Intriguing integrals: an Euler-inspired odyssey

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'Lisez Euler, lisez Euler, c'est notre maître à tous.' (Laplace)

This article began life as a talk given at the 2006 Mathematical Association Conference at Loughborough University. Many of the examples discussed had their roots in the work of Euler and the ‘Euler boxes’ below explore these connections. We use the standard references to Euler’s oeuvres by ‘Eneström numbers’ (from Eneström’s catalogue of Euler’s works) and dates of publication: a very convenient source is the almost complete Euler Archive, freely accessible at http://www.math.dartmouth.edu/~euler/.

1. Wallis’s product

We start by reprising a standard proof that Wallis’s infinite product (1655), \[ \prod_{n=1}^{\infty} \frac{2n\cdot2n}{2n-1.2n+1}, \] is equal to \( \frac{\pi}{2} \). Let \( I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta \) (\( = \int_0^{\pi/2} \cos^n \theta \, d\theta \)). Then, writing the integrand as \( \sin \theta \cdot \sin^{n-1} \theta \) and using parts, we obtain \( I_n = \frac{n-1}{n}I_{n-2} \) from which \( I_2n+1 = \frac{2n\cdot2n-2\ldots2}{2n+1.2n-1\ldots3} \) (since \( I_1 = 1 \)) and \( I_{2n} = \frac{2n-2\cdot2n-4\ldots2}{2n-2\cdot2n-4\ldots2} \) (since \( I_0 = \frac{\pi}{2} \)). The evaluation of Wallis’s product then follows by letting \( n \to \infty \) in the sandwich:

\[ \frac{2n+1}{2n+2} = \frac{I_{2n+2}}{I_{2n}} < \frac{I_{2n+1}}{I_{2n}} = \frac{2.2.4.4}{1.3.5\ldots} \frac{2n\cdot2n}{2n-1.2n+1} = \frac{2}{\pi} < 1. \]

Intriguingly, we can evaluate \( I_{2n} \) directly (but not apparently \( I_{2n+1} \)) as follows:

\[ I_{2n} = \int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{1}{4} \int_0^{2\pi} \cos^{2n} \theta \, d\theta = \frac{1}{4} \int_0^{2\pi} \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^{2n} \, d\theta \]

\[ = \frac{1}{2^{2n}} \int_0^{2\pi} \left( \binom{2n}{n} + \text{terms in } e^{i\theta} \right) \, d\theta \]

which equals \( \frac{\pi}{2^{2n+1}} \binom{2n}{n} \), since the terms involving \( e^{i\theta} \) integrate to 0.

Box 1: Euler and the gamma function

The gamma function was introduced by Euler in 1729 in correspondence with Goldbach which formed the basis of the more extensive article [E19, 1738]. Davis, in his beautiful article [1], reconstructs Euler’s solution to the problem of interpolating factorials. An initial clue seems to have been the realisation that setting \( n = \frac{1}{2} \) into the easily verified formula \( n! = \lim_{k \to \infty} \frac{k!(k+1)^n}{(n+1)(n+2)\ldots(n+k)} \) yields (after a
bit of manipulation) \( \frac{1}{2!} = \lim_{k \to \infty} \left( \frac{2k \cdot 2k}{1 \cdot 3 \cdot 5 \cdots 2k - 1 \cdot 2k + 1} \right)^{1/2} \left( \frac{k + 1}{2k + 1} \right)^{1/2} \)

which, on using Wallis’s product, suggests that \( \frac{1}{2!} = \frac{\sqrt{\pi}}{2} \). An audacious calculation letting \( m \to \infty \) in the Beta function, 
\[
B(m, n) = \int_0^1 x^{m-1} (1 - x)^{n-1} \, dx
\]
led Euler to the formula \( n! = \int_0^1 (- \ln x)^n \, dx \) (meaningful for \( n > -1 \)) which was the form in which he used the gamma function: it was Legendre who, putting \( t = - \ln x \) and shifting the origin so that \( \Gamma(t) = (x - 1)! \), introduced the modern form \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt \) for \( x > 0 \).

2. The Basel problem: what is the value of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)?

Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \), whence
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},
\]
it suffices to find \( \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \). A succinct evaluation of this sum (Proof 3 in Robin Chapman’s lovely compilation, [2]) runs as follows.

