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Normal Extensions of Representations of Abelian Semigroups

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Abstract. A commuting family of subnormal operators need not have a commuting normal extension. We study when a representation of an abelian semigroup can be extended to a normal representation, and show that it suffices to extend the set of generators to commuting normals. We also extend a result due to Athavale to representations on abelian lattice ordered semigroups.

1 Introduction

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *subnormal* if there exists a normal extension $N \in \mathcal{B}(\mathcal{K})$, where $\mathcal{H} \subseteq \mathcal{K}$ and $N|_{\mathcal{H}} = T$. There are many equivalent conditions for an operator being subnormal; for example, Agler showed a contractive operator T is subnormal if and only if for any $n \geq 0$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

One may refer to [9, Chapter II] for many other characterizations of subnormal operators.

A commuting pair of subnormal operators $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ might not have commuting normal extensions [1, 13], and a necessary and sufficient condition was given by Itô in [11]. Athavale obtained a necessary and sufficient condition for n commuting operators $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ to have commuting normal extensions in terms of operator polynomials [4, 6].

This paper considers the question as to when a contractive representation of a unital abelian semigroup can be extended to a contractive normal representation. Athavale's result can be applied to the set of generators and obtain a map that sends the semigroup into a family of commuting normal operators. Our first result shows that such a normal map guarantees the existence of a normal representation.

It is also observed that Athavale's result is equivalent to a certain representation being regular, and we further extend Athavale's result to abelian lattice ordered semigroups.

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2 Commuting Normal Extension

For an operator $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{K})$ extends T (S is called an extension of T) if it acts on a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $S|_{\mathcal{H}} = T$. In other words, S has the form

$$S = \begin{bmatrix} T & * \\ 0 & * \end{bmatrix}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *subnormal* if it has a normal extension. Among many characterizations of subnormal operators, Agler [2] showed a contractive operator T is subnormal if and only if for any $n \geq 0$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \geq 0.$$

However, a commuting pair of subnormal operators need not have a commuting pair of normal extension [1,13]. Itô [11] established a necessary and sufficient condition for a commuting family of subnormal operators to have commuting normal extensions. Athavale [4] generalized Agler's result to a family of commuting contractions.

Theorem 2.1 (Athavale) *Let $T = (T_1, T_2, \ldots, T_m)$ be a family of m commuting contractions. Then T has a commuting normal extension N if and only if for any $n_1, n_2, \ldots, n_m \geq 0$, we have*

$$\sum_{0 \leq k_i \leq n_i} (-1)^{k_1+k_2+\cdots+k_m} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} T_1^{*k_1} T_2^{*k_2} \cdots T_m^{*k_m} T_m^{k_m} \cdots T_1^{k_1} \geq 0. \tag{2.1}$$

One may observe that a family of m commuting contractions defines a contractive representation $T: \mathbb{N}^m \to \mathcal{B}(\mathcal{H})$ that sends each generator e_i to T_i. A commuting normal extension $N = (N_1, \ldots, N_m)$ can be seen as a contractive normal representation $N: \mathbb{N}^m \to \mathcal{B}(\mathcal{K})$ that extends T. Athavale's result gives a necessary and sufficient condition for the existence of a normal representation that extends T. If P is a unital abelian semigroup and $T: P \to \mathcal{B}(\mathcal{H})$ is a contractive representation, we can also ask the question when there exists a normal representation $N: P \to \mathcal{B}(\mathcal{K})$ that extends T.

Example 2.2 Consider $P = \mathbb{N}\backslash\{1\}$, which is a unital semigroup generated by 2 and 3. A contractive representation $T: P \to \mathcal{B}(\mathcal{H})$ is uniquely determined by $T(2), T(3)$, which satisfies $T(2)^3 = T(3)^2$. We use Theorem 2.1 to test if $T(2), T(3)$ has commuting normal extensions N_2, N_3. However, even if they do have such extensions, there is no guarantee that $N_2^3 = N_3^2$, and therefore, it is not clear if we can get a normal representation $N: P \to \mathcal{B}(\mathcal{K})$ that extends T. Nevertheless, since N_2, N_3 extend $T(2), T(3)$, respectively, we define a normal map $N: P \to \mathcal{B}(\mathcal{K})$ using N_2, N_3 such that $\{N(p)\}_{p \in P}$ is a family of commuting normal operators where $N(p)$ extends $T(p)$. As we shall soon see, in Theorem 2.6, the existence of such normal map guarantees a normal representation that extend T.

