A SEMIGROUP APPROACH TO LATTICES

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1. Introduction. In **(3**, p. 85**)** we defined a *Baer semigroup* to be a multiplicative semigroup with 0 having the property that the left (right) annihilator of every element is a principal left (right) ideal generated by an idempotent. We showed **(3**, Lemma 1(vi) and Theorem 5, p. 86**)** that with set inclusion as the partial order, the set of left annihilators and the set of right annihilators of elements of a Baer semigroup form dual isomorphic lattices with 0 and 1. However, the following questions were left open:

1. A Baer semigroup S is said to *coordinatize* the lattice L if L is isomorphic to the lattice of left annihilators of elements of S. Which lattices can be coordinatized by a Baer semigroup?

2. Is it possible to characterize complemented modular lattices in terms of properties of a suitable coordinatizing Baer semigroup?

Interestingly enough, we shall prove here that *every* lattice with 0 and 1 can be coordinatized by a Baer semigroup, and that the characterization of complemented modular lattices is an exact analogue of the one provided by D. J. Foulis **(2**, Theorem 10, p. 894**)** for orthocomplemented modular lattices. Following this we shall make a careful study of algebraic equivalence, showing that in a lattice L with 0 and 1, the interval $L(0, e)$ is isomorphic to $L(0, f)$ if and only if e is algebraically equivalent to f in some coordinatizing Baer semigroup.

Before proceeding, let us introduce some of the notation that will be used in the following. Let S be a multiplicative semigroup with 0. For each $x \in S$, let

$$L(x) = \{y{:}y \in S, yx = 0\}, \qquad R(x) = \{y{:}\ y \in S, xy = 0\},$$
$$L1(x) = \{e{:}\ e \in S, e = e^2, Se = L(x)\} \quad \text{and}$$
$$R1(x) = \{f{:}\ f \in S, f = f^2, fS = R(x)\}.$$

Unless otherwise specified, let us adopt the convention that e, f, g, h will always denote idempotents. The condition that S be a Baer semigroup can now be stated as follows: For each $x \in S$, both $L1(x)$ and $R1(x)$ are non-empty. In case S is a Baer semigroup, we agree to let $\mathfrak{L} = \mathfrak{L}(S)$ ($\mathfrak{R} = \mathfrak{R}(S)$) denote the lattice of left (right) annihilators of elements of S.

Let P be a partially ordered set. A mapping ϕ: $P \to P$ is called *monotone* if $a \leqslant b \Rightarrow a\phi \leqslant b\phi$ for all $a, b \in P$. Following R. Croisot **(1**, p. 454**)** we say that a monotone mapping ϕ: $P \to P$ is *residuated* if there exists a (necessarily

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unique) monotone mapping $\phi^+\colon P \to P$, called the *residual* of ϕ, such that $a\phi^+\phi \leqslant a \leqslant a\phi\phi^+$ for all $a \in P$. We shall frequently make use of the self-dual nature of this definition; for if ϕ is a residuated mapping on P having ϕ^+ as its residual mapping, then ϕ^+ is a residuated mapping on the dual of P having ϕ as its residual mapping. Thus any theorem about residuated mappings can be dualized to apply to residual mappings. Let $\mathfrak{S} = \mathfrak{S}(P)$ denote the set of all monotone residuated mappings on P and $\mathfrak{S}^+ = \mathfrak{S}^+(P)$ the set of mappings residual to some $\phi \in \mathfrak{S}(P)$. One then has (**1**, Prop. 1, p. 455) that $\mathfrak{S}(P)$ and $\mathfrak{S}^+(P)$ are anti-isomorphic semigroups with function composition as the semigroup operation and $\phi \to \phi^+$ as the anti-isomorphism; furthermore, each $\phi \in \mathfrak{S}(P)$ preserves arbitrary suprema whenever they exist in P, and each $\phi^+ \in \mathfrak{S}^+(P)$ preserves arbitrary infima whenever they exist. For a proof of this last assertion, see (**4**, p. 3). The point is that the pair (ϕ, ϕ^+) sets up a Galois connection between P and its dual. It follows that for each $\phi \in \mathfrak{S}(P)$, $\phi = \phi\phi^+\phi$ and $\phi^+ = \phi^+\phi\phi^+$.

Suppose now that $e, f \in L$ where L is a lattice with 0 and 1. The notation $M(e,f)$ denotes the fact that $a \leqslant f \Rightarrow (a \vee e) \wedge f = a \vee (e \wedge f)$, while $M^*(e,f)$ means that $a \geqslant f \Rightarrow (a \wedge e) \vee f = a \wedge (e \vee f)$. When we write $e \oplus f = 1$ we shall mean that e and f are complements with $M(e,f)$ and $M^*(f,e)$ true. Finally, it will prove useful to adopt the convention that Theorem n^* denotes the dual of Theorem n.

2. The coordinatization of a lattice.

In this section it will be shown that every lattice with 0 and 1 is isomorphic to the lattice of left annihilators of a suitable Baer semigroup. Since the proof of the next lemma is routine, it will be left for the reader.

