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HYPERBOLIC MANIFOLDS WITH THE STRONGLY SHADOWING PROPERTY

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Let f be a C^1 diffeomorphism of a compact smooth manifold M and $\Lambda \subset M$ a C^1 compact invariant submanifold with a hyperbolic structure as a subset of M. We show that the diffeomorphism $f|_\Lambda$ is Anosov if and only if Λ has the strongly shadowing property, and find hyperbolic sets which have the strongly shadowing property.

Let M be a compact smooth manifold and $f : M \to M$ a C^1 diffeomorphism. A closed invariant set $\Lambda \subset M$ is said to be *hyperbolic* for f if $T_\Lambda M$ has a continuous splitting (Whitney sum decomposition) $T_\Lambda M = E^s \bigoplus E^u$ satisfying :

(1) E^s and E^u are invariant under the derivative map Tf;

(2) there exist constants $c > 0$ and $0 < \lambda < 1$ such that for all $n \in \mathbb{Z}^+$,

$$\max\{\|Tf^n|_{E^s}\|, \|Tf^{-n}|_{E^u}\|\} < c\lambda^n.$$

If the Riemannian metric on M is such that in (2) we can take $c = 1$, then the metric is called *adapted* to Λ. We say that $\Lambda \subset M$ is a *hyperbolic manifold* for f if Λ is a C^1 compact invariant submanifold of M with a hyperbolic structure as a subset of M. If M is hyperbolic for f then f is called *Anosov*.

The simplest examples of Anosov diffeomorphisms are the toral hyperbolic automorphisms, that is, Anosov automorphisms of the group $T^n = \mathbb{R}^n/\mathbb{Z}^n$. They are induced by linear automorphisms $L : \mathbb{R}^n \to \mathbb{R}^n$ with determinant one, integer entries, and no eigenvalues of absolute value one.

Hirsch asks in [**2**], if $\Lambda \subset M$ is a hyperbolic manifold for f, does it follow that f restricted to Λ is Anosov (has a hyperbolic structure)? The answer given by Franks and Robinson in [**1**] was negative. But the following stronger version of the problem remains open

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CONJECTURE. [**7**, Problem 10]. If $f : M \to M$ is an Anosov diffeomorphism and $\Lambda \subset M$ is a C^1 compact invariant submanifold of M then $f|_\Lambda : \Lambda \to \Lambda$ is Anosov.

The conjecture was proved in some cases by Mane [**5**] and Zeghib [**8**]. Mane showed that the conjecture is true if $M = T^n$; and Zeghib proved that if f is a geodesic flow on a compact negatively curved manifold M then the conjecture is true.

In this paper, we give some partial results in this direction. First we recall the concept of the shadowing property (or pseudo orbit tracing property) and decribe the so-called Shadowing Lemma which is the main result about shadowing near a hyperbolic set of a diffeomorphism.

DEFINITION 1: Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary constants. A sequence $\xi = \{x_i\}$ in M is called an *δ-pseudo orbit* for f if $d\big(f(x_i), x_{i+1}\big) < \delta$ for all i; ξ is said to be *ε-shadowed* by a point $x \in M$ if $d\big(f^i(x), x_i\big) < \varepsilon$ for all i. We say that a subset Λ of M has the *shadowing property* for f if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit in Λ is ε-shadowed by a point x in M.

LEMMA 2. (Shadowing Lemma) *If Λ is a hyperbolic set for f, then there exists a neighbourhood U of Λ which has the shadowing property.*

REMARKS 3. If Λ is a hyperbolic set for f, then the set Λ has the shadowing property by the Shadowing Lemma. Note that, in general, the shadowing point x need not belong to Λ.

DEFINITION 4: A subset Λ of M has the *strongly shadowing property* for f if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit $\xi = \{x_i\}$ in Λ is ε-shadowed by a point x in Λ.

Here we show that if Λ is a hyperbolic manifold for f with the strongly shadowing property then $f|_\Lambda$ is Anosov.

THEOREM 5. *Let $\Lambda \subset M$ be a hyperbolic manifold for f which has the strongly shadowing property. Then $f|_\Lambda : \Lambda \to \Lambda$ is Anosov.*

PROOF: Since Λ is hyperbolic, there exists $c > 0$ such that if $d\big(f^n(x), f^n(y)\big) < c$ for $x \in \Lambda$, $y \in M$ and all $n \in \mathbb{Z}$ then $x = y$ (in this case, $c > 0$ is called an expansive constant of Λ). Choose $0 < \delta < c$ such that any δ-pseudo orbit in Λ is $c/2$-shadowed by a point x in Λ. Let $0 < \alpha < \delta/2$ be such that $d(x,y) < \alpha$ implies $d\big(f(x), f(y)\big) < \delta/2$. Let U be a compact neighbourhood of Λ satisfying $U \subset B(\Lambda, \alpha/2)$.