We also note that this semigroup $P = \mathbb{N}\backslash\{1\}$ is closely related to the so-called Neil algebra $\mathcal{A} = \{f \in A(\mathbb{D}) : f'(0) = 0\}$. Dilation on Neil algebra has been studied in [8,10]. Unlike $\mathbb{N}$ where every contractive representation has a unitary dilation due to Sz. Nagy's dilation, contractive representations of P may not have a unitary dilation.

Even so, for a contractive representation $T: P \to \mathcal{B}(\mathcal{H})$, we apply Ando's theorem to dilate $T(2), T(3)$ into commuting unitaries U_2, U_3, and therefore there exists a family $\{U_n\}_{n\in P}$ of commuting unitaries where $P_{\mathcal{H}} U_n|_{\mathcal{H}} = T(n)$ for each n [10, Example 2.4]. However, existence of such unitary maps does not guarantees a unitary dilation of T.

One of the main tools for the proof is the involution semigroup. Sz.Nagy used such a technique and proved a subnormality condition of a single operator due to Halmos [14], and Athavale also used this technique in [4]. We extend this technique to a more general setting.

Definition 2.3 A semigroup P is called an *involution semigroup* (or a $*$-*semigroup*) if there is an involution $*: P \to P$ that satisfies $p^{**} = p$ and $(pq)^* = q^* p^*$.

For example, any group G can be seen as an involution semigroup where $g^* = g^{-1}$. Any abelian semigroup can be seen as involution semigroup where $p^* = p$. A representation D of a unital involution semigroup P is a unital $*$-homomorphism. It is obvious that if $pp^* = p^* p$, then $D(p)$ is normal. Sz.Nagy established a condition that guarantees that a map on an involution semigroup has a dilation to a representation of the semigroup [14].

Theorem 2.4 *Let P be a $*$-semigroup and let $T: P \to \mathcal{B}(\mathcal{H})$ satisfy the following conditions:*

(i) $T(e) = I, T(p^*) = T(p)^*$.
(ii) *For any $p_1, \dots, p_n \in P$, the operator matrix $[T(p_i^* p_j)]$ is positive.*
(iii) *There exists a constant $C_a > 0$ for each $a \in P$ such that for all $p_1, \dots, p_n \in P$,*

$$\left[T(p_i^* a^* a p_j) \right] \le C_a^2 \left[T(p_i^* p_j) \right].$$

Then there exists a representation $D: P \to \mathcal{B}(\mathcal{K})$ that satisfies $T(p) = P_{\mathcal{H}} D(p)|_{\mathcal{H}}$ and $\|D(p)\| \le C_p$.

Now let P be a unital abelian semigroup and consider $Q = \{(p,q) : p, q \in P\}$. Q is a unital semigroup under the point-wise semigroup operation

$$(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2).$$

Define a involution operation of Q by $(p,q)^* = (q,p)$, which turns Q into an involution semigroup. Notice that since P is abelian, Q is also abelian. Moreover, any element $(p,q) = (0,q) + (0,p)^*$. If $D: Q \to \mathcal{B}(\mathcal{K})$ is a representation, then

$$D(0,p)^* D(0,p) = D(p,p) = D(0,p) D(0,p)^*,$$

and therefore $D(0,p)$ is normal.

Lemma 2.5 *Let $T \in \mathcal{B}(\mathcal{H})$ and $N \in \mathcal{B}(\mathcal{K})$ where $\mathcal{H}$ is a subspace of $\mathcal{K}$. Suppose $T = P_{\mathcal{H}} N|_{\mathcal{H}}$ and $T^* T = P_{\mathcal{H}} N^* N|_{\mathcal{H}}$; then N is an extension of T.*

Proof From the conditions, we have for any $h \in \mathcal{H}$, $\|Th\|^2 = \langle Th, Th \rangle = \langle T^* Th, h \rangle$. Since $T^* T = P_{\mathcal{H}} N^* N|_{\mathcal{H}}$, $\langle T^* Th, h \rangle = \langle N^* Nh, h \rangle = \|Nh\|^2$. On the other hand,

$\|Th\| = \sup_{\|k\|\le 1, k\in\mathcal{H}}\langle Th, k\rangle$. But $T = P_{\mathcal{H}}N|_{\mathcal{H}}$, and thus $\langle Th, k\rangle = \langle Nh, k\rangle$. Therefore,

$$\|Th\| = \sup_{\|k\|\le 1, k\in\mathcal{H}}\langle Th, k\rangle = \sup_{\|k\|\le 1, k\in\mathcal{H}}\langle Nh, k\rangle = \|P_{\mathcal{H}}Nh\|.$$