LEMMA 2.1. *Let L be a lattice with* 0 *and* 1. *For an arbitrary element e of L define*

$$x\theta_e = \begin{cases} x, & x \leqslant e, \\ e, & \text{otherwise}, \end{cases} \qquad y\theta_e{}^+ = \begin{cases} 1, & y \geqslant e, \\ y \wedge e, & \text{otherwise}, \end{cases}$$

$$x\psi_e = \begin{cases} 0, & x \leqslant e, \\ x \vee e, & \text{otherwise}, \end{cases} \qquad y\psi_e{}^+ = \begin{cases} y, & y \geqslant e, \\ e, & \text{otherwise}. \end{cases}$$

Then θ_e, ψ_e are idempotent elements of $\mathfrak{S}(L)$ with $\theta_e{}^+$, $\psi_e{}^+$, respectively, acting as their residual mappings.

LEMMA 2.2. *If L is a lattice with* 0 *and* 1, *then $\mathfrak{S}(L)$ is a Baer semigroup.*

Proof. By (**3**, Lemma 13(ii), p. 89) if $\phi, \psi \in \mathfrak{S}(L)$, then $\psi\phi = 0$ if and only if $1\psi \leqslant 0\phi^+$. If $\psi\phi = 0$, then evidently $\psi = \psi\theta_{0\phi^+}$; conversely, if $\psi = \psi\theta_{0\phi^+}$, then $1\psi = 1\psi\theta_{0\phi^+} \leqslant 1\theta_{0\phi^+} = 0\phi^+$ and $\psi\phi = 0$. Thus $L(\phi) = \mathfrak{S}\theta_{0\phi^+}$. A dual argument shows that $R(\phi) = \psi_{1\phi}\,\mathfrak{S}$. Therefore $\mathfrak{S}(L)$ is a Baer semigroup.

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Combining the above result with (**3**, Theorem 15, p. 90), we obtain

THEOREM 2.3. *For a poset* P *with* 0 *and* 1 *the following conditions are equivalent:*
(i) P *is a lattice.*
(ii) $\mathfrak{S}(P)$ *is a Baer semigroup.*
(iii) P *can be coordinatized by a Baer semigroup.*

There are two remarks that seem appropriate at this point. First of all, since every lattice can be embedded in a lattice with 0 and 1, the above theorem shows that lattice theory in general can be approached through the vehicle of Baer semigroups. Secondly, obvious modifications of the proof of Lemma 2.2 will produce the fact that a poset P and 0 and 1 is a join (meet) semilattice if and only if $\mathfrak{S}(P)$ has the property that the right (left) annihilator of every finite subset is a principal right (left) ideal generated by an idempotent.

3. Range-closed Baer semigroups. (The results in this section were obtained independently by J. C. Derdérian.) *In this section it will be assumed that* L *denotes a lattice with* 0 *and* 1. Generalizing a notion of D. J. Foulis (**2**, p. 890), we call an element ϕ of $\mathfrak{S}(L)$ *range-closed* if $a \leqslant 1\phi$ implies the existence of an element $b \in L$ such that $b\phi = a$; ϕ will be called *dual range-closed* if ϕ^+ is a range-closed element of $\mathfrak{S}(\tilde{L})$, where $\tilde{L}$ denotes the dual lattice of L. In the case of an arbitrary Baer semigroup S, an element y of S will be called *range-closed* if $Se \leqslant LR(y)$ implies the existence of an element Sg of $\mathfrak{L}(S)$ such that $LR(gy) = Se$; it will be called *dual range-closed* if $fS \leqslant RL(y)$ implies the existence of an element hS of $\mathfrak{R}(S)$ such that $RL(yh) = fS$. Recall (**3**, p. 94) that for each $y \in S$ there exists a residuated mapping ϕ_y: $\mathfrak{L}(S) \rightarrow \mathfrak{L}(S)$ defined by $(Se)\phi_y = LR(ey)$, where e is any idempotent generator of Se. The mapping ϕ_y^+ is given by the formula $(Se)\phi_y^+ = L(ye^r)$, where $e^r \in R1(e)$. It is also worth mentioning that by (**3**, Theorem 25, p. 94) the mapping $y \rightarrow \phi_y$ is a semigroup homomorphism of S into $\mathfrak{S}(\mathfrak{L}(S))$. It is now easily seen that y is a (dual) range-closed element of S if and only if ϕ_y is a (dual) range-closed element of $\mathfrak{S}(\mathfrak{L}(S))$, so that our definitions are compatible. An element y of S will be called *bilaterally range-closed* if it is both range-closed and dual range-closed. The Baer semigroup S will be called *range-closed*, *dual range-closed*, or *bilaterally range-closed* according to whether each element of S has the indicated property. Our first task will be to decide what it means to say that an element of a Baer semigroup is range-closed.

LEMMA 3.1. *Let* S *be a Baer semigroup and* $\mathfrak{S} = \mathfrak{S}(\mathfrak{L}(S))$. *If* $Sy \in \mathfrak{L}(S)$, *then* y *is range-closed. An idempotent element* e *is range-closed if and only if* $\mathfrak{S}\phi_e \in \mathfrak{L}(\mathfrak{S})$.