First we show that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. In fact, it is clear that $\Lambda \subset \bigcap_{n\in\mathbb{Z}} f^n(U)$ since Λ is invariant. To show that $\bigcap_{n\in\mathbb{Z}} f^n(U) \subset \Lambda$, we let $y \in \bigcap_{n\in\mathbb{Z}} f^n(U)$. Then we have $y \in f^n(U)$ for all $n \in \mathbb{Z}$, and so $f^n(y) \in U$ for all $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, choose $x_n \in \Lambda$ such that $d\big(x_n, f^n(y)\big) < \alpha$. Then the sequence $\{x_n : n \in \mathbb{Z}\}$ is an δ-pseudo orbit for f. In fact, we have

$$\begin{aligned} d(f(x_n), x_{n+1}) &\le d\big(f(x_n), f^{n+1}(y)\big) + d\big(f^{n+1}(y), x_{n+1}\big) \\ &< \frac{1}{2}\delta + \alpha < \delta, \end{aligned}$$

for all $n \in \mathbb{Z}$. Since Λ has the strongly shadowing property, there is $x \in \Lambda$ such that $\{x_n\}$ is $c/2$-shadowed by the point x. Then we have

$$\begin{aligned} d\big(f^n(x), f^n(y)\big) &\le d\big(f^n(x), x_n\big) + d\big(x_n, f^n(y)\big) \\ &< \frac{1}{2}c + \alpha < c, \end{aligned}$$

for all $n \in \mathbb{Z}$. Consequently we get $x = y$, and so $y \in \Lambda$.

Next we show that $f|_\Lambda$ is structurally stable. Let $g \in \mathrm{Diff}^1(\Lambda)$ be C^1 near to $f|_\Lambda$. Then we can find $\bar{g} \in \mathrm{Diff}^1(M)$ such that

$$\bar{g} \text{ is } C^1 \text{ near to } f \text{ and } \bar{g}|_\Lambda = g.$$

If we apply [**3**, Theorem 7.3] which says that the maximal hyperbolic sets enjoy a type of structural stability, then we can find a homeomorphism $h : \bigcap_{n\in\mathbb{Z}} f^n(U) \to \bigcap_{n\in\mathbb{Z}} \bar{g}^n(U)$ such that

(1) $\bar{g} \circ h = h \circ f$ on $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and

(2) h is C^0 near to the identity map on Λ.

Since $\bar{g}(\Lambda) = \Lambda$, we have $\Lambda \subset \bigcap_{n\in\mathbb{Z}} \bar{g}^n(U)$. Put $k = h^{-1}|_\Lambda$. Since Λ is a compact manifold and k is C^0 near to the identity map on Λ, k is surjective. Hence we get $h(\Lambda) = h\big(k(\Lambda)\big) = \Lambda$. This means that $f|_\Lambda$ is structurally stable.

If we apply [**6**, Theorem 5], we can see that $f|_\Lambda : \Lambda \to \Lambda$ is Anosov. □

COROLLARY 6. *Let $\Lambda \subset M$ be a hyperbolic manifold for f. Then $f|_\Lambda : \Lambda \to \Lambda$ is Anosov if and only if Λ has the strongly shadowing property.*

Hyperbolic manifolds which do not have the strongly shadowing property can be found in the example given by Franks and Robinson [**1**].

Now we wish to find hyperbolic sets which have the strongly shadowing property. Let $C(f)$ be the Birkhoff centre of f; that is,

$$C(f) = \overline{\{x \in M : x \in \omega(x) \cap \alpha(x)\}},$$

where $\omega(x)$ and $\alpha(x)$ denote the positive and negative limit set of x. Then $C(f)$ is a nonempty closed invariant subset of M. We say that a point $x \in M$ is called *nonwandering* if for any neighbourhood U of x and an integer $n_0 > 0$ there exists an integer $n > n_0$ with $f^n(U) \cap U \neq \emptyset$. A point $x \in M$ is said to be *chain recurrent* if for any $\varepsilon > 0$ there exists an ε-pseudo orbit for f form x to x. The set of nonwandering points and the set of chain recurrent points of f will be denoted by $\Omega(f)$ and $\mathrm{CR}(f)$, respectively. Then we have the following inclusions:

$$\overline{\mathrm{Per}(f)} \subset C(f) \subset \Omega(f) \subset \mathrm{CR}(f).$$

THEOREM 7. *If the set $C(f)$ is hyperbolic for f then it has the strongly shadowing property.*

For the proof of the theorem, we need the following results which are satisfied by hyperbolic sets.