Therefore, $\|Th\| = \|Nh\| = \|P_{\mathcal{H}}Nh\|$, and thus $\mathcal{H}$ is invariant for N. Hence, N is an extension of T. ∎

Theorem 2.6 *Let P be any unital abelian semigroup and let $T\colon P \to \mathcal{B}(\mathcal{H})$ be a unital contractive representation of P. Then the following are equivalent:*

(i) *There exists a contractive normal map $N\colon P \to \mathcal{B}(\mathcal{K})$ that extends T, where the family $\{N(p)\}_{p\in P}$ is a commuting family of normal operators.*

(ii) *There exists a contractive normal representation $N\colon P \to \mathcal{B}(\mathcal{L})$ that extends T.*

Proof (ii) $\Rightarrow$ (i) is trivial. For the other direction, let Q be the $*$-semigroup constructed before and let $\widetilde{T}\colon Q \to \mathcal{B}(\mathcal{H})$ defined by $\widetilde{T}(p,q) = T(p)^*T(q)$. For each $p \in P$, denote $N(p) = \left[\begin{smallmatrix} T(p) & X_p \\ 0 & Y_p \end{smallmatrix}\right]$. Pick $s_i = (p_i, q_i) \in Q$ and $t = (a, b) \in Q$. We will show that $\widetilde{T}$ satisfies all the conditions in Theorem 2.4.

The first condition of Theorem 2.4 is clearly valid. For the second condition, we have

$$\begin{aligned}\left[\widetilde{T}(s_i^* s_j)\right] &= \left[\widetilde{T}(q_i p_j, p_i q_j)\right] = \left[T(q_i)^*T(p_j)^*T(p_i)T(q_j)\right] \\ &= \operatorname{diag}\big(T(q_1)^*, T(q_2)^*, \dots, T(q_n)^*\big)\left[T(p_j)^*T(p_i)\right]\operatorname{diag}\big(T(q_1), \dots, T(q_n)\big)\end{aligned}$$

It suffices to show $\left[T(p_j)^*T(p_i)\right] \ge 0$. Notice that $\{N(p_i)\}$ is a commuting family of normal operators, and thus they also doubly commute (by Fuglede's Theorem):

$$\left[N(p_j)^*N(p_i)\right] = \left[N(p_i)N(p_j)^*\right] = \begin{bmatrix} N(p_1) \\ N(p_2) \\ \vdots \\ N(p_n)\end{bmatrix}\left[N(p_1)^* \; N(p_2)^* \; \cdots N(p_n)^*\right] \ge 0.$$

Then $N(p_i)$ extends $T(p_i)$, and therefore $P_{\mathcal{H}}N(p_j)^*N(p_i)|_{\mathcal{H}} = T(p_j)^*T(p_i)$. By projecting on $\mathcal{H}^n$, we get the desired inequality.

For the third condition, we have

$$\begin{aligned}\left[\widetilde{T}(s_i^* t^* t s_j)\right] &= \left[\widetilde{T}(q_i p_j ab, ab p_i q_j)\right] \\ &= \left[T(ab)^*T(q_i)^*T(p_j)^*T(p_i)T(q_j)T(ab)\right] \\ &= \operatorname{diag}\big(T(q_1)^*, T(q_2)^*, \dots, T(q_n)^*\big)\left[T(ab)^*T(p_j)^*T(p_i)T(ab)\right] \\ &\quad \operatorname{diag}\big(T(q_1), \dots, T(q_n)\big)\end{aligned}$$

Therefore, it suffices to show (with $C_t = 1$ in the condition) that

$$\left[T(ab)^*T(p_j)^*T(p_i)T(ab)\right] \le \left[T(p_j)^*T(p_i)\right]$$

Similar to the previous case, it suffices to show that

$$\left[N(ab)^*N(p_j)^*N(p_i)N(ab)\right] \le \left[N(p_j)^*N(p_i)\right].$$

Let $X = [N(p_j)^* N(p_i)] \geq 0$ and $D = \operatorname{diag}(N(ab), \dots, N(ab))$. Since D and X $*$-commute, D and $X^{1/2}$ also $*$-commute. We have

