not small.



It is clear that it will be certainly impossible to build a theory like quantum electrodynamics to describe strongly interacting hadrons.

We can, however, introduce initial point-like objects, and then, in fact, observe 'clouds', the radii of which are determined by the masses of the hadrons.

This is the basic idea of the theory of the strong interaction.

We need to construct a framework which would allow us to draw *pictures* representing a formal series for the hadron interaction amplitudes. From these pictures we will extract information without actually calculating the amplitudes, which would be, a priori, impossible. The Feynman diagrams can be considered as a 'laboratory of theoretical physics'.

1.4 Free particles

We start by considering free particle states and their propagation. There is a fantastic variety of hadrons with spins reaching up to $s = \frac{19}{2}$.

1.4.1 Particle states

$s = 0$. A free spinless (scalar) particle with a four-momentum p_μ is described by the wave function

$$s = 0 : \qquad \psi(x) \;=\; \frac{1}{\sqrt{2p_0}}\,\mathrm{e}^{-ipx}. \tag{1.9}$$

$s = \frac{1}{2}$. A spin-one-half particle has two states, $\lambda = 1, 2$,

$$s = \tfrac{1}{2} : \qquad \psi_\alpha^\lambda(x) = \frac{u_\alpha^{(\lambda)}}{\sqrt{2p_0}}\,\mathrm{e}^{-ipx}; \tag{1.10}$$

two states with definite parity are selected out of possible four spinors u_α by the Dirac equation, $(\hat{p} - m)u^\lambda$.

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$s = 1$. A spin one particle is described by a wave function ψ_μ bearing the Lorentz vector index:

$$s = 1 \ : \qquad \psi_\mu^\lambda(x) = \frac{e_\mu^{(\lambda)}}{\sqrt{2p_0}}\, \mathrm{e}^{-ipx}. \tag{1.11}$$

Here one has to single out three states, $\lambda = 1, 2, 3$, out of the four unit vectors. In the rest frame, the vector particle has three polarizations,

$$e_\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}; \qquad e_\mu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_\mu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_\mu^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

all having zero time component, $e_0^{(\lambda)} = 0$. In Lorentz invariant terms, the superfluous state is eliminated by the condition $e_\mu^{(\lambda)} p^\mu = 0$.

$s = 2$. A tensor particle has to have $2s + 1 = 5$ physical states. Its wave function is constructed with the help of a Lorentz tensor $T_{\mu\nu}$,

$$s = 2 \ : \qquad \psi_{\mu\nu} = \frac{T_{\mu\nu}}{\sqrt{2p_0}}\, \mathrm{e}^{-ipx}.$$

This tensor can be simply constructed as a product of two *vector* states, $e_\mu^{\lambda_1}$ and $e_\nu^{\lambda_2}$. Since $p^\mu e_\mu^{\lambda_i} = 0$, in the rest frame we have

$$\begin{aligned} &T_{00} = T_{i0} = T_{0k} = 0, \\ &t_{ik} = e_i^{\lambda_1} e_k^{\lambda_2} + e_i^{\lambda_2} e_k^{\lambda_1}, \quad i, k = 1, 2, 3. \end{aligned} \tag{1.12}$$

The symmetric 3×3 tensor (1.12) has $3 \cdot 4/2 = 6$ independent components. Combining two spin 1 particles, we obtain not only a spin 2 state but also one with spin zero; to exclude the latter we must make the tensor T traceless,

$$T_{ik} = t_{ik} - \tfrac{1}{3}\delta_{ik} \sum_{\ell=1}^{3} t_{\ell\ell}.$$

Finally, we have a symmetric Lorentz tensor, $T_{\mu\nu} = T_{\nu\mu}$,

$$T_{\mu\nu} = e_\mu^{\lambda_1} e_\nu^{\lambda_2} + e_\mu^{\lambda_2} e_\nu^{\lambda_1} - \frac{2}{3}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)\left(e^{\lambda_1} e^{\lambda_2}\right),$$

which on the mass shell, $p_\mu^2 = m^2$, is traceless, $T_{\mu\nu} \cdot g^{\mu\nu} = 0$, and 'orthogonal' to the four-momentum of the particle, $p^\mu T_{\mu\nu} = 0$.

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$s = \frac{3}{2}$. The wave function of a spin $\frac{3}{2}$ state bears simultaneously a spinor index α, and the vector index μ.

$$s = \tfrac{3}{2}\ : \qquad \psi_{\alpha,\mu}(x) = \frac{u_{\alpha,\mu}}{\sqrt{2p_0}}\, \mathrm{e}^{-ipx}.$$

Initially, $u_{\alpha,\mu}$ has 16 degrees of freedom (4 spinors $\times$ 4 vector components). The first condition,

$$(\hat{p} - m)u_{\alpha,\mu} \equiv \sum_{\beta} \left(\hat{p} - m\right)_{\alpha,\beta} u_{\beta,\mu} = 0, \tag{1.13a}$$

selects two spinors ($16 \to 8$), and the second one,

$$\gamma^{\mu} u_{\alpha,\mu} = 0, \tag{1.13b}$$

(four equations) leaves us with $8 - 4 = 4 = 2 \cdot \frac{3}{2} + 1$ states. From the pair of conditions (1.13) it conveniently follows that

$$p^{\mu} u_{\alpha,\mu} = 0.$$

Indeed, from (1.13b) we get

$$\gamma^{\mu} u_{\alpha,\mu} = 0 \implies 0 = (\hat{p} + m)\gamma^{\mu} u_{\alpha,\mu} = \gamma^{\mu} \underbrace{(-\hat{p} + m)u_{\alpha,\mu}}_{=0} + 2p^{\mu} u_{\alpha,\mu}.$$

There exists a special technology how to move further to higher spins.