The binomial expansion of \( (1 - t^2)^{-1/2} \) gives \( (1 - t^2)^{-1/2} = \sum_{n=0}^{\infty} a_n t^{2n} \) with \( a_0 = 1, \quad a_n = \frac{1.3 \ldots 2n - 1}{2.4 \ldots 2n} \quad (n \geqslant 1) \) and, on integrating from 0 to \( \pi \),
\[
\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{a_n}{2n + 1} \sin^{2n+1} x \quad (*).
\]
A further integration of (*) from 0 to \( \frac{\pi}{2} \) gives
\[
\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{a_n}{2n + 1} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \text{since} \quad a_n I_{2n+1} = \frac{1}{2n+1},
\]
on using the formula for \( I_{2n+1} \) from §1. It follows that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \frac{\pi^2}{8} = \frac{\pi^2}{6} \) — the result which emphatically marked Euler’s arrival on the mathematical scene!

If we divide both sides of (*) by \( 2 \sin x \) prior to integrating from 0 to \( \frac{\pi}{2} \), and name the resulting constant \( C \), we obtain:
\[
C = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} \, dx = \frac{1}{2} \left[ 1 \sum_{n=0}^{\infty} \frac{a_n I_{2n}}{2n+1} \right] = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{I_{2n}^2}{2n+1} = \frac{\pi}{4} + \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left( \frac{1.3 \ldots 2n - 1}{2.4 \ldots 2n} \right)^2
\]

since \( I_{2n} = \frac{\pi}{2} a_n \), on using the formula for \( I_{2n} \) from §1.
Box 2: Euler and the Basel problem

Euler’s breathtaking initial evaluation of $\sum \frac{1}{n^2}$ in [E41, 1735], based on applying the identity $\sum \frac{1}{\alpha^2} = \frac{a_{n-1}^2 - 2a_n a_{n-2}}{a_n^2}$ for the roots of a polynomial $a_n + a_{n-1}x + a_{n-2}x^2 + \ldots = 0$ to a power series such as $1 - \frac{x^2}{6} + \ldots = \frac{\sin x}{x} = 0$ (with roots $\pm n\pi, n = 1, 2, \ldots$), is the one usually quoted in the literature. But, as Ed Sandifer in one of his excellent monthly e-columns on the MAA website, [3], points out, Euler characteristically revisited the problem and, in a later paper [E63, 1741], he presented the proof given in §2! The only differences are notational: for Euler, $\arcsin x$ is the arc length on the unit circle corresponding to a ‘semi-chord’ of length $x$ (remember that radians were introduced as late as 1873 in an examination question at Queen’s College, Belfast set by James Thomson, the brother of William Thomson (Lord Kelvin)).

He thus preferred to work with $\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}} dx$ rather than its trigonometric equivalent $\int_0^{\pi/2} \sin^{2n+1} x dx$. The [E63, 1741] paper seems to have been seldom read: although it is discussed in detail by Ayoub in [4, pp.1079-80], Chapman in [2] cites a 1987 appearance of this argument in the literature.

3. Catalan’s constant

One of the glories (and frustrations) of integrals is the multiplicity of equivalent forms in which they occur. A particularly striking example of this is provided by the Catalan constant encountered in §2 as

$$C = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx.$$

On the one hand, using $t = \tan \frac{1}{2}x$, ‘the world’s sneakiest substitution’ according to [5], we obtain

$$C = \int_0^1 \frac{\tan^{-1} t}{t} dt$$

$$= \int_0^1 \left( t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots \right) dt,$$ using the Maclaurin series for $\tan^{-1} t$

$$= 1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2},$$ which is the usual definition of $C$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n a_n I_{2n+1}}{2n + 1},$$ since $a_n I_{2n+1} = \frac{1}{2n + 1}$ from the line following (*) in §2

3.1. Catalan’s constant

The Catalan constant $C$ is defined by

$$C = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx.$$
\[
\begin{align*}
&= \int_0^{\pi/2} \sinh^{-1}(\sin x) \, dx, \\
&\quad \text{since } (1 + t^2)^{-1/2} = \sum_{n=0}^\infty (-1)^n a_n t^{2n} \text{ and } \sinh^{-1} x = \sum_{n=0}^\infty (-1)^n a_n x^{2n+1} \frac{1}{2n + 1}.
\end{align*}
\]