$$D^* X D = X^{1/2} D^* D X^{1/2} \leq \|N(ab)\| X.$$

Since N is contractive, this shows that $D^* X D \leq X$. Therefore, all conditions in Theorem 2.4 are met, and thus there exists a contractive representation $S: Q \to \mathcal{B}(\mathcal{L})$ such that $\widetilde{T}(p,q) = P_{\mathcal{H}} S(p,q)|_{\mathcal{H}}$. Denote $M(p) = S(0,p)$. Then $M: P \to \mathcal{B}(\mathcal{L})$ is a representation of P, and moreover,

$$T(p)^* T(p) = P_{\mathcal{H}} S(p,p)|_{\mathcal{H}} = P_{\mathcal{H}} M(p)^* M(p)|_{\mathcal{H}}.$$

By Lemma 2.5, we know $M(p)$ extends $T(p)$, and therefore M is a normal extension. ■

Remark 2.7 When the semigroup is $P = \mathbb{N}^k$, Theorem 2.6 is trivial. For a normal map $N: \mathbb{N}^k \to \mathcal{B}(\mathcal{K})$, one can define a normal representation by sending each generator e_i to $N(e_i)$. However, it is not clear how we can derive a normal representation from a normal map when the semigroup does not have nice generators. For example, we have seen this issue in Example 2.2 where the semigroup $P = \mathbb{N}\backslash\{1\}$ is finitely generated. This result shows that finding a commuting family of normal extensions for $\{T(p)\}_{p \in P}$ is equivalent of finding a normal representation that extends T.

Corollary 2.8 *Let P be a commutative unital semigroup generated by $\{p_i\}_{i \in I}$, and let $T: P \to \mathcal{B}(\mathcal{H})$ a unital contractive representation. Then the family $\{T(p_i)\}_{i \in I}$ has commuting normal extensions $\{N_i\}_{i \in I}$ if and only if there exists a normal representation $N: P \to \mathcal{B}(\mathcal{K})$ such that each $N(p)$ extends $T(p)$.*

Proof The backward direction is obvious. Now assuming $\{T(p_i)\}_{i \in I}$ has commuting normal extension $\{N_i\}_{i \in I}$. For each element $p \in P$, write p as a finite product of $\{p_i\}_{i \in I}$ and define $N(p)$ to be the corresponding product of $T(p_i)$. Since N_i commutes with one another, we obtain a normal map $\overline{N}: P \to \mathcal{B}(\mathcal{L})$ where $\{\overline{N}(p)\}_{p \in P}$ is a family of commuting normal operators where $\overline{N}(p)$ extends $T(p)$. Theorem 2.6 implies the existence of the desired normal representation N. ■

Remark 2.9 Corollary 2.8 shows that for a contractive representation $T: P \to \mathcal{B}(\mathcal{H})$, it suffices to extends the image of T on a set of generators. Since Athavale's result still holds for an infinite family of operators (Corollary 3.6), we may use condition (2.1) to check if the set of generators have a commuting normal extension. However, when the semigroup has too many generators, condition (2.1) is hard to check. We will give another equivalent condition for an abelian lattice ordered group in the next section.

3 Normal Extensions For Lattice Ordered Semigroups

A lattice ordered semigroup P is a unital normal semigroup inside a group $G = P^{-1}P$ that induces a lattice order. Given a unital normal semigroup $P \subseteq G = P^{-1}P$, there is a natural partial order on G given by $x \leq y$ when $x^{-1}y \in P$. If any two elements g, h in

G has a least upper bound (also called the join $g \vee h$) and greatest lower bound (also called the meet $g \wedge h$) under this partial order, the partial order is called a *lattice order*.

Example 3.1 (Examples of Lattice Ordered Semigroups)

(i) $\mathbb{N}^k$ is a lattice ordered semigroup inside $\mathbb{Z}^k$ for any $k \in \mathbb{N} \bigcup \{\infty\}$.
(ii) $\mathbb{R}^+$ is a lattice ordered semigroup inside $\mathbb{R}$. Notice that $\mathbb{R}^+$ is not countably generated.
(iii) More generally, if the partial order induced by P is a total order, or equivalently, $G = P \bigcup P^{-1}$, then P is also a lattice ordered semigroup in G.
(iv) If P_i are lattice ordered semigroups inside G_i, then their product $\prod P_i$ is also a lattice ordered semigroup inside $\prod G_i$.
(v) If X is a topological space and $C^+(X)$ contains all the non-negative continuous function on X, then $C^+(X)$ is a lattice ordered semigroup inside $C(X)$, where the group operation is point-wise addition.
(vi) Even though our focus is on abelian lattice ordered semigroups, there are non-abelian lattice ordered semigroups. Consider an uncountable totally ordered set X, and define G to be the set of all order preserving bijections on X. G is a group under composition. Define $P = \{\alpha \in G : \alpha(x) \geq x\}$; then P is a non-abelian lattice ordered semigroup in G [3].