1.4.2 Particle propagators

Relativistic propagation of free particles is described by Green functions

$$G(x) = \int \frac{d^4p}{(2\pi)^4 i}\, \mathrm{e}^{-ipx} G(p).$$

In the momentum space, the Green functions of scalar, fermion and vector particles are

$$s = 0\ : \qquad G(p) = \frac{1}{m^2 - p^2}, \tag{1.14a}$$

$$s = \tfrac{1}{2}\ : \qquad G(p) = \frac{1}{m - \hat{p}} = \frac{1}{m^2 - p^2} \cdot (m + \hat{p}), \tag{1.14b}$$

$$s = 1\ : \qquad G_{\mu\nu}(p) = \frac{1}{m^2 - p^2} \cdot \left(\frac{p_{\mu} p_{\nu}}{m^2} - g_{\mu\nu} \right). \tag{1.14c}$$

The factors in the numerator that accompany the pole $1/(m^2 - p^2)$ in the propagators of particles with spin originate from a summation over physical polarization states:

$$\sum_{\lambda=1}^{2} u^{\lambda}(p)\ \bar{u}^{\lambda}(p) \;=\; m + \hat{p},$$
$$\sum_{\lambda=1}^{3} e^{\lambda}_{\mu}(p)\ e^{*\lambda}_{\nu}(p) \;=\; \frac{p_\mu p_\nu}{m^2} - g_{\mu\nu}.$$

Analogously, propagators of particles with higher spins will contain the structures

$$s = \tfrac{3}{2}\ : \qquad G_{\mu\nu}(p) \;\propto\; (p_\mu + m\gamma_\mu)(m - \hat{p})(p_\nu + m\gamma_\nu), \tag{1.14d}$$
$$s = 2\ : \qquad G_{\mu\nu,\mu'\nu'} \propto \left(g_{\mu\mu'} - \frac{p_\mu p_{\mu'}}{m^2}\right)\left(g_{\nu\nu'} - \frac{p_\nu p_{\nu'}}{m^2}\right) + \text{perm.} \tag{1.14e}$$

The numerators (1.14) are polynomials in p, and for large p values the Green functions are growing as $p^{2(s-1)}$. This growth is unavoidable and leads to a large number of unpleasant problems.

One might imagine a scenario with mass degeneracy, such that a scalar and a vector state together would be described by a propagator

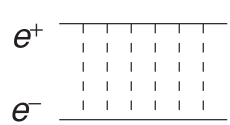
$$G_{1+0}(p) = \frac{-g_{\mu\nu}}{m^2 - p^2},$$

free of the increasing term $p_\mu p_\nu$ present in (1.14c). This, however, is not a solution, since such a scalar particle would be a *ghost* – a state with a negative transition probability, $-g_{00} = -1$. This would violate the rule according to which the amplitude near the pole has a definite sign, following from the unitarity.

Do we really need to ascribe a bare field and its own interaction to each of few hundred existing hadrons? Possibly, one can treat all these hadrons by expressing them by means of a few fundamental objects.

1.5 Hadrons as composite objects

In the language of field theory, we introduce some fundamental fields and describe them in terms of the wave function ψ. We construct the interaction Hamiltonian which may have *bound states*. The known example of appearance of such a composite particle in relativistic field theory is *positronium* – a bound state of an electron and a positron.



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The QED coupling being small, the binding energy of positronium is much smaller than the electron mass, $\epsilon \ll m$. Consequently, in the average e^+ and e^- are at a relatively large distance $\Delta r \gg m^{-1}$ apart. However, even here there is a possibility to produce additional pairs etc., so that the field-theoretical nature of the state is rather rich and complex.

1.5.1 Scattering of composite states

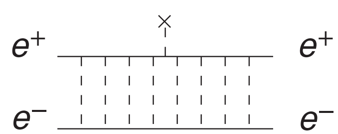
How to describe the scattering of a bound system – positronium – in an external field?

In quantum mechanics the scattering amplitude has a structure

$$f \sim \int \mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{r}_c}\psi_f(\mathbf{r}_{12})[V(\mathbf{r}_1)+V(\mathbf{r}_2)]\,\mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}_c}\psi_i(\mathbf{r}_{12}), \tag{1.15}$$

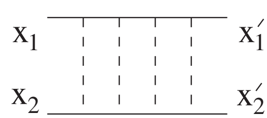
where $\mathbf{r}_c$ is the centre-of-mass coordinate, and $\psi(\mathbf{r}_{12})$ the relative motion wave function.

In terms of diagrams we have



(1.16)

The non-relativistic Green function of the e^+e^- system,

$G(\mathbf{r}'_c, \mathbf{r}'_{12}, t'; \mathbf{r}_c, \mathbf{r}_{12}, t) =$ 

,

$$\mathbf{r}_c = \tfrac{1}{2}(\mathbf{x}_1+\mathbf{x}_2), \qquad \mathbf{r}'_c = \tfrac{1}{2}(\mathbf{x}'_1+\mathbf{x}'_2), \tag{1.17a}$$

$$\mathbf{r}_{12} = \mathbf{x}_1-\mathbf{x}_2, \qquad \mathbf{r}'_{12} = \mathbf{x}'_1-\mathbf{x}'_2, \tag{1.17b}$$

can be expressed as a sum over eigenstates of the product of the final and (conjugate) initial wave functions:

$$G(\mathbf{r}'_c, \mathbf{r}'_{12}, t'; \mathbf{r}_c, \mathbf{r}_{12}, t) = \sum_n \psi_n(\mathbf{r}'_c, \mathbf{r}'_{12}, t')\,\psi_n^*(\mathbf{r}_c, \mathbf{r}_{12}, t),$$

where $t \equiv x_{10} = x_{20}$, $t' \equiv x'_{10} = x'_{20}$. Among these terms there is one which corresponds to the bound state D:

$$G = \sum_{\mathbf{p}_c} \psi_D(\mathbf{r}'_c, \mathbf{r}'_{12}, t')\,\psi_D^*(\mathbf{r}_c, \mathbf{r}_{12}, t) \;+\; [\text{continuous } e^+e^- \text{ spectrum}].$$

