1

Introduction

In this chapter we will give the reader a general overview of our themes and motivations. We will give a brief outline of what functional approximations are for, especially multivariate ones, what kind of approximation methods are typical, and which ones we shall study in this book and why.

In many mathematical and scientific applications, approximation of functions of possibly very many variables (unknowns) is often needed because the theoretically known function is in fact too complicated to evaluate (especially when it would have to be evaluated very many times and/or the number of unknowns is very high) or in fact not known at all in the application except at a few points. Those could be predefined or not initially available. On top of this, the given data may be inaccurate or noisy, to a level often estimated in advance. Therefore the question arises how to approximate univariable or multivariable functions efficiently in such circumstances.

Interpolation and quasi-interpolation are both highly useful means of approximating functions and data in multivariable dimensions, say, for the notation in this book, in \( \mathbb{R}^n \). They are useful methods that approximate from spaces spanned for example by polynomials, piecewise polynomials, trigonometric polynomials and exponentials or radial basis functions.

Interpolation and generally approximation using radial basis functions are very good examples since they have become a well-known and appreciated tool for approximating multivariate functions, especially when the dimensions \( n \) are really large, for which polynomial interpolation and approximation from polynomial spaces become difficult and very much depend on the geometry of the data. By contrast, the success of radial basis function approximation is linked in particular to the available variety of approximations of different kinds from the vector spaces spanned by the translates of the radial basis functions.

We point out that the most important choice at the beginning of the approximation procedure is the selection of the space of approximants.
Important features are its approximation power (highly useful convergence properties and error estimates are available, as will be seen with a special emphasis in this book), its applicability in basically any space dimension – any number of variables – and its simplicity in formulating and stating the approximation (or interpolation) problem.

Despite wishing to avoid pointwise collocation and its potential disadvantages, and therefore using quasi-interpolation in the end, much of the theoretical analysis began using interpolants from radial basis function spaces. That is, we wish to meet the approximand (the function \( f \) that is being approximated) at a number (sometimes infinite, sometimes gridded, in practice of course finite) of given points \( \xi \in \mathbb{R}^n \).

Returning to the general concepts, during the past few decades quasi-interpolation has become a particularly popular approach in approximation theory, especially for smoothing purposes (e.g. when given data are noisy) and in contrast to interpolation schemes. The approach we take in this book uses a quasi-Lagrange function in linear combinations with coefficients that are derived from the approximand by linear operators, most often point evaluation, in such a way that certain low-degree polynomials are not just approximated but recovered exactly. Often this type of reproduction is only required for the leading monomial term of the approximation. On top of this, we do not always require point evaluation of approximands to form the quasi-interpolants, but also admit more general operators applied to the approximands (e.g. local integrals or derivative evaluations) before they are used to formulate the quasi-interpolating approximation.

These fundamental ideas can replace pointwise interpolation, and the approach should lead to approximation orders when sufficiently smooth general functions are approximated under suitable conditions (such as localness of the said quasi-Lagrange functions).

But now let us return for a moment specifically to the approach using radial basis functions.

We will speak about more general approaches to quasi-interpolation later on; this, however, is a good way to start explaining these ideas. This is because we note another highly important feature of radial basis functions, namely that for large well-known classes of radial basis functions the interpolation problem is uniquely solvable, sometimes with no or at most some very easily verifiable conditions on the mentioned points and their geometry, and thus completely different from the far more complicated case of multivariable polynomial interpolation, for example, where the geometry of the interpolation points is of central importance to the unique solvability of the interpolation problem. This is true, for example, for the multiquadric or inverse multiquadric radial basis function specified below, which in many applications and parts of the literature is the prime example of approximation
with radial basis functions, others being inverse multiquadrics, Gaussian and Poisson kernels and thin-plate splines. We note already at this point that other than in straight polynomial interpolation, for example, the linear spaces used here to form the approximants depend on the initial points, also called ‘centres’ as well, because we take the radial basis functions and shift them by these centres (thus they are radially symmetric about these points, which explains their name).

Initial examples of radial basis function quasi-interpolants that will turn out to be useful are the aforementioned multiquadrics \( \phi(r) = \sqrt{r^2 + c^2} \) for a real parameter \( c \) that is explicitly allowed to be zero, or thin-plate splines \( \phi(r) = r^2 \log r \). The former case is an example (of which there are many) where unique interpolants are guaranteed when the points are distinct and there are at least two of them. This is the simplest conceivable condition. There are some easy linear side conditions, and conditions requiring that the data points do not lie all on a straight line, that give the same property to thin-plate splines.