If P is a lattice ordered semigroup inside G, then every element $g \in G$ has a unique positive and negative part, g_+, g_-, in the sense that $g = g_-^{-1} g_+$ and $g_+ \wedge g_- = e$. This notion of positive and negative part is essential in defining a regular dilation. For a lattice ordered semigroup P inside G, a representation $T : P \to \mathcal{B}(\mathcal{H})$ has a dilation $U : G \to \mathcal{B}(\mathcal{K})$ if U is a unitary representation of G on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that for any $p \in P$,

$$T(p) = P_{\mathcal{H}} U(p)|_{\mathcal{H}}.$$

Such a dilation is called regular if for any $g \in G$,

$$T(g_-)^* T(g_+) = P_{\mathcal{H}} U(g)|_{\mathcal{H}}.$$

There is a dual version of regular dilation that call such a dilation $*$-regular if for any $g \in G$,

$$T(g_+) T(g_-)^* = P_{\mathcal{H}} U(g)|_{\mathcal{H}}.$$

These two definitions are equivalent in the sense that T is $*$-regular if and only if $T^* : P^{-1} \to \mathcal{B}(\mathcal{H})$ where $T^*(p^{-1}) = T(p)^*$ is regular [12, Proposition 2.5]. We call a representation T *regular* if it has a regular dilation.

A well known result due to Sarason shows that such a Hilbert space $\mathcal{K}$ can be decomposed as $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$, where under such such decomposition,

$$U(p) = \begin{bmatrix} * & 0 & 0 \\ * & T(p) & 0 \\ * & * & * \end{bmatrix}.$$

Regular dilations were first studied by Brehmer [7] where he gave a necessary and sufficient condition for a representation of $\mathbb{N}^\Omega$ to be regular.

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Theorem 3.2 (Brehmer) *Let Ω be a set, and denote $\mathbb{Z}^\Omega$ to be the set of $(t_\omega)_{\omega\in\Omega}$ where $t_\omega \in \mathbb{Z}$ and $t_\omega = 0$ except for finitely many ω. Also, for a finite set $V \subset \Omega$, denote $e_V \in \mathbb{Z}^\Omega$ to be 1 at those $\omega \in V$ and 0 elsewhere. If $\{T_\omega\}_{\omega\in\Omega}$ is a family of commuting contractions, we define a contractive representation $T: \mathbb{Z}_+^\Omega \to \mathcal{B}(\mathcal{H})$ by*

$$T(t_\omega)_{\omega\in\Omega} = \prod_{\omega\in\Omega} T_\omega^{t_\omega}.$$

Then T is regular if and only if for any finite $U \subseteq \Omega$, the operator

$$\sum_{V\subseteq U} (-1)^{|V|} T(e_V)^* T(e_V) \geq 0.$$

Recently, the author extended this result to an arbitrary lattice ordered semigroup (not necessarily abelian) [12].

Theorem 3.3 *Let P be a lattice ordered semigroup in G and let $T: P \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Define $\widetilde{T}: G \to \mathcal{B}(\mathcal{H})$ by $\widetilde{T}(g) = T(g_-)^* T(g_+)$. Then T is regular if and only if for any $p_1, \ldots, p_n \in P$ and $g \in P$ where $g \wedge p_i = e$ for all $i = 1, 2, \ldots, n$, we have*

$$\big[T(g)^* \widetilde{T}(p_i p_j^{-1}) T(g)\big] \leq \big[\widetilde{T}(p_i p_j^{-1})\big].$$

Although it is observed that condition (2.1) implies a representation $T: \mathbb{N}^m \to \mathcal{B}(\mathcal{H})$ has regular dilation [5], the converse is not true. However, we will prove that Athavale's result is equivalent of saying that a certain representation T^∞ is regular. First of all, define $\mathbb{N}^{m\times\infty}$ by taking the product of infinitely many copies of $\mathbb{N}^m$; in other words, $\mathbb{N}^{m\times\infty}$ is the abelian semigroup generated by $(e_{i,j})_{1\leq i\leq m, j\in\mathbb{N}}$. Consider $T^\infty: \mathbb{N}^{m\times\infty} \to \mathcal{B}(\mathcal{H})$ where T^∞ sends each generator $e_{i,j}$ to T_i.