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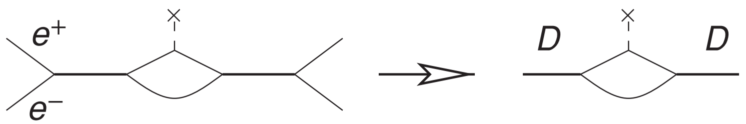
In the mixed space–energy representation, the stationary state Green function,

$$G(\mathbf{r}, \mathbf{r}'; E) \;=\; \sum_n \frac{\psi_n(\mathbf{r}')\, \psi_n^*(\mathbf{r})}{E_n - E},$$

contains the *pole* in energy, corresponding to the positronium state. Let us single out this pole from the sum:

$$G(\mathbf{r}, \mathbf{r}'; E) \quad \Longrightarrow \quad G_D(\mathbf{r}, \mathbf{r}'; E) \;=$$

Then, the interaction diagram (1.16) will reduce to

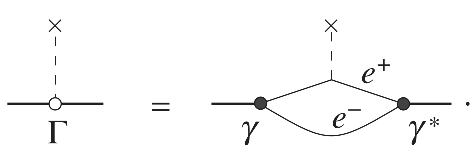


where we have cut off the electron ends since we are interested in a bound state at infinity, not a fermion pair. To calculate the interaction amplitude, we have to replace the positronium lines by the initial and final state wave functions. This way we arrive at the expression similar to (1.15) for the non-relativistic scattering amplitude.

In other words, the notion of the Green function of the positronium (or of the deuteron, for that matter) is unnecessary. It can, however, be introduced by separating from the product of the wave functions the dependence on the total four-momentum of the bound system,

$$\psi_D(x')\psi_D^*(x) \;=\; \mathrm{e}^{-iE(t'-t)+i\mathbf{p}_c(\mathbf{r}'_c-\mathbf{r}_c)}\psi(\mathbf{r}')\psi^*(\mathbf{r}). \tag{1.18}$$

Attributing the exponent to the Green function of the free positronium, the wave function of the relative motion will have the meaning of an *exact vertex* γ describing the transition between the positronium and the free e^+e^- pair, which will determine the interaction of the bound state as a whole with the external field:



An interesting feature of γ is that this vertex does not contain any input bare value: it derives from the structure of the bound state.

In the non-relativistic theory, describing a bound state in terms of its proper Green function G_D and its proper interaction vertex Γ is merely a formal unification. However, the existence of two possibilities to construct a theory of particle interactions (with different results!) is important for

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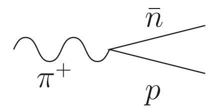
us in the context of strong interactions, where a pion, for example, can be looked upon as consisting of a pair of nucleons, 

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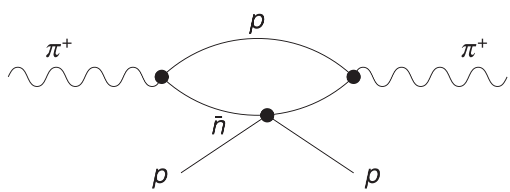
1.5.2 Quarks

Thus, we start with the hypothesis (which may turn out to be wrong) that there exists a small number of fundamental objects. From the point of view of *hadrons*, these objects can be chosen almost arbitrarily. *Almost*, since one cannot build up a spin $s = \frac{1}{2}$ particle from spinless π-mesons, or strange hadrons like K-mesons and the Λ baryon out of non-strange nucleons.

In the *Sakata model*, the three lightest baryons were chosen as building blocks: the pair of nucleons, p, n, and the strange baryon Λ $(S = -1)$: This model treats mesons as bound states: $\pi = N\bar{N}$, $K = N\bar{\Lambda}$, etc.,



though it remains unclear what to do with a rich variety of baryons.

Gell-Mann put forward a deeper idea based on the existence of eight very similar baryons (n, p, Λ, $\Sigma^{\pm,0}$, $\Xi^{0,-}$) with close mass values. The underlying $SU(3)$ symmetry became the origin of the idea of *quarks*.

Tempted to consider all the hadrons as *composite* objects, one introduces three quarks, $q_{1,2,3} = u, d, s$, which, for some mysterious reason, are not observable as free particles but are confined* inside the mesons $(q\bar{q})$ and the baryons (qqq). Fractional electric charges of the quarks may not look too attractive. Still, it would be nice if they existed in nature, and this possibility should not be disregarded a priori.

Thus, we assume that there are some fundamental particles that we introduce as bare fields into the theory. These particles may or may not show up in the physical spectrum of the theory (it would be beautiful if not). If so, all the observable hadrons are composite objects, and all reactions between them can be represented as the strong interaction between the underlying quarks.

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* This could be possible, for example, if the quark binding energy were so strong as to provide masses of the bound states *much smaller* than the large masses of the quarks.

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1.6 Interacting particles

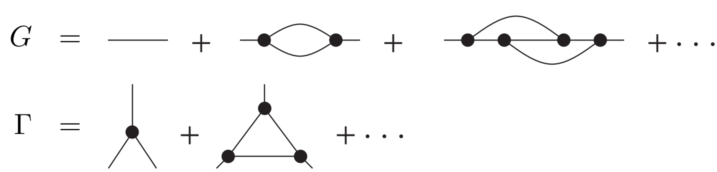
How to construct the interaction between our fundamental objects? Let us see what options quantum field theory (QFT) can offer us.

1.6.1 $\lambda\varphi^3$

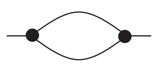
Let us look at the simplest QFT that describes a scalar field φ with three-particle interaction. Given the interaction constant λ, and depicting by a straight line the free particle propagator,

$$G(p) = \frac{1}{m^2 - p^2} = \text{———},$$

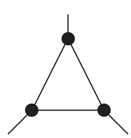
we can draw Feynman diagrams for the exact particle Green function and for interaction amplitudes,



Evaluating the self-energy diagram, in the region of large virtual momenta we will have a logarithmically divergent integral,

$$\Sigma(p) =$$

$$= \lambda^2 \int \frac{d^4k}{(2\pi)^4 i} \frac{1}{m^2 - k^2} \frac{1}{m^2 - (p-k)^2} \sim \int_m \frac{d^4k}{k^4}.$$

This turns out to be the *only* divergence in the theory! The integrals with more than two propagators in the loop, converge:


$$\sim \lambda^3 \int \frac{d^4k}{k^6} \sim \frac{\lambda^3}{p^2},$$

with p^2 the characteristic virtuality of the external lines. This shows that (apart from the mass renormalization) the self-interaction effects vanish in the large momentum region.