LEMMA 8. [**3**, Stable Manifold Theorem]. Let $\Lambda \subset M$ be a hyperbolic set for f. Then there exist constants $\varepsilon > 0$ and $0 < \lambda < 1$ such that for any $x \in \Lambda$

(1) $W^s_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \varepsilon \text{ for all } n \geq 0\}$ is a C^1 submanifold of M with $T_x W^s_\varepsilon(x) = E^s_x$;

(2) if $y, z \in W^s_\varepsilon(x)$ then $d(f^n(y), f^n(z)) \leq \lambda^n d(y,z)$ for all $n \geq 0$.

LEMMA 9. [**3**]. *Let $\Lambda \subset M$ be a hyperbolic set for f and $\varepsilon > 0$ a constant as in Lemma 8. Then,*

(1) *there exists a constant $\delta > 0$ such that;*
   - (i) *if $d(x,y) < \delta$, $x, y \in \Lambda$, then $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is a single point set;*
   - (ii) *the map $\theta : U_\delta(\Lambda) \to M$ given by $\theta(x,y) = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is continuous, where $U_\delta(\Lambda) = \{(x,y) \in \Lambda \times \Lambda : d(x,y) < \delta\}$; and*

(2) *there exists a constant $\delta' > 0$ such that if $d(x,y) < \delta'$, $x, y \in \Lambda$, then $d(x, \theta(x,y)) < \delta$ and $d(y, \theta(x,y)) < \delta$.*

The following lemma is the main theorem of Kato's paper [**4**].

LEMMA 10. *If $\Lambda \subset M$ is a hyperbolic set for f then there exists a neighbourhood U of Λ having the following property: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any map $g : M \to M$ which is injective on a subset ξ of $U \cap g^{-1}(U)$ and $d_0(f,g) < \delta$ there exists a map $h : U \to M$ satisfying the conditions*

(1) *$h \circ g = f \circ h$ on ξ; and*

(2) *$d_0(h, 1d_U) < \varepsilon$*

PROOF OF THE THEOREM 7: Let us apply Lemmas 8, 9 and 10 for the hyperbolic set $C(f)$. Then we obtain positive constants $\varepsilon, \lambda < 1, \delta, \delta'$ and a neighbourhood U of $C(f)$ in M which satisfy the results of the above lemmas and the inclusion $B(C(f), \varepsilon) \subset U$.

First we show that if $d(x,y) < \delta$, $x, y \in C(f)$, then the point $\theta(x,y)$ belongs to $C(f)$, where θ is the map obtained in Lemma 9. Let $\alpha > 0$ be arbitrary, $\theta(x,y) = z$ and $\theta(y,x) = w$. Since $z \in W_\varepsilon^s(x)$ and $x \in \omega(x)$, there exists $a > 0$ satisfying

$$d\big(f^a(z), f^a(x)\big) < \frac{1}{2}\alpha \quad \text{and} \quad d\big(f^{a+1}(x), x\big) < \frac{1}{2}\alpha.$$

Since $x \in \alpha(x)$ and $w \in W_\varepsilon^u(x)$, we can choose $b > 1$ such that

$$d\big(f(x), f^{-b}(x)\big) < \frac{1}{2}\alpha \quad \text{and} \quad d\big(f^{-b+1}(x), f^{-b+1}(w)\big) < \frac{1}{2}\alpha.$$

Since $w \in W_\varepsilon^s(y)$ and $y \in \omega(y)$, there is $c > 0$ satisfying

$$d\big(f^c(w), f^c(y)\big) < \frac{1}{2}\alpha \quad \text{and} \quad d\big(f^{c+1}(y), y\big) < \frac{1}{2}\alpha.$$

Since $y \in \alpha(y)$ and $z \in W_\varepsilon^u(y)$, we can get $d > 1$ such that

$$d\big(f(y), f^{-d}(y)\big) < \frac{1}{2}\alpha \quad \text{and} \quad d\big(f^{-d+1}(y), f^{-d+1}(z)\big) < \frac{1}{2}\alpha.$$

Then the sequence

$$\big\{z, f(z), \dots, f^{a-1}(z), f^a(x), x, f^{-b}(x), f^{-b+1}(w), \dots, f^{-1}(w), \\ w, f(w), \dots, f^{c-1}(w), f^c(y), y, f^{-d}(y), f^{-d+1}(z), \dots, f^{-1}(z), z\big\}$$

is a periodic α-pseudo orbit for f contained in U.

To show that $z \in C(f)$, we let $r > 0$ be arbitrary and $l > 0$ a constant which satisfy the results of Lemma 10. By the first step of the proof, we can choose a periodic $l/2$-pseudo orbit $\{z = z_0, z_1, \dots, z_n = z\}$ in U from z to z. Let $g : M \to M$ be a continuous map satisfying

(1) $g(z_i) = z_{i+1}$, for $0 \le i \le n-2$, and $g(z_{n-1}) = z$; and
(2) $d_0(f, g) < l$

If we apply Lemma 10, then we can choose a continuous map $h : U \to M$ satisfying the conditions

(1) $f \circ h = h \circ g$ on ξ; and
(2) $d_0(h, 1d_U) < r$.

