A CLASS OF TWO GENERATOR TWO RELATION FINITE GROUPS

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1. Introduction

A group which is minimally generated by n generators and defined by n relations is said to have zero deficiency. The class of finite groups known to have zero deficiency is small, consisting of cyclic groups, certain metacyclic groups [4] and classes of groups given in [1], [2] and [3].

In this paper we give a class of 2 generator 2 relation finite groups which does not appear to be contained in the known classes. The class of groups considered is denoted $\{G(\alpha,\beta,\gamma)\}$, with presentation

$$G(\alpha,\beta,\gamma)=\{a,b \mid bab^{-1}=a^{\alpha}b^{\beta},\ a^{-1}b^{\beta}a=b^{\gamma\beta}\},$$

which is shown to be finite for $\alpha,\gamma>1$. Also we show that each element of $G(\alpha,\beta,\gamma)$ may be written as $a^r b^s$ for suitable r and s but that $G(\alpha,\beta,\gamma)$ is in general not metacyclic.

2. Finiteness of $G(\alpha, \beta, \gamma)$

The relations are

(1) $$bab^{-1}=a^{\alpha}b^{\beta}, \qquad \alpha>1 \qquad \text{and}$$

(2) $$a^{-1}b^{\beta}a=b^{\gamma\beta}, \qquad \gamma>1.$$

From (2) we have immediately

(3) $$a^{-r}b^{s\beta}a^{r}=b^{t}, \quad t=s\beta\gamma^{r} \quad \text{for} \quad r>0 \text{ giving}$$

(4) $$\begin{aligned} b^{\beta+1}a &= b(b^{\beta}a)=(ba)b^{\gamma\beta} = a^{\alpha}b^{1+\beta+\gamma\beta} \\ &= b^{\beta}(ba)=(b^{\beta}a^{\alpha})b^{1+\beta} = a^{\alpha}b^{1+\beta+\beta\gamma^{\alpha}} \quad \text{whence} \end{aligned}$$

(5) $$b^{\gamma\beta(\gamma^{\alpha-1}-1)}=1,$$

and since $a\, b^{\gamma\beta}a^{-1}=b^{\beta}$ then

$$b^{\beta(\gamma^{\alpha-1}-1)} = 1. \tag{6}$$

Consider the normal subgroup $B = \langle b^{\beta} \rangle$ of $G(\alpha, \beta, \gamma)$. Let $G_1 = G/B, a_1 = aB$, $b_1 = bB$. Then G_1 has presentation

$$\{a_1, b_1 \,|\, b_1 a_1 b_1^{-1} = a_1^{\alpha},\ b_1^{\beta} = 1\}$$

and is therefore metacyclic of order $\beta(\alpha^{\beta} - 1)$.
Since each element of G_1 has the form

$$a_1^{r_1} b_1^{s_1}\ (0 \leqq r_1 < \alpha^{\beta} - 1,\ 0 \leqq s_1 < \beta),$$

each element of G has the form

$$a^r b^s (0 \leqq r < \alpha^{\beta} - 1,\ 0 \leqq s < \beta\,(\gamma^{\alpha-1} - 1)).$$

Hence G is finite of order $\leqq (\alpha^{\beta} - 1)\ \beta(\gamma^{\alpha-1} - 1)$.

3. An example of $G(\alpha, \beta, \gamma)$

To show that $G(\alpha, \beta, \gamma)$ is in general not metacyclic we look more closely at

$$G(m, m-1, m).$$

Consider the set, G, of ordered pairs (α, β) with $0 \leqq \alpha,\ \beta < n$ where $n = m^{m-1} - 1$.

Define multiplication by

$$(\alpha, \beta)\,(\gamma, \delta) = (x, y) \text{ where}$$

$$x \equiv \alpha + \gamma m^{\beta} \text{ modulo } n \text{ and} \tag{7}$$

$$y \equiv \delta + \beta m^{\gamma} \text{ modulo } n.$$

Then G becomes a group generated by (1,0) and (0,1) with

$$(0,1)\,(1,0)(0,1)^{-1} = (1,0)^m(0,1)^{m-1} \text{ and}$$

$$(1,0)^{-1}(0,1)^{m-1}(1,0) = (0,1)^{m(m-1)} \text{ whence}$$

G is a factor group of $G(m, m-1, m)$.
Let $a = (0,1)$, $b = (1,0)$ then the elements

$$a^{-1}bab^{-1} = a^{\alpha-1}b^{\beta} = (m-1,\ m-1) \text{ and}$$

$$b^{-\beta}a^{-1}b^{\beta}a = b^{(\gamma-1)\beta} = (0,\ (m-1)^2)$$

lie in G'. Since

$$(m-1,\ m-1)^t = (t(m-1),\ t(m-1)) \text{ and}$$

$$(0,\ (m-1)^2)^t = (0,\ t(m-1)^2),$$

these elements generate trivially interesting cyclic subgroups of orders $n/(m-1)$ and $n/(m-1)^2$. Hence G' is not cyclic and so $G(m,\ m-1,\ m)$ cannot be metacyclic.

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References

[1] I. D. MacDonald, 'On a class of finitely presented groups', *Canad. J. Math.* 14 (1962), 602–613.

[2] J. Mennicke, 'Einige endlicke Gruppen mit drei Erzeugenden und drei Relationen', *Archiv der Math.* 10 (1959), 409–418.

[3] J. W. Wamsley, 'A class of three generator three relation finite groups', *Canad. J. Math.* 22 (1970), 36–40.

[4] J. W. Wamsley, 'The deficiency of metacyclc groups', *Proc. Amer. Math. Soc.* 24 (1970), 724–726.

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