Proof. Suppose that $Sy \in \mathfrak{L}(S)$. Then (**3**, Lemma 1(v), p. 86) $Sy = LR(y)$ so $Sf \leqslant LR(y)$ implies that $f = xy$ for some $x \in S$. Let g be an idempotent generator of $LR(x)$. Then by (**3**, Lemma 24, p. 93)

$$(Sg)\phi_y = LR(gy) = LR(xy) = LR(f) = Sf.$$

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In order to prove the second assertion we need only show that for every range-closed idempotent $\phi \in \mathfrak{S}$, $\mathfrak{S}\phi = L(\psi_{1\phi})$. Note first that by **(3**, Lemma 13, p. 89**)** $\xi\psi_{1\phi} = 0 \Leftrightarrow 1\xi \leqslant 1\phi$. But $a \leqslant 1\phi \Rightarrow a = b\phi$ for suitable $b \in L \Rightarrow a\phi = b\phi\phi = b\phi = a$, so $1\xi \leqslant 1\phi \Rightarrow \xi = \xi\phi$. Conversely, if $\xi = \xi\phi$, then $1\xi = 1\xi\phi \leqslant 1\phi$, thus completing the proof.

LEMMA 3.2. *An element ϕ of $\mathfrak{S}(L)$ is range-closed if and only if $a\phi^+\phi = a \wedge 1\phi$ for all $a \in L$.*

Proof. Suppose first that ϕ is range-closed. Then $a \wedge 1\phi \leqslant 1\phi$ implies that $a \wedge 1\phi = b\phi$ for some $b \in L$. Using the fact that ϕ^+ is a meet homomorphism and that $1\phi\phi^+ = 1$, we see that

$$a \wedge 1\phi = b\phi = b\phi\phi^+\phi = (a \wedge 1\phi)\phi^+\phi = (a\phi^+ \wedge 1\phi\phi^+)\phi = a\phi^+\phi.$$

The reverse implication is obvious.

W once more generalize a notion of Foulis **(2**, p. 892**)** and call $\phi \in \mathfrak{S}(L)$ *totally range-closed* if ψ range-closed implies $\psi\phi$ range-closed for all $\psi \in \mathfrak{S}(L)$; it will be called *dual totally range-closed* if ψ^+ dual range-closed implies $\psi^+\phi^+$ dual range-closed. Let $\mathfrak{S}_{\mathrm{TRC}}(L)$ ($\mathfrak{S}_{\mathrm{DTRC}}(L)$) denote the set of (dual) totally range-closed elements of $\mathfrak{S}(L)$. Since the identity map is a bilaterally range-closed element of both $\mathfrak{S}(L)$ and $\mathfrak{S}^+(L)$, it is immediate that every (dual) totally range-closed element of $\mathfrak{S}(L)$ is (dual) range-closed. Furthermore, both $\mathfrak{S}_{\mathrm{TRC}}(L)$ and $\mathfrak{S}_{\mathrm{DTRC}}(L)$ are semigroups with respect to function composition. It will prove convenient to let $\mathfrak{S}_{\mathrm{SRC}}(L) = \mathfrak{S}_{\mathrm{TRC}}(L) \cap \mathfrak{S}_{\mathrm{DTRC}}(L)$ and call an element of $\mathfrak{S}_{\mathrm{SRC}}(L)$ *strongly range-closed.*

LEMMA 3.3. *Let $\phi \in \mathfrak{S}(L)$. The following conditions are then equivalent:*

(i) $\phi \in \mathfrak{S}_{\mathrm{TRC}}(L)$.

(ii) *For each $f \in L$ there exists a range-closed idempotent ξ_f such that $1\xi_f = f$ and $\xi_f\phi$ is range-closed.*

(iii) *For all $e, f \in L$, $(e\phi^+ \wedge f)\phi = e \wedge f\phi$.*

Proof. (i) $\Rightarrow$ (ii). Take ξ_f to be the range-closed idempotent θ_f described in Lemma 2.1.

(ii) $\Rightarrow$ (iii). By Lemma 3.2,

$$\begin{aligned}(e\phi^+ \wedge f)\phi = (e\phi^+ \wedge 1\xi_f)\phi = e\phi^+\xi_f^+\xi_f\phi = e(\xi_f\phi)^+(\xi_f\phi)\\ = e \wedge 1\xi_f\phi = \mathrm{e} \wedge f\phi.\end{aligned}$$

(iii) $\Rightarrow$ (i). Assume that $(e\phi^+ \wedge f)\phi = e \wedge f\phi$ for all $e, f \in L$. If ψ is range-closed, we then have

$$e(\psi\phi)^+(\psi\phi) = e\phi^+\psi^+\psi\phi = (e\phi^+ \wedge 1\psi)\phi = e \wedge 1\psi\phi,$$

so $\psi\phi$ is range-closed.

Making use of Lemma 3.3 and its dual, we have

THEOREM 3.4. *An element ϕ of $\mathfrak{S}(L)$ is strongly range-closed if and only if the following two conditions hold:*
(i) $(e\phi^+ \wedge f)\phi = e \wedge f\phi$ *for all* $e, f \in L$.
(ii) $(e\phi \vee f)\phi^+ = e \vee f\phi^+$ *for all* $e, f \in L$.

Suppose now that $e \oplus f = 1$. It was shown in **(3,** Lemma 16, p. 90**)** that the mapping $\phi_{e,f}$ given by $a\phi_{e,f} = (a \vee e) \wedge f$ is an idempotent element of $\mathfrak{S}(L)$ whose residual mapping $\phi_{e,f}{}^+$ is defined by $a\phi_{e,f}{}^+ = (a \wedge f) \vee e$. Furthermore, since $a \leqslant f \Leftrightarrow a = a\phi_{e,f}$ and $b \geqslant e \Leftrightarrow b = b\phi_{e,f}{}^+$, we see that $\phi_{e,f}$ is bilaterally range-closed. The next theorem shows that this is the only way a bilaterally range-closed idempotent can arise in $\mathfrak{S}(L)$.