On the other hand, \( C = \frac{1}{2} \int_0^{\pi/2} x \csc x \, dx \), so, integrating by parts,

\[
C = -\frac{1}{2} \int_0^{\pi/2} \ln \left( \frac{x}{2} \right) \, dx \quad \text{(or } -\int_0^{\pi/4} \ln \theta \, d\theta) \]

\[
= -\int_0^1 \frac{\ln t}{1 + t^2} dt, \quad \text{on substituting } t = \tan \frac{1}{2} x
\]

\[
= \frac{1}{2} \int_0^\infty \frac{u}{\cosh u} \, du, \quad \text{on substituting } t = e^{-u}
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \sinh^{-1}(\tan v) \, dv, \quad \text{on substituting } \sinh u = \tan v.
\]

Thus, comparing our two endpoints, we have the rather remarkable equation

\[
2C = \int_0^{\pi/2} \sinh^{-1}(\sin x) \, dx + \int_0^{\pi/2} \sinh^{-1}(\cos x) \, dx = \int_0^{\pi/2} \sinh^{-1}(\tan x) \, dx.
\]

Also, using the trigonometric identity \( \tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x} \), we can get another series for \( C \), this time in terms of \( I_{2n+1} \) (compare §2):

\[
C = \int_0^{\pi/2} \frac{\ln \tan \frac{x}{2}}{2} \, dx
\]

\[
= -\frac{1}{4} \int_0^{\pi/2} \ln \left( \frac{1 - \cos x}{1 + \cos x} \right) \, dx
\]

\[
= \frac{1}{2} \int_0^{\pi/2} \left( \sum_{n=0}^\infty \frac{\cos^{2n+1} x}{2n + 1} \right) \, dx, \quad \text{on using the Maclaurin series for } \ln \left( \frac{1 + x}{1 - x} \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^\infty \frac{I_{2n+1}}{2n + 1}.
\]

Double integrals, first introduced by Euler in [E391, 1768], also reap dividends in the form of a series for \( C \) involving both \( I_{2n} \) and \( I_{2n+1} \):

\[
C = \sum_{n=0}^\infty \frac{(-1)^n}{(2n + 1)^2}
\]

\[
= \int_0^1 \int_0^1 (1 - s^2 t^2 + s^4 t^4 - \ldots) \, ds \, dt
\]

\[
= \int_0^1 \int_0^1 \frac{1}{1 + s^2 t^2} \, ds \, dt
\]
\[
= \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2} x \ sec^2 \frac{1}{2} y}{4 + 4 \ tan^2 \frac{1}{2} x \ tan^2 \frac{1}{2} y} \, dx \, dy,
\]

using that sneaky substitution \( s = \tan \frac{1}{2} x, \ t = \tan \frac{1}{2} y \) again
\[
= \frac{1}{2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{1 + \cos x \cos y} \, dx \, dy,
\]
on simplifying using the double-angle formulae for \( \cos x, \cos y \)
\[
= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n I_n^2,
\]
on reversing the ‘expand and integrate term-wise’ move made above: here, as in §1,
\[
I_n = \int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \, dx.
\]

There is an interesting switch here from the absolutely convergent alternating series for \( C \) at the start to the conditionally convergent alternating series at the end!

Adamchik has a useful compilation of other series and integral representations for \( C \) on his website, [6]. Numerical evaluation gives \( C = 0.915965594677... \) but it is not known whether or not \( C \) is an irrational number: the plot thickens if we consider the series
\[
S_i = \sum_{n=0}^{\infty} \frac{1}{(4n + i)^2}, \ i = 1, 2, 3, 4.
\]
For \( S_2 = \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} = \frac{\pi^2}{32} \) and \( S_4 = \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{96} \), while
\[
S_1 + S_3 = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} = \frac{\pi^2}{8} \text{ and } S_1 - S_3 = C, \text{ Thus } S_1 = \frac{\pi^2}{16} + \frac{C}{2},
\]
\( S_3 = \frac{\pi^2}{16} - \frac{C}{2} \) so that the series \( S_1, S_3 \) and \( C \) are all equally mysterious!