The absence of divergences is clear from dimensional considerations. Let us look, e.g. at the particle number density operator, which has the dimension

$$\left[\varphi \frac{\partial \varphi}{\partial t}\right] = \left[\delta^{(3)}(\mathbf{r})\right] = \left[m^3\right].$$

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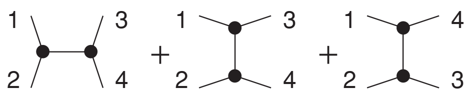
Therefore, the field $\varphi(x)$ itself has the dimension of mass, $[\varphi] = [m]$. Since the *action* is dimensionless ($\hbar = 1$),

$$\mathcal{L} = \int \lambda \cdot \varphi^3(x)\, d^4x, \quad [\mathcal{L}] = [m^0],$$

we get $[\lambda] = [m]$. The coupling constant having the dimension of mass, at large momenta p (where the finite mass is unimportant) this gives us the real dimensionless expansion parameter $\sim\lambda^2/p^2$, vanishing in the ultraviolet region. Such theories are referred to as 'superconvergent'.

From the Born diagrams for the two particle scattering amplitude,



$$= \frac{\lambda^2}{m^2 - s} + \frac{\lambda^2}{m^2 - t} + \frac{\lambda^2}{m^2 - u},$$

we see that the interaction disappears when the energy and momentum transfer invariants (s, t, u) become large.

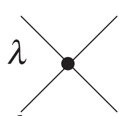
The widespread opinion according to which the $\lambda\varphi^3$ QFT is a bad one is owing to the non-positive definiteness of the energy density, $dE/dV \propto \lambda\varphi^3$, because of which this theory has no vacuum state.

Unfortunately, all this has nothing to do with Nature. It is, however, a useful QFT model for those cases when the spin is of no importance.

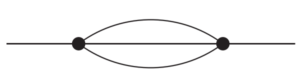
1.6.2 $\lambda\varphi^4$

Let us consider the next, more complicated example: the quartic interaction between spinless fields.

Pions are *pseudoscalar* particles, and therefore the transition $\pi \to \pi\pi$ is forbidden by parity conservation. This makes the $\lambda\varphi^4$ QTF closer to reality; it can be used to model interaction between pions. Now the coupling constant λ is *dimensionless*, so that we should expect logarithmic ultraviolet divergences, as in the case of QED.

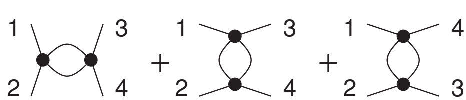


The simplest correction to the Green function G now diverges *quadratically*:



$$\sim \quad (d^4k)^2 \cdot \frac{1}{[k^2]^3}.$$

What concerns the effective charge, the first correction to the two-particle scattering amplitude λ consists of three graphs,



$$\sim \lambda^2 \int \frac{d^4k}{k^4} \sim \lambda^2 \ln \Lambda^2_{\mathrm{UV}}.$$

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The situation is similar to that in electrodynamics, and the renormalization procedure is carried out in the same way. And, similarly, the 'zero charge problem' appears: in both theories the renormalized coupling tends to zero when the ultraviolet cutoff Λ is taken to infinity:

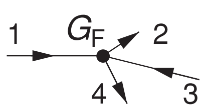
$$e_c^2 = \frac{e_0^2}{1 + \frac{e_0^2}{3\pi} \ln \frac{\Lambda^2}{m^2}}, \tag{1.19a}$$

$$\lambda_c^2 = \frac{\lambda_0^2}{1 + \frac{\lambda_0^2}{4\pi} \ln \frac{\Lambda^2}{m^2}}. \tag{1.19b}$$

In the QED context this was an 'academic' problem since the coupling was small, $e_c^2 \ll 1$, and the real contradiction appeared at fantastically large momenta and could be ignored. Not so when $\lambda = \mathcal{O}(1)$; the theory becomes unreliable at low momentum scales $p \gtrsim \Lambda \sim m$. Obviously, we are unable to get any information from such a theory.

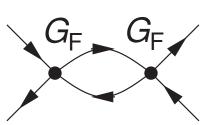
1.6.3 Four-fermion interaction

It would be nice to start constructing the theory from *fermions*, since from fermions one can build bosons, but not vice versa. One can imagine a quartic interaction between fermions, in analogy with the $\lambda\varphi^4$ model for scalars. The vertex may look as follows (Fermi interaction):



$$= G_{\mathrm{F}}\, (\bar{u}_2 \mathbf{O} u_1)(\bar{u}_4 \mathbf{O} u_3),$$

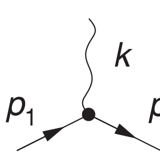
where the operator $\mathbf{O}$ in each of the fermion brackets may contain Dirac matrices, $\mathbf{O} \propto 1,\ \gamma_\mu,\ \gamma_5\gamma_\mu,\ \sigma_{\mu\nu}$, etc. Here the coupling constant has a *negative* mass dimension, $[G_{\mathrm{F}}] = [m^{-2}]$, so that the interaction grows with energy, and the theory becomes non-renormalizable:



$$\sim G_{\mathrm{F}}^2 \int \frac{d^4k}{(\hat{k})^2} \sim \Lambda_{\mathrm{UV}}^2.$$

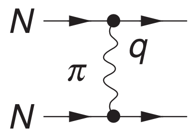
1.6.4 A nucleon and a pion

Interaction between a fermion (nucleons) and a spinless field can be modelled as $\bar{\psi}\psi\,\varphi$:



$$= g\big(\bar{u}(p_2) i\gamma_5 u(p_1)\big).$$

(Here we have introduced in the vertex the factor $i\gamma_5$ to match the fact that the pion is a pseudoscalar.) In this theory nucleons interact via pion exchange,

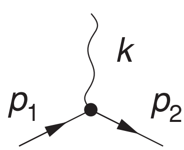


$$A = g^2(\bar{u}i\gamma_5 u)\frac{1}{m^2 - q^2}(\bar{u}i\gamma_5 u).$$

The coupling g is dimensionless, and the essential divergences are just logarithmic. This is quite a nice theory, it differs from electrodynamics only in that the π-meson field that carries the interaction between fermions, unlike the photon, has a finite mass, $m_\pi \neq 0$.