THEOREM 3.5. *Let* $\phi = \phi^2 \in \mathfrak{S}(L)$. *The necessary and sufficient condition that* ϕ *be bilaterally range-closed is that* $0\phi^+ \oplus 1\phi = 1$ *and* $\phi = \phi_{0\phi^+,1\phi}$.

Proof. Sufficiency was proved in the remarks preceding the theorem, so we now assume that ϕ is bilaterally range-closed. Since $1\phi \wedge 0\phi^+ \leqslant 1\phi$, there exists an element a of L such that $a\phi = 1\phi \wedge 0\phi^+$. But then $a\phi \leqslant 0\phi^+$ implies that $a\phi = a\phi\phi = 0$. Thus $1\phi \wedge 0\phi^+ = 0$, and dually $1\phi \vee 0\phi^+ = 1$. Furthermore, if we make use of Lemma 3.1 and its dual, we see that

$$a\phi = a\phi\phi^+\phi = a\phi\phi^+\phi^+\phi = (a \vee 0\phi^+) \wedge 1\phi.$$

Hence

$$a \leqslant 1\phi \Rightarrow a = (a \vee 0\phi^+) \wedge 1\phi;$$

i.e., $M(0\phi^+, 1\phi)$ holds. The fact that $M^*(1\phi, 0\phi^+)$ holds will follow by a dual argument. We conclude that $0\phi^+ \oplus 1\phi = 1$ and $\phi = \phi_{0\phi^+,1\phi}$.

LEMMA 3.6. *Let* $\phi, \psi \in \mathfrak{S}(L)$ *with* $1\phi = e$, $1\psi = f$, *and* $0\psi^+ = f'$. *Suppose that* ϕ *is range-closed and* ψ *is bilaterally range-closed. Then* $\phi\psi$ *is range-closed if and only if* $M^*(e, f')$.

Proof. We shall make repeated use of Lemma 3.1 and its dual. Suppose first that $\phi\psi$ is range-closed. Then

$$a\psi^+\phi^+\phi\psi = a(\phi\psi)^+(\phi\psi) = a \wedge 1\phi\psi = a \wedge e\psi.$$

If $a \geqslant f'$, then $a = b\psi^+$ for some $b \in L$ and

$$b \wedge e\psi = b\psi^+\phi^+\phi\psi = a\phi^+\phi\psi = (a \wedge e)\psi.$$

This shows that

$$(a \wedge e)\psi\psi^+ = (b \wedge e\psi)\psi^+ = b\psi^+ \wedge e\psi\psi^+ = a \wedge e\psi\psi^+;$$

i.e., $(a \wedge e) \vee f' = a \wedge (e \vee f')$. Suppose conversely that $M^*(e, f')$ holds.

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Then

$$a\psi^+\phi^+\phi\psi\psi^+ = (a\psi^+ \wedge e) \vee f' = a\psi^+ \wedge (e \vee f') = a\psi^+ \wedge e\psi\psi^+ = (a \wedge e\psi)\psi^+.$$

Hence

$$a(\phi\psi)^+(\phi\psi) = a\psi^+\phi^+\phi\psi = a\psi^+\phi^+\phi\psi\psi^+\psi = (a \wedge e\psi)\psi^+\psi = (a \wedge e\psi) \wedge f = a \wedge e\psi = a \wedge 1\phi\psi,$$

so that $\phi\psi$ is range-closed.

An immediate consequence of the above result is the fact that if each $e \in L$ admits a complement f such that $e \oplus f = 1$, then L is a complemented modular lattice if and only if every bilaterally range-closed element of $\mathfrak{S}(L)$ is strongly range-closed. If we recall that by Lemma 2.1, for each $a \in L$ there exists a range-closed idempotent $\theta_a \in \mathfrak{S}(L)$ such that $1\theta_a = a$ and a dual range-closed idempotent ψ_a such that $0\psi_a^+ = a$, the next two lemmas follow immediately from Lemma 3.6 and its dual.

LEMMA 3.7. *Let ϕ be bilaterally range-closed. Then ϕ is strongly range-closed if and only if $M(a, 1\phi)$ and $M^*(b, 0\phi^+)$ for all $a, b \in L$.*

LEMMA 3.8. *An idempotent ϕ is strongly range-closed if and only if the following three conditions hold:*

(i) *There exist elements e, f such that $e \oplus f = 1$ and $\phi = \phi_{e,f}$.*
(ii) *$M(a, f)$ for all $a \in L$.*
(iii) *$M^*(b, e)$ for all $b \in L$.*

It now follows that if L is a complemented modular lattice, then $\mathfrak{S}_{SRC}(L)$ is a bilaterally range-closed Baer semigroup which coordinatizes L. We now investigate the lattice of left annihilators of elements of a range-closed Baer semigroup.

