Note on γ-Matrices Efficient at an Isolated Point

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In a recent paper [1] γ-matrices were constructed summing the binomial series at isolated points *to its right value.* It is assumed that the reader is able to refer to this paper and is familiar with its notation and terminology.

The object of this note is to give the construction of γ-matrices which sum the binomial series at an isolated point *to an arbitrary given value,* while inside the circle of convergence the generalised sum is necessarily the right value.

We first consider the series Σz^k at $z = z_0$, where $|z_0| > 1$. If a is an arbitrary complex number, and $b = a - 1/(1 - z_0)$, we construct the matrix

$$G \equiv (g_{n,k}), \qquad n, k = 0, 1, 2, \ldots,$$

given by $g_{n,k} = 1$ for $k < n$, $= 1/(1 - z_0)$ for $k = n$, $= b/2^{k-n} z_0^k$ for $k > n$.

Obviously G is a γ-matrix, since $g_{n,k} \to 1$ as $n \to \infty$, and

$$\sum_k |g_{n,k} - g_{n,k+1}| < 1 + 2/|1 - z_0| + 2|b| \Sigma |2z_0|^{-k} \quad \text{for every } n.$$

Applying G to the series Σz_0^k, we have

$$S_n = \sum_k g_{n,k} z_0^k = 1 + z_0 + z_0^2 + \ldots + z_0^{n-1} + z_0^n/(1 - z_0) + b/2 + b/4 + \ldots$$
$$= 1/(1 - z_0) + b = a,$$

so that $S_n \to a$.

For other values of z, $S_n(z) = \sum_k g_{n,k} z^k$ converges only if $|z| < 2|z_0|$, and then

$$S_n(z) = \frac{1}{1 - z} + \frac{(z_0 - z) z^n}{(1 - z)(1 - z_0)} + \frac{b z^{n+1} z_0}{(2z_0 - z) z_0^{n+1}},$$

which tends to a limit only if $z = z_0$ or $|z| < 1$.

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[1] P. Vermes, "On γ-matrices and their application to the binomial series," *Proc. Edinburgh Math. Soc.* (2), 8 (1947), 1-13 (11-13).

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In the above construction we used the matrix $H(1, z_0)$ of the previous paper, replacing its zero elements by suitable numbers. Given the series $\Sigma c_k z^k$ representing $(1-z)^{-p}$, where p is a positive integer, we replace the zero elements of the matrix $H(p, z_0)$ in a similar way. We then have

$$g_{n,k} = (1-z_0)^{-p} \sum_{j=0}^{n-k} \binom{p}{j} (-z_0)^j \text{ for } k \leqq n, \quad = b/c_k\, z_0^{k} 2^{k-n} \text{ for } k > n,$$

where $b = a - 1/(1-z_0)^p$, and G has the required properties.

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