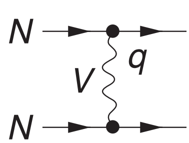
1.6.5 A nucleon and a vector meson

We can also make the nucleon interact with a massive vector meson field V_μ, in a QED-like manner: $\bar{\psi}\,\gamma^\mu\,\psi\,V_\mu$.



$$= \quad g\,\big(\bar{u}(p_2)\gamma^\mu u(p_1)\big)\cdot e^\lambda_\mu(k),$$

with $e^\lambda_\mu(k)$ the polarization vectors of the field V $(\lambda = 1, 2, 3)$. However, the situation here is potentially dangerous. Recall that the Green function of a massive vector field contains the term with momenta in the numerator:



$$G(q) = \frac{1}{m^2 - q^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}\right).$$

We must ensure that the term $q_\mu q_\nu/m^2$ drops out in the physical amplitudes, otherwise renormalizability of the theory would be lost. This is the case in QED, owing to the conservation of the electromagnetic current.

Conservation of current makes electrodynamics with a *massive photon* a perfectly legitimate renormalizable QFT, which construction can be borrowed to model strong interaction of point-like protons and neutrons with an *electrically neutral* vector meson. Such a theory would not be too bad; it would cause no objection apart from the 'zero-charge problem'

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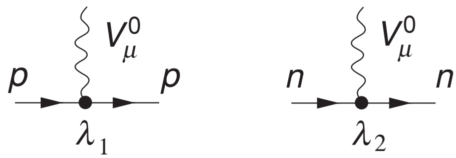
that plagues it (together with the previously considered $NN\pi$ model).



(1.20)

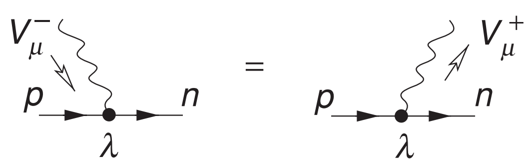
However, in reality *neutral* vector mesons occupy no special place in the hadron world; *charged* mesons are plentiful and one sees no reason to discriminate between them.

1.6.6 Charged vector mesons

Assume that we want to describe a charged meson. Let us consider three vector mesons, V^0 and $V^\pm$ (like a triplet of ρ-mesons), and discuss how they might interact with p and n.

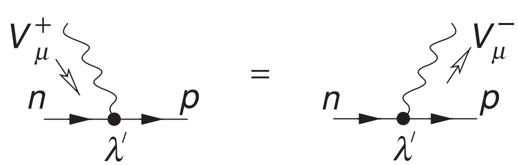
With the neutral meson V^0 everything is simple: it may be emitted (absorbed) either by a proton or a neutron as shown in (1.20), with λ_1 and λ_2 the corresponding coupling constants.

How will a charged meson interact with nucleons? A negative particle can be absorbed by a proton, which will turn into a neutron when absorbing a negative-charge meson V^-, or emitting V^+:



(1.21a)

Another possible process is absorption of V^+ (emission of V^-) by a neutron:



(1.21b)

Its amplitude is identical to that of (1.21a) if the theory is T-invariant. (Thus we have in principle three different coupling constants: two for the neutral meson in (1.20) and one for the charged, $\lambda = \lambda'$ in (1.21).)

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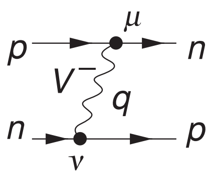
Now we have a look at the diagram for scattering of nucleons via V exchange:



$$G(q) = \frac{1}{m^2 - q^2}\Big(- g_{\mu\nu} + \frac{q_\mu q_\nu}{m^2}\Big).$$

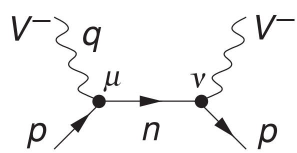
Will the 'bad term' now disappear? Convolution of the nucleon vertex with the meson momentum $q = k_n - k_p$ produces

$$q^\mu \cdot \big(\bar{u}_n \gamma_\mu u_p\big) = \bar{u}_p(\hat{k}_n - \hat{k}_p)u_n = (M_n - M_p) \cdot \big(\bar{u}_n u_p\big) \neq 0,$$

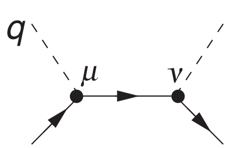
where we have used the Dirac equation for the on-mass-shell nucleons, $(\hat{k} - M)u = 0$. The result is not zero. And even if we set $M_p = M_n$ in zeroth approximation, this will not help in higher orders.

In the case of a neutral meson, electric charge of the fermion was preserved, and, as a result, the vector vertex of V^0 emission turned out to be identical to the conserved electromagnetic current, ensuring $q^\mu A_\mu = 0$. In graphs with $V^\pm$ emission this is no longer true.

Consider, for example, the elastic V^-p scattering amplitude,

$M_{\mu\nu}$ =  . (1.22)

We would like to have $q^\mu M_{\mu\nu} = 0$, with q_μ the vector meson momentum. In QED Compton scattering there were *two* graphs whose sum satisfied this property:



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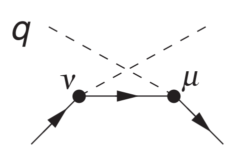

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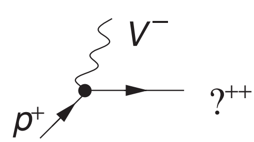
In our new context, the second contribution is absent: in a crossed diagram, an emission of V^- by a proton implies virtual exchange of a non-existent doubly charged nucleon.



We come to the conclusion that such a theory is always non-renormalizable. There is, however, a beautiful way to correct the situation provided by the *Yang–Mills theories*.

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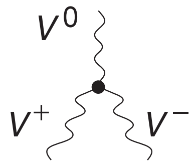

An incorporation of a specially chosen three-linear interaction between mesons allows one to construct a renormalizable theory of *massless* vector fields, $m_V = 0$. By adding another scalar field φ, vector mesons can be made *massive*, $m_V \neq 0$, without losing the renormalizability. This way the Glashow–Weinberg–Salam theory of weak interactions is constructed, with scalar 'Higgs' providing masses to the intermediate vector bosons Z^0 and $W^{\pm}$.