Lemma 3.4 *As defined above, T^∞ is regular if and only if T satisfies condition (2.1).*

Proof It suffices to verify that condition (2.1) is equivalent to Brehmer's condition on $\mathbb{N}^{m\times\infty}$ in Theorem 3.2. For any finite set $U \subseteq \{1, 2, \ldots, m\} \times \mathbb{N}$, denote by n_i the number of $u \in U$ whose first coordinate is i. For any subset $V \subseteq U$, denote by k_i the number of $v \in V$ whose first coordinate is i. It is clear that $0 \leq k_i \leq n_i$. Notice that $T(e_V) = T_1^{k_1} T_2^{k_2} \cdots T_m^{k_m}$, and among all subsets of U, there are exactly $\binom{n_1}{k_1} \cdots \binom{n_m}{k_m}$ subsets V that have k_i elements whose first coordinate is i. Therefore,

$$\sum_{V\subseteq U} (-1)^{|V|} T(e_V)^* T(e_V) = \sum_{0\leq k_i\leq n_i} (-1)^{k_1+k_2+\cdots+k_m} \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} T_1^{*k_1} T_2^{*k_2} \cdots T_m^{*k_m} T_m^{k_m} \cdots T_1^{k_1}.$$

Hence, Brehmer's condition holds if and only if T satisfies condition (2.1). ∎

Notice that condition (2.1) cannot be generalized directly to arbitrary abelian lattice ordered semigroups when the semigroup lacks generators. However, Lemma 3.4 motivates us to consider T^∞ in an abelian lattice ordered semigroup. For a lattice ordered semigroup P inside a group G, define $P^\infty = \prod_{i=1}^\infty P$ to be the abelian semigroup generated by infinitely many identical copies of P. Inside the n-th copy of P^∞, we

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denote p by $p \otimes \delta_n$. A typical element of P^∞ can be denoted by $\sum_{i=1}^N p_i \otimes \delta_i$ for some large enough N. Then P^∞ is naturally a lattice ordered semigroup inside the group G^∞, where

$$\Big(\sum_{i=1}^N p_i \otimes \delta_i\Big) \wedge \Big(\sum_{i=1}^N q_i \otimes \delta_i\Big) = \sum_{i=1}^N p_i \wedge q_i \otimes \delta_i.$$

Our main result shows that T^∞ being regular is equivalent to having a normal extension.

Theorem 3.5 *Let $T: P \to \mathcal{B}(\mathcal{H})$ be a contractive representation of an abelian lattice ordered semigroup. Define $T^\infty: P^{\mathbb{N}} \to \mathcal{B}(\mathcal{H})$ by $T^\infty(p,n) = T(p)$ for any n. Then the following are equivalent:*

(i) *T has a contractive normal extension to a representation $N: P \to \mathcal{B}(\mathcal{K})$. In other words, there exists a contractive normal representation $N: P \to \mathcal{B}(\mathcal{K})$ such that for all $p \in P$, $T(p) = N(p)|_{\mathcal{H}}$.*

(ii) *T^∞ is regular.*

Proof (i) $\Rightarrow$ (ii): First of all notice that the family $\{N(p)\}_{p\in P}$ $*$-commutes due to Fuglede's theorem. Define N^∞ by sending $N^\infty(p,n) = N(p)$ for all $p \in P, n \in \mathbb{N}$. Then for any $s,t \in P^\infty$, $N^\infty(s), N^\infty(t)$ are a finite product of operators in $\{N(p)\}_{p\in P}$ and therefore they also $*$-commute. In particular, N^∞ is Nica-covariant and therefore is regular [12, Theorem 4.1]. Since N extends T, N^∞ also extends T^∞, and therefore for any $s,t \in P^\infty$,

$$P_{\mathcal{H}} N^\infty(t)^* N^\infty(s)|_{\mathcal{H}} = T^\infty(t)^* T^\infty(s).$$

Thus, N^∞ satisfies the condition in Theorem 3.3, and by projecting onto $\mathcal{H}$, T^∞ also satisfies this condition and thus is regular.