**Box 3: Euler and the evaluation of \( \zeta (n) \)**

Writing \( \zeta (n) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) and \( L(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^s} \), Euler was able in [E41, 1735] to extend his evaluation of \( \zeta (2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \) at first laboriously to \( \zeta (4), \zeta (6), \ldots, \zeta (12) \) and \( L(3), L(5), L(7) \): Ayoub’s very readable survey article, [4], and Weil’s magisterial account of Euler’s work in number theory, [7], are well worth reading on the twists and turns of Euler’s quest for insight. Ultimately Euler was to find the infinite product representations for \( \sin x \) and \( \cos x \) and use these to establish the general
formulae $\zeta(2n) = \frac{B_{2n}}{(2n)!} 2^{2n-1} \pi^{2n}$ and $L(2n + 1) = \frac{E_{2n}}{(2n)!} 2^{2n+2}$: here $(B_n)$ are the (rational) Bernoulli numbers encountered, for example, in the Maclaurin series $\tan x = \sum_{n=0}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} x^{2n-1}$ and $(E_n)$ are the (integral) Euler numbers encountered, for example, in the Maclaurin series $\sec x = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n)!} x^{2n}$.

However, $L(2) (= C)$ and $\zeta(3)$ resisted all of Euler's efforts to evaluate them: $L(2)$ remains a mystery and it was only as recently as 1978 that Apéry succeeded in proving that $\zeta(3)$ is irrational.

4. Euler's constant

Euler's constant, $\gamma$, is defined as $\gamma = \lim_{n \to \infty} \gamma_n$ where $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$, $n \geq 1$. Figure 1 gives a pictorial 'look and see' proof that the limit defining $\gamma$ exists. The shaded regions (between the upper rectangles and the graph of $y = \frac{1}{x}$, together with the rectangle $n - 1 \leq x \leq n$, $0 < y < \frac{1}{n}$) have total area $\gamma_n$ and these have been translated horizontally to lie within the strip $0 < x, y$. Since the area for $\gamma_n$ is a subset of that for $\gamma_{n-1}$, $(\gamma_n)$ is a decreasing sequence; from the convexity of the graph, $\gamma_n$ occupies more than half of the initial strip so $(\gamma_n)$ is bounded below by $\frac{1}{2}$ and hence converges with limit $\gamma > \frac{1}{2}$.

![Figure 1: Existence of Euler's constant](https://www.cambridge.org/core/core/terms, https://doi.org/10.1017/S0025557200182063)
Clearly, since, \( \ln(n + 1) - \ln n \to 0 \) as \( n \to \infty \),

\[
\gamma = \lim_{n \to \infty} \left[ 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + 1) \right]
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{1}{k} - \ln \left( \frac{k + 1}{k} \right) \right] \quad (A)
\]

\[
= \lim_{n \to \infty} \sum_{k=2}^{n+1} \left[ \frac{1}{k-1} + \ln \left( \frac{k-1}{k} \right) \right] = \lim_{n \to \infty} \sum_{k=2}^{n+1} \left[ \frac{1}{k} + \ln \left( \frac{1}{k} \right) \right] \quad (B)
\]

\[
= \lim_{n \to \infty} \left\{ 1 + \sum_{k=2}^{n+1} \left[ \frac{1}{k} + \ln \left( \frac{1}{k} \right) \right] \right\}, \text{ since } \frac{1}{n+1} \to 0 \text{ as } n \to \infty. \quad (C)
\]

Putting \( x = \frac{1}{k} \) into the respective Maclaurin series for \( \ln(1 + x) \) in (A), \( \ln(1 - x) \) in (B) and \( \ln(1 - x) \) in (C) and rearranging gives

\[
\gamma = \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \quad (A)
\]

\[
\gamma = \sum_{k=2}^{\infty} \left( \frac{k-1}{k} \right) (\zeta(k) - 1) \quad (B)
\]

\[
\gamma = 1 - \sum_{k=2}^{\infty} \frac{1}{k} (\zeta(k) - 1). \quad (C)
\]