We postpone the discussion of the dynamics of Yang–Mills fields to the last lecture. Now let us turn to general features of relativistic particle scattering.

1.7 General properties of S-matrix: unitarity and crossing

Can we learn anything about the strong interactions given that there is no hope of employing perturbation theory?

Suppose we have some relativistic quantum field describing the objects that interact strongly, with a large *coupling constant* $g \sim 1$. In spite of the inapplicability of the perturbative methods, there is nevertheless a number of general statements that can be made.

1.7.1 S-matrix

First of all, in order to describe interaction processes we introduce the S-matrix whose elements S_{ab} quantify the transition from the initial state a to some final state b,

$$S = \mathrm{I} + i\,T; \quad S_{ab} = \delta_{ab} + i\,T_{ab}. \tag{1.23}$$

Here I is a symbolic representation of the absence of interaction; δ_{ab} means that in the final state we find the incoming particles with unperturbed momenta. T is called the reaction matrix and takes care of the interaction. It contains the δ-function to ensure the energy–momentum conservation and the product of the factors $1/\sqrt{2p_0}$ that originate from the relativistic normalization of the wave functions of incoming (i) and outgoing particles (j):

$$T_{ab} = (2\pi)^4\,\delta^4\left(\sum_{i\in a} p_i - \sum_{j\in b} k_j\right)\prod_{i\in a}\frac{1}{\sqrt{2p_{0i}}}\prod_{j\in b}\frac{1}{\sqrt{2k_{0j}}}\cdot\mathcal{M}_{ab}. \tag{1.24}$$

So defined, the scattering amplitude $\mathcal{M}_{ab}$ is *Lorentz invariant*. Typically, the initial state consists of two particles with four-momenta p_1 and p_2.

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To obtain the probability of the reaction one squares (1.24). Dropping the factor

$$(2\pi)^4 \delta^4(0) \ = \ V \cdot T$$

which formally represents the full volume of the space–time, we get the measurable *probability density* of the reaction. The wave function normalization factors of the outgoing particles participate in forming the Lorentz invariant phase space volume element of the final state,

$$d\Gamma_j = \left(\frac{1}{\sqrt{2k_{0j}}} \right)^2 \frac{d^3\mathbf{k}_j}{(2\pi)^3} \ = \ \frac{d^3\mathbf{k}_j}{2(2\pi)^3 k_{0j}} = \frac{d^4 k_j}{(2\pi)^4} \cdot 2\pi \delta_+\big(k_j^2 - m_j^2\big), \tag{1.25}$$

where δ_+ selects among the two solutions of the on-mass-shell condition the *positive energy* (physical) one: $k_{0j} = \sqrt{m_j^2 + \mathbf{k}_j^2}$. The initial state normalization factors combine with the relative velocity of the incoming particles,

$$j \equiv |\mathbf{v}_1 - \mathbf{v}_2| = \left| \frac{\mathbf{p}_1}{p_{01}} - \frac{\mathbf{p}_2}{p_{02}} \right|,$$

to form the flux factor,

$$J = (\sqrt{2p_{10}})^2 (\sqrt{2p_{20}})^2 \cdot j = 4|p_{20}p_{1z} - p_{10}p_{2z}|\,, \tag{1.26}$$

where we have chosen the direction $\mathbf{z}$ as the collision axis. The combination (1.26) is invariant under boosts along the z axis. Choosing the centre of mass system of reference (cms) in which the incoming momenta are equal and opposite, $\mathbf{p}_1 = -\mathbf{p}_2 = (0, 0, p_c)$,

$$\begin{aligned} p_c = p_c(s) &= \frac{\sqrt{s^2 - 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}}{2\sqrt{s}} \\ &= \frac{\sqrt{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}}{2\sqrt{s}}, \end{aligned} \tag{1.27}$$

we get the Lorentz invariant flux

$$J \ = \ 4p_c(s)\sqrt{s}. \tag{1.28}$$

Finally, the differential cross section of the process $p_1, p_2 \to \{k_j\}$ reads

$$d\sigma(a \to b) \ \equiv \ \frac{1}{J} |\mathcal{M}_{ab}|^2 (2\pi)^4 \delta^4 \bigg(p_1 + p_2 - \sum_{j\in b} k_j \bigg) \cdot \frac{1}{[n!]} \prod_{j\in b} d\Gamma(k_j). \tag{1.29}$$

Among n produced particles there may be identical ones. The symmetry factor eliminates multiple counting of physically indistinguishable

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configurations produced by a permutation of identical particles. When in the final state b there are n_s particles of the type s,

$$\frac{1}{[n!]} \equiv \prod_s \frac{1}{n_s!}, \quad \sum_s n_s = n.$$

1.7.2 Unitarity

The S-matrix (1.23) is *unitary*,

$$S\,S^\dagger = 1 \quad \Longrightarrow \quad T_{ab} - T^\dagger_{ab} = i\left(T\,T^\dagger\right)_{ab},$$

or, deciphering a symbolic matrix multiplication,

$$\frac{1}{i}\left(T_{ab} - T^*_{ba}\right) = \sum_c T_{ac}\,T^*_{cb}\,. \tag{1.30a}$$

If the interaction is invariant with respect to the *time reversal*, T, which is the case for the strong interaction of hadrons, then the matrix $\mathcal{S}$ is *symmetric*, $T_{ab} = T_{ba}$, and the unitarity relation (1.30a) takes the form

$$\frac{1}{i}(T_{ab} - T^*_{ab}) \;= 2\,\mathrm{Im}\,T_{ab} = \sum_c T_{ac}\,T^*_{bc}\,. \tag{1.30b}$$

This expression implies an integration over momenta of all the particles in the intermediate state c. Therefore, the symmetry factor is present on the r.h.s. of (1.30), analogously to the differential cross section case (1.29).