LEMMA 3.9. *If S is a range-closed Baer semigroup, then $L = \mathfrak{L}(S)$ is a complemented modular lattice.*

Proof. Let $Se \in L$ and $f \in R1(e)$. By Lemma 3.1*, f is dual range-closed and consequently it is a bilaterally range-closed idempotent. It follows that $0\phi_f^+ \oplus 1\phi_f = 1$; i.e., that $Se \oplus Sf = 1$. Now for $Sg \in L$, ϕ_g is range-closed and $\phi_g\,\phi_f$ is also range-closed. By Lemma 3.6, $M^*(Sg, Se)$, whence L is both complemented and modular.

Combining Lemma 3.9 with the remark immediately preceding it, we have

THEOREM 3.10. *Let L be a partially ordered set with 0 and 1. The following conditions are equivalent:*

(i) *L is a complemented modular lattice.*
(ii) *L can be coordinatized by a bilaterally range-closed Baer semigroup.*
(iii) *L can be coordinatized by a range-closed Baer semigroup.*
(iii*) *L can be coordinatized by a dual range-closed Baer semigroup.*

4. Strongly regular Baer semigroups. Let x be an element of a Baer semigroup S. Let us call x *left regular* if $Sx \in \mathfrak{L}(S)$, *right regular* if $xS \in \mathfrak{R}(S)$, and *strongly regular* if it is both left and right regular. The Baer semigroup S will be called *left, right, or strongly regular* if every element of S has the specified property. Now if x is left regular there must exist an idempotent e such that $Sx = Se$. Then $e = yx$ for some $y \in S$ and $x = xe$, so $x = xyx$. Thus left regularity implies regularity in the sense of von Neumann. In the case of a ring with identity, these two notions of regularity coincide; however, if L is a three element chain, one can show that $\mathfrak{S}(L)$ is regular in the sense of von Neumann but it is neither left nor right regular. Our definition of strong regularity is a generalization of the notion of **-regularity* introduced by D. J. Foulis in (**2**, p. 893). In fact Foulis proved (**2**, Theorem 10, p. 894) that an orthomodular lattice is modular if and only if it can be coordinatized by a *-regular Baer *-semigroup. This leads us to conjecture that a poset with 0 and 1 is a complemented modular lattice if and only if it can be coordinatized by a strongly regular Baer semigroup. We begin with the observation that Lemma 3.1 can now be stated as follows: *Every left regular element of a Baer semigroup is range-closed. An idempotent e is range-closed if and only if it induces a left regular residuated mapping ϕ_e.* It follows from this that every left regular Baer semigroup S is range-closed. Then by Theorem 3.10, $\mathfrak{L}(S)$ is a complemented modular lattice. Thus, in order to prove our conjecture, it will suffice to prove that if L is a complemented modular lattice, $\mathfrak{S}_{\mathrm{SRC}}(L)$ is strongly regular. Our goal in this section will be to establish this fact.

Let us first consider a slightly different question. Let ϕ be a bilaterally range-closed element of $\mathfrak{S}(L)$, where L is a lattice with 0 and 1. We would like to find a bilaterally range-closed element ψ such that $\phi = \phi\psi\phi$ and $\psi = \psi\phi\psi$. Suppose for the moment that such a ψ can be found. Evidently $\psi\phi$ and $\phi\psi$ are idempotents. Let $a \leqslant 1\psi\phi$. Then $a \leqslant 1\phi$ so that $a = b\phi$ for some $b \in L$, and $a = b\phi = b\phi\psi\phi = a\psi\phi$. By symmetry and duality, it follows that both $\phi\psi$ and $\psi\phi$ are bilaterally range-closed. By Theorem 3.5, we can write $\psi\phi = \phi_{e',e}$ and $\phi\psi = \phi_{f',f}$ where $e' \oplus e = f' \oplus f = 1$. We also see that

$$e = 1\psi\phi \leqslant 1\phi = 1\phi\psi\phi = 1\phi\phi_{e',e} \leqslant 1\phi_{e',e} = e,$$

so $1\phi = e$. Similarly, $1\psi = f$, $0\psi^+ = e'$, and $0\phi^+ = f'$. Even more can be said. For if we make use of the fact that ψ is bilaterally range-closed, we have that

$$\psi = \psi\psi^+\psi = \psi(\psi^+\phi^+\psi^+)\psi = (\psi\psi^+)\phi^+(\psi^+\psi),$$

whence

$$a\psi = [(a \vee 0\psi^+)\phi^+] \wedge 1\psi = [(a \vee e')\phi^+] \wedge f$$

where $e' \oplus 1\phi = 0\phi^+ \oplus f = 1$. This suggests the following definition for ψ: assume the existence of elements e, f such that $e \oplus 1\phi = 0\phi^+ \oplus f = 1$. Then let

$$a\psi = [(a \vee e)\phi^+] \wedge f \quad \text{and} \quad a\psi^+ = [(a \wedge f)\phi] \vee e.$$

We would, of course, like to show that $\psi \in \mathfrak{S}(L)$ with ψ^+ as its residual mapping. In the next few lemmas the terminology will be as introduced here.

LEMMA 4.1. *$f\phi = 1\phi$ and $e\phi^+ = 0\phi^+$.*

Proof. Since $f \vee 0\phi^+ = 1$, $1\phi = (f \vee 0\phi^+)\phi = f\phi \vee 0\phi^+\phi = f\phi$. The fact that $e\phi^+ = 0\phi^+$ follows dually.