Numerical evaluation (which Euler did to 5 decimal places using the series in (A) in the paper [E43, 1740] where he introduced \( \gamma \)) gives \( \gamma = 0.577215664901... \) but, as with in §3, it is a deep unsolved problem as to whether or not \( \gamma \) is irrational. Euler’s constant also features in some intriguing integrals: one of my favourites is \( \int_{0}^{\infty} e^{-x} \ln x \, dx = -\gamma \). To see this, write

\[
\int_{0}^{\infty} e^{-x} \ln x \, dx = \lim_{n \to \infty} \int_{0}^{n} \left( 1 - \frac{x}{n} \right)^{n} \ln x \, dx = \lim_{n \to \infty} n \int_{0}^{1} t^{n} \ln(n - nt) \, dt
\]

where \( t = 1 - \frac{x}{n} \). But this latter integral is

\[
n \int_{0}^{1} \left( t^{n} - \frac{1}{n+1} \right) \ln(n - nt) \, dt + \frac{n}{n+1} \int_{0}^{1} \ln(n - nt) \, dt
\]

\[
= \frac{-n}{n+1} \int_{0}^{1} t^{n+1} \ln(n - nt) \, dt + \frac{n}{n+1} \int_{0}^{1} \ln(n - nt) \, dt \quad (\text{integrating each term by parts})
\]

\[
= \frac{-n}{n+1} \int_{0}^{1} \left( t + t^{2} + \ldots + t^{n} \right) dt + \frac{n}{n+1} (\ln n - 1)
\]

\[
= \frac{-n}{n+1} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n+1} - \ln n \right)
\]

\[
\to -\gamma \text{ as } n \to \infty.
\]
Can we avoid $\ln n$ in the definition of $\gamma$?

Let $H_n$ be the ‘harmonic sum’ $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ so that $\gamma_n = H_n - \ln n$ and consider the double sequence $a_{m,n} = H_m + H_n - H_{mn}$. Then $a_{m,n} = \gamma_m + \ln m + \gamma_n + \ln n - (\gamma_m + \ln mn) = \gamma_m + \gamma_n - \gamma_{mn}$. Clearly $a_{m,n} = a_{n,m}$ and

$$a_{m,n} - a_{m,n+1} = H_{m(n+1)} - H_{mn} + H_n - H_{n+1}$$

$$= \frac{1}{mn+1} + \ldots + \frac{1}{mn+n} - \frac{1}{n+1}$$

$$> \frac{m}{mn+m} - \frac{1}{n+1} = 0.$$ 

Thus $(a_{m,n})$ decreases both with increasing $m$ and increasing $n$. Also, for fixed $m$, $\lim_{n \to \infty} a_{m,n} = \gamma_m + \gamma - \gamma = \gamma_m$ so that $\lim_{m,n \to \infty} a_{m,n} = \gamma$. It follows that, for any increasing function $f : \mathbb{N} \to \mathbb{N}$, the ‘log-free’ sequence $(a_{m,f(n)})$ of rational numbers converges to $\gamma$. Pólya and Szegő in [8] and Scott in [9] both note this behaviour for the particular case $f(n) = n^2$.

G. H. Hardy’s exquisite sum

The following exquisite result of G. H. Hardy features as an exercise in [10, p. 277]. Since the sequence $(\gamma_n - \gamma)$ decreases to 0, the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} (\gamma_n - \gamma)$ converges and it is natural to seek its sum.

$$\sum_{n=1}^{2N} (-1)^{n-1} (\gamma_n - \gamma) = (\gamma_1 - \gamma_2) + (\gamma_3 - \gamma_4) + \ldots + (\gamma_{2N-1} - \gamma_{2N})$$

$$= \left( \ln \frac{2}{1} - \frac{1}{2} \right) + \left( \ln \frac{4}{3} - \frac{1}{4} \right) + \ldots + \left( \ln \frac{2N}{2N-1} - \frac{1}{2N} \right)$$

$$= \ln \frac{2 \cdot 4 \cdots 2N}{1 \cdot 3 \cdots 2N-1} - \frac{1}{2} \left( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{N} \right)$$

$$= \frac{1}{2} \ln \prod_{n=1}^{N} \frac{2n.2n}{2n-1.2n+1} + \frac{1}{2} \ln (2N+1) - \frac{1}{2} (\gamma_N + \ln N)$$

$$\to \frac{1}{2} \ln \frac{\pi}{2} - \frac{1}{2} \ln 2 - \frac{1}{2} \gamma \text{ as } N \to \infty,$$

on using Wallis’s product (§1) in the first term. The sum of the series is thus $\frac{1}{2} \ln \frac{\pi}{e^2}$, a beautiful formula which Euler himself would surely have enjoyed!