(ii) $\Rightarrow$ (i): Let $U: G^\infty \to \mathcal{B}(\mathcal{K})$ be a regular unitary dilation of T^∞, and decompose $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$ so that under such decomposition, for each $w \in P^\infty$,

$$U(w) = \begin{bmatrix} * & 0 & 0 \\ * & T(w) & 0 \\ * & * & * \end{bmatrix}.$$

Fix $p \in P$, let $U_i(p) = U(p \otimes \delta_i)$. Under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$, let

$$U_i(p) = \begin{bmatrix} A_i & 0 & 0 \\ B_i & T(p) & 0 \\ C_i & D_i & E_i \end{bmatrix}.$$

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First by regularity of U, for any $i \neq j$,

$$\begin{aligned} T(p)^*T(p) &= P_{\mathcal{H}} U(p \otimes \delta_i - p \otimes \delta_j)|_{\mathcal{H}} \\ &= P_{\mathcal{H}} U_j(p)^* U_i(p)|_{\mathcal{H}} \\ &= P_{\mathcal{H}} \left[\begin{bmatrix} A_j^* & B_j^* & C_j^* \\ 0 & T(p)^* & D_j^* \\ 0 & 0 & E_j^* \end{bmatrix} \begin{bmatrix} A_i & 0 & 0 \\ B_i & T(p) & 0 \\ C_i & D_i & E_i \end{bmatrix}\right]_{\mathcal{H}} \\ &= P_{\mathcal{H}} \left[\begin{bmatrix} * & * & * \\ * & T(p)^*T(p) + D_j^* D_i & * \\ * & * & * \end{bmatrix}\right]_{\mathcal{H}}. \end{aligned}$$

Therefore, each $D_j^* D_i = 0$ whenever $i \neq j$. When $i = j$, since U is a unitary representation, $U_i(p)$ is a unitary, and thus $D_i^* D_i = I - T(p)^* T(p)$. Now fix $\epsilon > 0$ and denote

$$\Lambda_\epsilon = \left\{ \lambda = (\lambda_i)_{i=1}^\infty \in c_{00} : \sum_{i=1}^\infty \lambda_i = 1, 0 \le \lambda_i \le 1, \|\lambda\|_2 < \epsilon \right\}.$$

This set is non-empty, since we can let $\lambda_i = \frac{1}{n}$ for $1 \le i \le n$, and 0 otherwise. This gives $\|\lambda\|_2 = \frac{1}{\sqrt{n}}$, which can be arbitrarily small as $n \to \infty$. For each $\lambda \in \Lambda_\epsilon$, let $N_\lambda = \sum_{i=1}^\infty \lambda_i U_i(p)$, which converges, since λ has finite support. Denote

$$\mathcal{N}_\epsilon = \{N_\lambda : \lambda \in \Lambda_\epsilon\}$$

Notice that $P_{\mathcal{H}} N_\lambda|_{\mathcal{H}} = \sum_{i=1}^\infty \lambda_i T(p) = T(p)$. Therefore, under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$,

$$N_\lambda = \begin{bmatrix} A_\lambda & 0 & 0 \\ B_\lambda & T(p) & 0 \\ C_\lambda & D_\lambda & E_\lambda \end{bmatrix}.$$

Here, $D_\lambda = \sum_{i=1}^\infty \lambda_i D_i$, and thus

$$D_\lambda^* D_\lambda = \sum_{i,j=1}^\infty \overline{\lambda_i} \lambda_j D_i^* D_j = \sum_{i=1}^\infty |\lambda_i|^2 D_i^* D_i.$$

Here, we used the fact that $D_i^* D_j = 0$ whenever $i \neq j$. Note that each $D_i^* D_i = I - T(p)^* T(p)$, which is contractive. Hence,

$$\|D_\lambda^* D_\lambda\| \le \|\lambda\|_2^2 < \epsilon^2.$$

Each N_λ is a convex combination of U_i, and thus is contained in the convex hull of U_i, which is also contained in the unit ball in $\mathcal{B}(\mathcal{K})$. Observe that each $\mathcal{N}_\epsilon$ is also convex. Therefore, the convexity implies their SOT* and WOT closures agree (here, $SOT^* - \lim T_n = T$ if T_n and T_n^* converges to T and T^* respectively in SOT.). Hence,

$$\overline{\mathcal{N}_\epsilon}^{SOT^*} = \overline{\mathcal{N}_\epsilon}^{WOT} \subseteq \overline{\operatorname{conv}}^{WOT}\{U_i\} \subseteq b_1(\mathcal{B}(\mathcal{K})).$$