LEMMA 4.2. $(a\phi \vee e)\phi^+ = a \vee 0\phi^+$; $(a\phi^+ \wedge f)\phi = a \wedge 1\phi$.

Proof. By duality we shall only consider the first formula. By Lemma 3.2*,

$$a \vee 0\phi^+ = a\phi\phi^+ = [(a\phi \vee e) \wedge 1\phi]\phi^+$$
$$= (a\phi \vee e)\phi^+ \wedge 1\phi\phi^+ = (a\phi \vee e)\phi^+.$$

LEMMA 4.3. *ψ is a bilaterally range-closed element of $\mathfrak{S}(L)$ with ψ^+ as its residual map.*

Proof. By Lemma 4.2,

$$a\psi\psi^+ = \{[(a \vee e)\phi^+] \wedge f\}\psi^+ = \{[((a \vee e)\phi^+) \wedge f]\phi\} \vee e$$
$$= [(a \vee e) \wedge 1\phi] \vee e = a \vee e \geqslant e.$$

Here we have used the fact that $M^*(1\phi, e)$ holds. A dual argument shows that $a\psi^+\psi = a \wedge f \leqslant a$.

LEMMA 4.4. (i) $\phi\psi = \phi_{0\phi^+,f}$; (ii) $\phi\psi\phi = \phi$; (iii) $\psi\phi = \phi_{e,1\phi}$; (iv) $\psi = \psi\phi\psi$.

Proof. (i) By Lemma 4.2, $a\phi\psi = [(a\phi \vee e)\phi^+] \wedge f = (a \vee 0\phi^+) \wedge f$.
(ii) By (i), $a\phi^+\psi^+\phi^+ = a\phi^+\phi^+_{0\phi^+,f} = (a\phi^+ \wedge f) \vee 0\phi^+ = a\phi^+$.
(iii) $a\psi\phi = \{[(a \vee e)\phi^+] \wedge f\}\phi = (a \vee e) \wedge 1\phi$.
(iv) $a\psi\phi\psi = a\psi\phi_{0\phi^+,f} = (a\psi \vee 0\phi^+) \wedge f = a\psi$ since $1\psi = f$.

Combining the above results, we have

THEOREM 4.5. *Let ϕ be a bilaterally range-closed element of $\mathfrak{S}(L)$. The necessary and sufficient condition that there exist a bilaterally range-closed element ψ such that $\phi = \phi\psi\phi$ and $\psi = \psi\phi\psi$ is that there exist elements $e, f \in L$ such that $e \oplus 1\phi = 0\phi^+ \oplus f = 1$. Indeed, there is a one–one correspondence between the elements ψ and ordered pairs (e, f) such that $e \oplus 1\phi = 0\phi^+ \oplus f = 1$.*

Let us now relate all of this to the case where L is a complemented modular lattice. Theorem 4.5 then asserts that for $\phi \in \mathfrak{S}(L)$ strong regularity is equivalent to being strongly range-closed, whence $\mathfrak{S}_{\mathrm{SRC}}(L)$ is a strongly range-closed Baer semigroup which coordinatizes L. We summarize our results in the next theorem.

THEOREM 4.6. *Let L be a poset with 0 and 1. The following conditions are then equivalent:*

(i) *L is a complemented modular lattice.*
(ii) *L can be coordinatized by a strongly regular Baer semigroup.*
(iii) *L can be coordinatized by a left regular Baer semigroup.*
(iii*) *L can be coordinatized by a right regular Baer semigroup.*
(iv) *L can be coordinatized by a strongly range-closed Baer semigroup.*
(v) *L can be coordinatized by a range-closed Baer semigroup.*
(v*) *L can be coordinatized by a dual range-closed Baer semigroup.*

5. Algebraic equivalence. *In this section S will denote a Baer semigroup.* We shall say that two elements Se, Sf of $\mathfrak{L}(S)$ are *algebraically equivalent* and write $Se \sim_a Sf$ if there exist idempotents e_1, f_1 and elements x, y of S such that $Se = Se_1$, $Sf = Sf_1$, $xy = e_1$, and $yx = f_1$ The elements x, y are said to *implement* the algebraic equivalence of Se and Sf. If e and f are elements of a lattice with 0 and 1, we shall say that e is *algebraically equivalent* to f if the corresponding elements of $\mathfrak{L}(\mathfrak{S}(L))$ are algebraically equivalent, under the natural isomorphism described in (**3**, Theorem 15, p. 90). Before proceeding, we might mention that by (**5**, Hilfsatz 1.4, p. 159) two principal left ideals of a continuous regular ring are perspective if and only if they are algebraically equivalent. Since algebraic equivalence is clearly an equivalence relation, one would hope that a careful study of this type of equivalence might eventually give some clue as to just when perspectivity is transitive in a complemented modular lattice.

LEMMA 5.1. *Let e, f be idempotent elements of S with $Se, Sf \in \mathfrak{L}(S)$. If $xy = e$ and $yx = f$, set $x_1 = exf$ and $y_1 = fye$. Then*

$$x_1 y_1 = e, \quad y_1 x_1 = f, \quad x_1 = ex_1 f, \quad y_1 = fy_1 e, \quad LR(x_1) = Sf, \quad \text{and } LR(y_1) = Se.$$

Furthermore, if x is idempotent, so is x_1.