So for multiquadrics and any other radial basis function that immediately gives non-singularity to the interpolation problem with finitely many distinct centres \( \xi \in \Xi \) (such as inverse multiquadrics, Gaussian or Poisson kernels, or multiquadrics with \( c = 0 \) ‘linears’; see Buhmann, 2003), we formulate the search for an interpolant of the type

\[
s(x) = \sum_{\xi \in \Xi} \lambda_\xi \phi(||x - \xi||), \quad x \in \mathbb{R}^n,
\]

by solving the linear system of equations

\[
\sum_{\xi \in \Xi} \lambda_\xi \phi(||\zeta - \xi||) = f(\zeta), \quad \zeta \in \Xi,
\]

with the interpolation matrix

\[
\{\phi(||\zeta - \xi||)\}_{\zeta, \xi \in \Xi}.
\]

In the case of extra conditions such as with thin-plate splines, this would be the search for an interpolant

\[
 s(x) = \sum_{\xi \in \Xi} \lambda_\xi \phi(||x - \xi||) + p(x), \quad x \in \mathbb{R}^n,
\]

with a polynomial of degree one in the thin-plate spline case and extra conditions

\[
\sum_{\xi \in \Xi} \lambda_\xi q(\xi) = 0
\]

for all at most linear polynomials \( q \). Such extra conditions are not unknown for univariate splines either; even straightforward natural cubic splines
\[ s(x) = \sum_{\xi \in \Xi} \lambda_{\xi} \phi(|x - \xi|) + cx + d, \quad x \in \mathbb{R}, \]

with \( \phi(r) = r^3 \) enabling us to express the spline as a radial basis function, and with real constants \( c \) and \( d \), demand side conditions for uniqueness, namely second derivatives of \( s \) vanishing at the first and last knot.

While interpolation is sometimes explicitly preferred or demanded, as it reproduces the data values at the interpolation points exactly, in many circumstances quasi-interpolation is to be preferred because it has other reproducing properties, is easier to compute, and possesses a tendency to smooth data, which is frequently desired. Quasi-interpolation forms the approximant \( Qf \) from the (data) function (approximand) \( f \) by building sums of kernel functions, which we call ‘quasi-Lagrange functions’ \( \psi \), shifted as \( \psi(\cdot - \xi) \) or in other ways depending on the data points \( \psi_{\xi} \in \mathbb{R}^n \) multiplied by the function values \( f(\xi) \), their derivatives, integrals or other linear operators \( \lambda_{\xi} f \) applied to them.

The interpolation property is now replaced by other conditions, typically of \( \psi \) or \( \psi_{\xi} \) being from the space spanned by \( \phi(\|\cdot - \xi\|) \) but restricted to locally supported or decaying functions, and that the quasi-interpolation operator be exact (reproducing) on certain function spaces, so that approximants are linear combinations of such \( \psi \) functions. For these, polynomials of a fixed maximal total degree in \( n \) unknowns is a typical example. The latter opens the door to powerful convergence results for sufficiently smooth approximands \( f \) by Taylor series or related arguments.

In summary, the difference between interpolation and quasi-interpolation is that an approximation from a finite- or infinite-dimensional linear space \( S \) spanned by basis functions \( \phi_{\xi}, \xi \in \Xi \), need not satisfy pointwise interpolation conditions such as

\[ s(\zeta) = f(\zeta) \quad \text{for all } \zeta \in \Xi, \]

but the approximants are essentially from the same spaces.

We now have new means to approximate certain approximands well that are smooth enough, from certain Sobolev spaces, etc. The basis functions are no longer Lagrange functions but ‘bell-shaped’ quasi-Lagrange functions. Therefore, in particular, approximations can be computed without solving linear (interpolation equations) systems of equations. Moreover, it will turn out that convergence estimates are simpler to achieve with quasi-interpolation when they are employed in place of interpolation, as in particular there are no complicated estimates of operator norms (Lebesgue constants).

We have used the elements \( \xi \in \Xi \), for the time being, as indices for the basis functions, but in due course it will be seen that they are inherently related to the linear spaces spanned by radial basis functions. However, giving up the explicit
interpolation conditions raises the need for other conditions that fix the approximant and avoid trivial choices. In particular, we note explicitly that formulating quasi-interpolants via linear combinations

\[ \sum_{\xi \in \Xi} f(\xi) \phi(\|x - \xi\|) \]

is normally useless because the radial basis functions are not at all local with respect to space (except for the special cases of Gaussian or Poisson kernels or compactly supported radial basis functions). The above approximation would provide no locality, and normally approximations to smooth functions that are locally described by Taylor expansions, for instance, will not be any good. Spline approximations, however, do give compact support (localness) and some polynomial recovery, but are much harder to formulate in more than one or two dimensions, partly due to their piecewise structure. We shall see that, nonetheless, at least in one dimension quasi-interpolation is related to approximation by piecewise polynomials.

In contrast to the standard approach of interpolation, incidentally, approximations created by quasi-interpolants are normally not unique, which usually has little impact on practical applications.

It is also often much more difficult to estimate the desired Lebesgue constant (operator norms) for interpolation, whereas for quasi-interpolants this is usually straightforward.

We should point out that also other approximation methods from spaces spanned by radial basis functions are possible, that is, neither interpolation nor quasi-interpolation but wavelets (Buhmann, 2003), prewavelets, compression or smoothing splines with generalised cross-validation (Wahba and Wendelberger, 1980), for example.

We address some of these ideas at the end of the book.

Having given the reasons why we consider the approximations by quasi-interpolation to be useful and interesting, we shall proceed to some general remarks, univariate approximations by quasi-interpolation, and then come to multivariable methods.