Let $h(z) = \overline{z}$. Then we have

$$f^n(\overline{z}) = f^n\big(h(z)\big) = h\big(g^n(z)\big) = hg(z_{n-1}) = h(z) = \overline{z}.$$

This means that $z \in C(f)$.

Next we show that $C(f)$ has the strongly shadowing property. Let $\beta > 0$ be arbitrary. Then we can choose positive constants ε, $\lambda < 1$, δ and δ' satisfying the results of Lemmas 8 and 9, and assume that $2\delta/(1-\lambda) < \beta$ and $2\varepsilon < \delta' < \delta$. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be an ε-pseudo orbit in $C(f)$, and let $\xi_n^+ = \{x_0, x_1, \dots, x_n\}$. Set $y_0 = x_0$ and define y_k recursively by

$$y_k = \theta\big(x_k, f(y_{k-1})\big), \quad 0 \le k \le n.$$

Since $y_{k-1} \in W_\varepsilon^s(x_{k-1})$ and $d\big(x_k, f(x_{k-1})\big) < \varepsilon$, we have

$$\begin{aligned} d\big(x_k, f(y_{k-1})\big) &< d\big(x_k, f(x_{k-1})\big) + d\big(f(x_{k-1}), f(y_{k-1})\big) \\ &< \varepsilon + \varepsilon < \delta', \quad 1 \le k \le n. \end{aligned}$$

Hence our definition is valid, and the points y_k belong to $C(f)$. Since $y_k \in W_\varepsilon^s(x_k) \cap W_\varepsilon^u\big(f(y_{k-1})\big)$, by Lemma 9, we have

$$d(x_k, y_k) < \delta \quad \text{and} \quad d\big(y_k, f(y_{k-1})\big) < \delta,$$

for any $1 \le k \le n$. Put $\overline{y} = f^{-n}(y_n)$. Then the set ξ_n^+ is β-shadowed by the point $\overline{y} \in C(f)$. In fact, we have

$$\begin{aligned} d\big(f^k(\overline{y}), y_k\big) = d\big(f^{k-n}(y_n), y_k\big) &\le d\big(y_k, f^{-1}(y_{k+1})\big) + d\big(f^{-1}(y_{k+1}), f^{-2}(y_{k+2})\big) \\ &\qquad + \cdots\cdots\cdots + d\big(f^{k-n+1}(y_{n-1}), f^{k-n}(y_n)\big) \\ &= \sum_{i=1}^{n-k} d\big(f^{-i+1}(y_{k+i-1}), f^{-i}(y_{k+i})\big) \\ &= \sum_{i=1}^{n-k} d\big(f^{-i}(f(y_{k+i-1})), f^{-i}(y_{k+i})\big) \\ &\le \sum_{i=1}^{n-k} \lambda^i d\big(f(y_{k+i-1}), y_{k+i}\big) \\ &< \sum_{i=1}^{\infty} \lambda^i \delta = \frac{\delta}{1-\lambda}, \end{aligned}$$

for any $1 \le k \le n$. Consequently we have

$$\begin{aligned} d\big(f^k(\overline{y}), x_k\big) &\le d\big(f^k(\overline{y}), y_k\big) + d(y_k, x_k) \\ &< \frac{\delta}{1-\lambda} + \delta < \beta \end{aligned}$$

for all $1 \le k \le n$. Similarly we can show that every finite pseudo orbit $\xi_n = \{x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n\}$ of ξ is β-shadowed by a point $\overline{y_n}$ in $C(f)$, for each

$n \geq 1$. Let $\lim_{n\to\infty} \overline{y_n} = \overline{x}$. Then it is easy to show that ξ is β-shadowed by the point $\overline{x}$ in $C(f)$. This means that $C(f)$ has the strongly shadowing property. ∎

If the chain recurrent set $\mathrm{CR}(f)$ is hyperbolic then we have $\mathrm{CR}(f) = C(f)$. Hence we get the following corollary. Note that $\Omega(f) \neq C(f)$ even if $\Omega(f)$ is hyperbolic.

Corollary 11. *If $\mathrm{CR}(f)$ is hyperbolic then it has the strongly shadowing property.*

Theorem 12. *If $C(f)$ (or $\mathrm{CR}(f)$) is a hyperbolic manifold for f then $f|_{C(f)}$ (or $f|_{\mathrm{CR}(f)}$) is Anosov, respectively.*

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