Proof. Notice that $x_1 y_1 = exfye = exyxye = e^4 = e$. Similarly, one has $y_1 x_1 = f$. It is obvious that $x_1 = ex_1 f$ and $y_1 = fy_1 e$. In order to show that $Sf = LR(x_1)$, it is enough to show that $R(f) = R(x_1)$. But this is immediate since $fz = 0 \Rightarrow x_1 z = x_1 fz = 0$ and $x_1 z = 0 \Rightarrow fz = y_1 x_1 z = 0$. A similar argument shows $Se = LR(y_1)$. Finally, if $x = x^2$, we have

$$x_1^2 = exfexf = exyxxyxf = exyxyxf = e^3xf = exf = x_1,$$

thus completing the proof.

LEMMA 5.2. *Let e, f be idempotent elements of S. If $Se, Sf \in \mathfrak{L}$ with $Se \sim_a Sf$, then the intervals $\mathfrak{L}(0, Se)$ and $\mathfrak{L}(0, Sf)$ are isomorphic.*

Proof. By Lemma 5.1 we may assume the existence of elements x, y of S such that $xy = e$, $yx = f$, $x = exf$, and $y = fye$. Let α denote the restriction of the induced mapping ϕ_x to $\mathfrak{L}(0, Se)$ and β the restriction of ϕ_y to $\mathfrak{L}(0, Sf)$. Since

$$(Se)\alpha = LR(ex) = LR(x) = Sf$$

and $(Sf)\beta = Se$, we see that

$$\alpha\colon \mathfrak{L}(0, Se) \to \mathfrak{L}(0, Sf) \quad \text{and} \quad \beta\colon \mathfrak{L}(0, Sf) \to \mathfrak{L}(0, Se)$$

are each monotone mappings. The proof is completed by noting that if $Sa \leqslant Se$, then

$$(Sa)\alpha\beta = (Sa)\phi_x\,\phi_y = (Sa)\phi_{xy} = (Sa)\phi_e = LR(a) = Sa.$$

Similarly, if $Sb \leqslant Sf$, then $(Sb)\beta\alpha = Sb$. It follows that α is an isomorphism of $\mathfrak{L}(0, Se)$ onto $\mathfrak{L}(0, Sf)$ having β as its inverse.

It is of some interest to note that a "converse" of the above lemma is also true.

THEOREM 5.3. *Let L be a lattice with* 0 *and* 1. *The interval $L(0, e)$ is isomorphic to $L(0,f)$ if and only if $e \sim_a f$.*

Proof. In view of Lemma 5.2 and the proof of Lemma 2.2, it suffices to show that if $L(0, e)$ is isomorphic to $L(0,f)$, then $\mathfrak{S}\theta_e \sim_a \mathfrak{S}\theta_f$, where θ_e and θ_f are as defined in Lemma 2.1. Accordingly, let $\alpha\colon L(0, e) \to L(0,f)$ be an isomorphism with $\beta\colon L(0,f) \to L(0, e)$ as its inverse. Define ϕ and ψ by the formulas $\phi = \theta_e\,\alpha$ and $\psi = \theta_f\,\beta$. Clearly, $\phi\psi = \theta_e$ and $\psi\phi = \theta_f$, so we need only to show that ϕ and ψ are elements of $\mathfrak{S}(L)$. By symmetry, it suffices to deal with ϕ. We claim that ϕ^+ is defined as follows: $a\phi^+ = 1$ if $a \geqslant f$ and $a\phi^+ = (a \wedge f)\beta$ otherwise. If $a < e$, then

$$a\phi\phi^+ = a\theta_e\,\alpha\phi^+ = a\alpha\phi^+ = a\alpha\beta = a \geqslant a;$$

otherwise,

$$a\phi\phi^+ = a\theta_e\,\alpha\phi^+ = e\alpha\phi^+ = f\phi^+ = 1 \geqslant a.$$

Hence $a \leqslant a\phi\phi^+$ for all $a \in L$. Now if $b \geqslant f$, then

$$b\phi^+\phi = 1\phi = 1\theta_e\,\alpha = e\alpha = f \leqslant b;$$

otherwise,

$$b\phi^+\phi = (b \wedge f)\beta\phi = (b \wedge f)\beta\theta_e\,\alpha = (b \wedge f)\beta\alpha = b \wedge f \leqslant b.$$

This completes the proof that ϕ^+ is residual to ϕ.

For the remainder of this section it will be assumed that L is a complemented modular lattice. We shall first show that if we wish to study algebraic equivalence in such a lattice, we may restrict our attention to $\mathfrak{S}_{\mathrm{SRC}}(L)$. Following this, we shall characterize both perspectivity and projectivity in terms of suitable algebraic equivalences. It will prove convenient to adopt the convention that for each $e \in L$, e' represents some (fixed) complement of e.