In terms of the invariant amplitude $\mathcal{M}$ defined in (1.24),

$$\begin{aligned}\frac{\mathcal{M}_{ab} - \mathcal{M}^*_{ab}}{i} = &\sum_n \frac{1}{[n!]} \int \mathcal{M}_{an}(\{p\}_a;\{k\}_n)\,\mathcal{M}^*_{bn}(\{p\}_b;\{k\}_n) \\ &\cdot (2\pi)^4\delta\left(\sum p^a_i - \sum_{\ell=1}^n k_\ell\right) \prod_{\ell=1}^n \left\{\delta_+\left(k^2_\ell \,-\, m^2_\ell\right)\frac{d^4k_\ell}{(2\pi)^3}\right\},\end{aligned} \tag{1.31}$$

where $\{p\}$ marks the set of momenta in the initial (a) and final states (b), and $\{k\}_n$ – momenta of n intermediate state particles.

If we take $a \equiv b$, the 'optical theorem' emerges which relates the imaginary part of the forward scattering amplitude to the total cross section:

$$2\,\mathrm{Im}\,A_{aa} \;=\; J\cdot\sigma^a_{\mathrm{tot}}. \tag{1.32}$$

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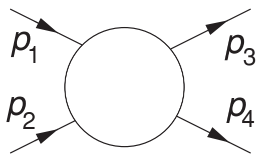
1.7.3 Mandelstam plane for $2 \to 2$ scattering

Consider a two-particle interaction amplitude $1 + 2 \to 3 + 4$. How many Lorentz invariant variables characterize the process? We have three momentum four-vectors, that is $3 \times 4 = 12$ independent components. Four on-mass-shell conditions, $p_i^2 = m_i^2$, one per each participating particle, leave us with $12 - 4 = 8$. Finally, we must subtract six parameters (three rotations and three Lorentz boosts) which characterize the reference frame and do not affect the invariant amplitude, $8 - 6 = 2$.



In a general case of the reaction $n_1 \to n_2$, the counting goes as follows,

$$4(n_1 + n_2 - 1) - (n_1 + n_2) - 6 = 3(n_1 + n_2) - 10. \tag{1.33}$$

For example, a $2 \to 3$ process depends on five independent Lorentz invariant combinations of momenta.

A convenient way to characterize $2 \to 2$ processes is provided by the *Mandelstam variables*

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{1.34a}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{1.34b}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2. \tag{1.34c}$$

The variables (1.34) are not independent but satisfy an easy-to-verify kinematic relation:

$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{1.35}$$

This relation makes it convenient to represent the kinematics of the process on the *Mandelstam plane*, exploiting the property of an equilateral triangle as shown in Fig. 1.1. Where is the physical region of the reaction

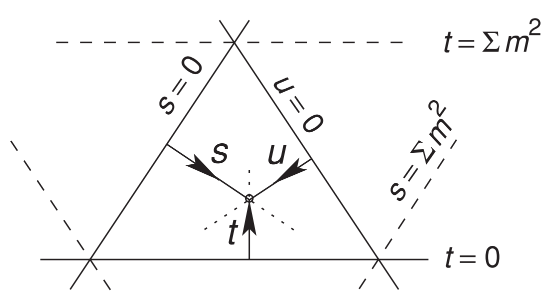


Fig. 1.1 Mandelstam plane.

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on the Mandelstam plane? The meaning of the Mandelstam invariants (1.34) is the most transparent in the centre-of-mass reference frame (cms) of the reaction. Here $\mathbf{p}_1 + \mathbf{p}_2 = 0$, so that the variable s in (1.34a),

$$s = (p_{1\mu} + p_{2\mu})^2 \equiv (p_{10} + p_{20})^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = (E_{1c} + E_{2c})^2 = E_c^2,$$

becomes the square of the total energy of the colliding particles.

t and u are invariant momentum transfers. In particular, t defined in (1.34b) can be represented as

$$\begin{aligned} t = (p_{3\mu} - p_{1\mu})^2 &\equiv (E_3 - E_1)^2 - (\mathbf{p}_3 - \mathbf{p}_1)^2 \\ &= (E_3 - E_1)^2 - (\mathrm{p}_3 - \mathrm{p}_1)^2 - 2\mathrm{p}_1\mathrm{p}_3(1 - \cos\Theta), \end{aligned}$$

where $\mathrm{p}_i = |\mathbf{p}_i|$ and Θ is the scattering angle:

$$\cos\Theta = \frac{\mathbf{p}_1 \cdot \mathbf{p}_3}{\mathrm{p}_1\mathrm{p}_3}.$$

In the centre-of-mass frame, the moduli of three-momenta of the incoming particles, $\mathrm{p}_1 = \mathrm{p}_2 = p_c$, and of the produced ones, $\mathrm{p}_3 = \mathrm{p}_4 = p'_c$, are given by (1.27) as a function of the energy and of the masses of the particles in the initial and final state, correspondingly.

In the case of *elastic* scattering when $m_3 = m_1$ and $m_4 = m_2$ (as, e.g. in the reaction $\pi N \to \pi N$), one has $\mathrm{p}_3 - \mathrm{p}_1 = E_{3c} - E_{1c} = 0$, and

$$t = -2p_c^2(1 - \cos\Theta_c)\,. \tag{1.36a}$$

If *all* the masses are equal, $m_1 = m_2 = m_3 = m_4$, then the cms expression for the variable u (1.34c) becomes simple also:

$$u = -2p_c^2(1 + \cos\Theta_c)\,. \tag{1.36b}$$

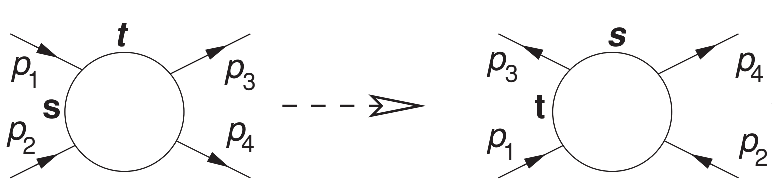
In this case the physical region of the reaction $1 + 2 \to 3 + 4$,

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0, \tag{1.37}$$

is shown by the shaded area in Fig. 1.2.