The Banach Alaoglu theorem gives $b_1(\mathcal{B}(\mathcal{K}))$ is WOT-compact, and therefore $\overline{\mathcal{N}_\epsilon}^{WOT}$ is a decreasing nest of WOT-compact sets. By the Cantor intersection theorem,

$$\bigcap_{\epsilon > 0} \overline{\mathcal{N}_\epsilon}^{SOT^*} = \bigcap_{\epsilon > 0} \overline{\mathcal{N}_\epsilon}^{WOT} \neq \varnothing.$$

Pick $N(p) \in \bigcap_{\epsilon>0} \overline{\mathcal{N}_\epsilon}^{SOT^*}$. Then for any $\epsilon > 0$, we can choose a net $(N_\lambda)_{\lambda\in I_\epsilon}$, where $I_\epsilon \subseteq \Lambda_\epsilon$, such that $SOT^* - \lim_{I_\epsilon} N_\lambda = N(p)$, and thus $SOT^* - \lim_{I_\epsilon} N_\lambda^* = N(p)^*$. Now both N_λ, N_λ^* are uniformly bounded by 1, since they are all contractions. Hence, their product is SOT-continuous:

$$SOT - \lim_{\Lambda} N_\lambda^* N_\lambda = N(p)^* N(p),$$
$$SOT - \lim_{\Lambda} N_\lambda N_\lambda^* = N(p) N(p)^*$$

But since U_i are commuting unitaries and thus $*$-commute, N_λ is normal. Hence, $N(p)^*N(p) = N(p)N(p)^*$, and $N(p)$ is normal.

Consider $N(p) \in \mathcal{B}(\mathcal{K})$ under the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{H} \oplus \mathcal{K}_-$, each entry must be the WOT-limit of $(N_\lambda)_{\lambda\in I_\epsilon}$, and therefore it has the form

$$N(p) = \begin{bmatrix} A(p) & 0 & 0 \\ B(p) & T(p) & 0 \\ C(p) & D(p) & E(p) \end{bmatrix}.$$

Since $(D_\lambda)_{\lambda\in I_\epsilon}$ WOT-converges to $D(p)$, and for each $\lambda \in \Lambda_\epsilon$, $\|D_\lambda\| < \epsilon$. Therefore, $\|D(p)\| < \epsilon$ for every $\epsilon > 0$ and thus $D(p) = 0$. Hence, $\mathcal{H}$ is invariant for $N(p)$, whence $N(p)$ is a normal extension for $T(p)$.

The procedure above gives a normal map $N\colon P \to \mathcal{B}(\mathcal{K})$ where each $N(p)$ is a normal contraction that extends $T(p)$. Notice that $N(p)$ is a WOT-limit of convex combinations of $\{U_i(p)\}_{i\in\mathbb{N}}$, where the family $\{U_i(p)\}_{i,p}$ is commuting since P is abelian. Any convex combination of $\{U_i(p)\}_{i\in\mathbb{N}}$ also commutes with any convex combination of $\{U_i(q)\}_{i\in\mathbb{N}}$. Therefore, $\{N(p)\}_{p\in P}$ is also a commuting family of normal operators. By Theorem 2.6, there exists a normal representation $N\colon P \to \mathcal{B}(\mathcal{L})$ that extends T. ∎

As an immediate corollary, Theorem 2.1 can be extended to any family of commuting contractions $\{T(\omega)\}_{\omega\in\Omega}$ by considering Brehmer's condition on $\mathbb{N}^{\Omega\times\infty}$.

Corollary 3.6 *Let $\{T_i\}_{i\in I}$ be a family of commuting contractions. Then there exists a family of commuting normal contractions $\{N_i\}_{i\in I}$ that extends $\{T_i\}_{i\in I}$ if and only if for any finite set $F \subseteq I$, $\{T_i\}_{i\in F}$ satisfies condition (2.1).*

It is known that isometric representations of lattice ordered semigroups are automatically regular [12, Corollary 3.8]. Therefore, if $T\colon P \to \mathcal{B}(\mathcal{H})$ is an isometric representation, then $T^\infty\colon P^\infty \to \mathcal{B}(\mathcal{H})$ is also an isometric representation, and thus T has a subnormal extension.

Corollary 3.7 *Every isometric representation of an abelian lattice ordered semigroup has a contractive subnormal extension.*

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