THEOREM 5.4. *Given $e, f \in L$, the intervals $L(0, e)$ and $L(0,f)$ are isomorphic if and only if $e \sim_a f$ with the equivalence implemented by elements of $\mathfrak{S}_{\mathrm{SRC}}(L)$.*

Proof. Let $\alpha\colon L(0, e) \to L(0,f)$ be an isomorphism with $\beta\colon L(0,f) \to L(0, e)$ as its inverse. Set $\phi = \phi_{e',e}\,\alpha$ and $\psi = \phi_{f',f}\,\beta$. Evidently $\phi\psi = \phi_{e',e}$ and

$\psi\phi = \phi_{f',f}$ so that we need only show that $\phi, \psi \in \mathfrak{S}_{\mathrm{SRC}}(L)$. By symmetry, we shall only deal with ϕ. We claim that its residual mapping ϕ^+ is defined by the formula $a\phi^+ = (a \wedge f)\beta \vee e'$. To see this, note that

$$a\phi\phi^+ = (a\phi_{e',e}\alpha\beta) \vee e' = a\phi_{e',e} \vee e' = a \vee e' \geqslant a,$$

while

$$a\phi^+\phi = [(a \wedge f)\beta \vee e']\phi_{e',e}\alpha = (a \wedge f)\beta\phi_{e',e}\alpha = (a \wedge f)\beta\alpha = a \wedge f \leqslant a.$$

This shows that ϕ is a bilaterally range-closed element of $\mathfrak{S}(L)$ and since L is modular, we see that ϕ is indeed strongly range-closed. The converse is obvious.

We are now ready to look at perspectivity and projectivity in L. Two elements $e, f \in L$ are called *perspective* (and denoted $e \sim f$) if they have a common complement in L. They are called *projective* if there exist finitely many elements $g_0, g_1, \ldots, g_n$ such that $g_0 = e, g_n = f$, and $g_i \sim g_{i+1}$ for $i = 0, 1, \ldots, n - 1$.

LEMMA 5.5. *Suppose that $\phi, \psi \in \mathfrak{S}(L)$ have the property that*

$$\phi\psi = \phi_{e',e}, \quad \psi\phi = \phi_{f',f}, \quad \phi = \phi_{e',e}\phi\phi_{f',f} \quad \textit{and}\ \psi = \phi_{f',f}\psi_{e',e}.$$

Then

$$1\psi = e, \quad 1\phi = f, \quad 0\phi^+ = e', \quad \textit{and}\ 0\psi^+ = f'.$$

Proof. We have $e = 1\phi\psi \leqslant 1\psi = 1\psi\phi_{e',e} \leqslant e$. The remaining assertions follow similarly.

LEMMA 5.6. *Let $\phi, \psi \in \mathfrak{S}_{\mathrm{SRC}}(L)$ with $\psi = \psi^2$. If $\phi\psi = \phi_{e',e}$ and $\psi\phi = \phi_{f',f}$, then $e \sim f$.*

Proof. By virtue of Lemma 5.1, we may assume that $\phi = \phi_{e',e}\phi\phi_{f',f}$ and $\psi = \phi_{f',f}\psi\phi_{e',e}$. Then by Lemma 5.5, $1\psi = e$, $1\phi = f$, and $0\psi^+ = f'$. Since $\psi \in \mathfrak{S}_{\mathrm{SRC}}(L)$, we have from Theorem 3.5 that 1ψ and $0\psi^+$ are complements. Hence f' is a complement of both f and e.

THEOREM 5.7. *Two elements $e, f \in L$ are perspective if and only if they are algebraically equivalent with the equivalence implemented by strongly range-closed idempotents.*

Proof. If g is a common complement for e and f, then $\phi_{g,e}\phi_{g,f} = \phi_{g,f}$ and $\phi_{g,f}\phi_{g,e} = \phi_{g,e}$. The converse follows from Lemma 5.6.

THEOREM 5.8. *Two elements $e, f \in L$ are projective if and only if $e \sim_a f$ with the equivalence implemented by mappings, each of which is the product of a finite number of strongly range-closed idempotents.*

Proof. If e is projective to f, the required algebraic equivalence can be obtained by repeated applications of Theorem 5.7. Suppose conversely that $\phi \in \mathfrak{S}_{\mathrm{SRC}}(L)$, $\phi\psi = \phi_{e',e}$, and $\psi\phi = \phi_{f',f}$ where $\psi = \psi_1\psi_2 \ldots \psi_k$ with each ψ_i

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a strongly range-closed idempotent. We may clearly suppose that $\phi = \phi_{e',e}\phi\phi_{f',f}$ and $\psi = \phi_{f',f}\psi\phi_{e',e}$. Then let

$$\theta_0 = \phi\psi_1\psi_2 \ldots \psi_k, \quad \theta_1 = \psi_k\phi\psi_1\psi_2 \ldots \psi_{k-1}, \quad \ldots, \quad \theta_{k-1} = \psi_2 \ldots \psi_k\phi\psi_1,$$
$$\text{and} \quad \theta_k = \psi_1\psi_2 \ldots \psi_k\phi.$$

Routine computation shows that each θ_j is idempotent and therefore of the form $\theta_j = \phi_{g_j',g_j}$. By Lemma 5.6,

$$e = g_0 \sim g_1 \sim \ldots \sim g_{k-1} \sim g_k = f,$$

thus showing that e is projective to f.

COROLLARY. *Let ϕ be the product of a finite number of strongly range-closed idempotents. Then any complement of $0\phi^+$ is projective to 1ϕ.*

Proof. Let f be a complement of $0\phi^+$ and e a complement of 1ϕ. By Lemma 4.4, there exists a strongly range-closed element ψ such that $\phi\psi = \phi_{0\phi^+,f}$ and $\psi\phi = \phi_{e,1\phi}$. A glance at the proof of the above theorem now establishes the corollary.

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