1.7.4 Crossing symmetry

One and the same diagram can be viewed differently. Let us 'rotate' our scattering diagram by 90°:



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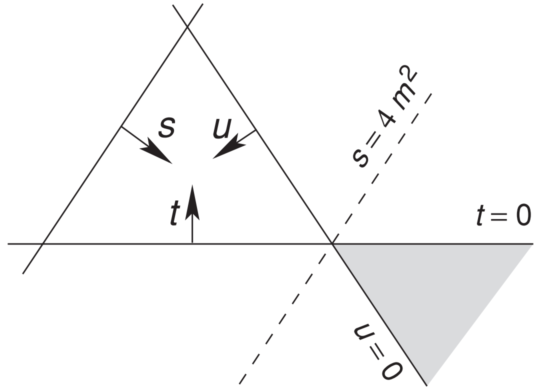

Fig. 1.2 Physical region of scattering of equal-mass particles.

The 'new' picture can be interpreted as an interaction between two particles with momenta p_1 and $-p_3$, producing particles p_4 and $-p_2$. To make this interpretation valid, we must take the energy components of the momenta p_3 and p_2 to be *negative*: $p_{30} \le -m_3$ and $p_{20} \le -m_2$. But as you know, in the relativistic theory the propagation of a negative energy particle 3 with a momentum p_3 corresponds to the propagation of its *antiparticle* ($\bar{3}$) with the four-momentum $\bar{p}_3 = -p_3$. Therefore, in this region of momenta the very same diagram describes another physical process, namely a collision between the particle 1 and the antiparticle $\bar{3}$ (having a four-momentum $\bar{p}_3 = -p_3$), which results in the production of particles 4 and $\bar{2}$ in the final state. This is called a *t-channel* reaction, since here the invariant

$$t = (p_1 - p_3)^2 = (p_1 + \bar{p}_3)^2 \ge (m_1 + m_3)^2 \tag{1.38a}$$

has the meaning of the cms energy of colliding particles 1 and $\bar{3}$.

Analogously, in the region of momenta $p_{40} \le -m_4$, $p_{20} \le -m_2$ we obtain the amplitude of a *u-channel* process, $1 + \bar{4} \to 3 + \bar{2}$,

$$u = (p_1 - p_4)^2 = (p_1 + \bar{p}_4)^2 \ge (m_1 + m_4)^2 \,. \tag{1.38b}$$

Imagine that we have calculated the necessary diagrams and know the scattering amplitude as a function of the invariants in the physical region (1.37) of the reaction $1 + 2 \to 3 + 4$. If the amplitude were an analytic function of its variables s and t, we would be able to *analytically continue* the result into the physical region of either of the two crossing reactions (1.38a) or (1.38b). As we will shortly see, this is indeed the case: the analyticity is a direct consequence of *causality*.

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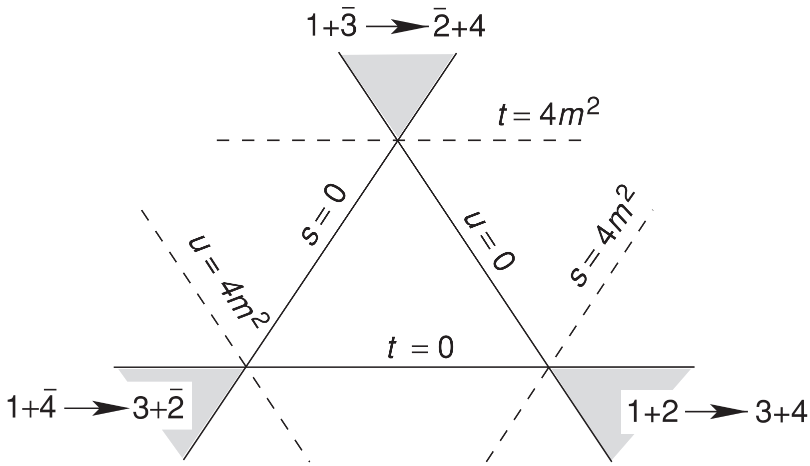

Fig. 1.3 Physical regions of crossing reactions on the Mandelstam plane.

Therefore, one function describes three different scattering processes that are related by crossing:

$$
\begin{aligned}
&s\text{-channel}: && 1+2 \to 3+4, && s = (p_1+p_2)^2 \geq (m_1+m_2)^2;\\
&t\text{-channel}: && 1+\bar{3} \to \bar{2}+4, && t = (p_1+\bar{p}_3)^2 \geq (m_1+m_3)^2;\\
&u\text{-channel}: && 1+\bar{4} \to 3+\bar{2}, && u = (p_1+\bar{p}_4)^2 \geq (m_1+m_4)^2.
\end{aligned}
$$

Physical regions of the crossing reactions are displayed in Fig. 1.3 for the simplest case of equal particle masses.

It is important to remember that the *unitarity* seriously restricts the scattering amplitude. Moreover, these restrictions are different in each of the three crossing channels. Thus, one function has to satisfy three specific unitarity relations in complementary physical regions on the Mandelstam plane.

In non-relativistic quantum mechanics an interaction is described by means of a potential which can be chosen practically arbitrarily. Not so in the relativistic theory. If we were to introduce here a notion of 'relativistic potential', the latter would be severely restricted by the unitarity conditions in the cross-channels. This is a specifically relativistic feature since the crossing itself is of relativistic nature.

In the next lecture we will demonstrate that the causality ensures that the scattering amplitudes are analytic functions of momenta. An analytic function is identified by its singularities. The structure of these singularities may be studied, as it turns out, with the help of a (formally senseless) series of Feynman diagrams *as if* in the perturbation-theory framework.

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This statement holds even for strongly interacting objects, in which case the very applicability of diagrammatic expansion is highly questionable.

Let us formulate straight away our main hypothesis:

> *Analytic properties of the exact amplitude coincide with those of the corresponding perturbation-theory diagrams.*

To check that, we shall show that all singularities of Feynman graphs (their position, nature and strength) have a clear physical origin and are closely related to unitarity. This statement does not depend on the particular particle content of the theory or on specific properties of the interaction. The only important thing is to have the input objects – bare particles – to be point-like, that is to be included into some quantum field theory (QFT) scheme.