A concrete example of Riemann’s rearrangement theorem

Riemann’s famous rearrangement theorem asserts that, for every conditionally convergent series of real numbers, given any real number $L$, there exists a rearrangement of the terms of the series with sum $L$. For the
conditionally convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, we can use $\gamma$ to make Riemann's abstract theorem concrete as follows. Let $(P_n), (Q_n)$ be two strictly increasing sequences of positive integers and let $T_n$ be the sum of the first $P_n$ positive terms together with the first $Q_n$ negative terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$: $T_n$ thus corresponds to a specific rearrangement of the terms. Then

$$T_n = \left(1 + \frac{1}{3} + \ldots + \frac{1}{2P_n - 1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2Q_n}\right),$$

$$= \left(1 + \frac{1}{2} + \ldots + \frac{1}{2P_n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2P_n}\right) = \gamma_{2P_n} + \ln(2P_n) - \frac{1}{2}(\gamma_{P_n} + \ln P_n) - \frac{1}{2}(\gamma_{Q_n} + \ln Q_n),$$

$$= \gamma_{2P_n} - \frac{1}{2}\gamma_{P_n} - \frac{1}{2}\gamma_{Q_n} + \ln\left(\frac{P_n}{Q_n}\right).$$

Suppose further that $\frac{P_n}{Q_n} \to \frac{1}{4}e^{2\lambda} = \alpha$; then $T_n \to L$. Indeed, the specific choice $Q_n = n$, $P_n = \lfloor n\alpha \rfloor$ works for $\alpha > 1$ while $Q_n = \lceil n\alpha \rceil$, $P_n = n$ works for $0 < \alpha < 1$. (This is an instance of a more general result due to Pringsheim about specific rearrangements of conditionally convergent series which may be found in [10, p. 75-76].)

**Box 4: Euler and $\gamma$**

Euler introduced his eponymous constant (which he called $C$) almost *en passant* in [E43, 1740] by deriving the equivalent of formula (A) above and using it to calculate $\gamma$ correct to 5 decimal places. This paper also contains some related series, which he tackled by the ‘$\gamma_n$-method’ we used in our discussion of Riemann's rearrangement theorem, such as

$$\ln r = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r-1} - \frac{1}{r+1} + \frac{1}{r+2} + \ldots + \frac{1}{2r-1} - \frac{1}{2r} + \ldots,$$

a generalisation of the series for $\ln 2$, and the more intricate

$$\frac{\pi^2}{6} - \ln 2 - \gamma = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \ldots.$$  

Euler returned to this topic in [E583, 1785] where he focused on calculating $\gamma$ to higher precision. He noted the occurrence of $\gamma$ in integrals such as $\gamma = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right) dx$ and he derived formulae (B) and (C) above, together with the asymptotic expansion $\gamma_n - \gamma \sim \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$  

which he obtained by applying the Euler-Maclaurin summation formula to $f(x) = \frac{1}{x}$ between $x = 1$ and $x = n$. He then used the latter series to calculate $\gamma$ correct to 15 decimal places. The Bernoulli numbers form something of a *leitmotiv* in Euler's work on analysis as he strove to understand their multiple occurrences in the formulae for $\sum_{n=1}^{N} n^k$, in the
formulae for \( \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \), in the Euler-Maclaurin summation formula and in the Maclaurin series for functions such as \( \frac{x}{e^{x} - 1} \). It was Hardy who opined in [11] that ‘The most famous constant in analysis, after \( e \) and \( \pi \), is Euler’s constant \( \gamma \) …’. But, while there is a plethora of attractive popular books on \( \pi \), I think there is only one such on \( e \), [12], and I am certain there is only one on \( \gamma \), [13]!

5. The ‘sinc’ integral, \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \)

The ‘sinc’ integral, \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx \), was one of countless ‘one-off’ tricky indefinite integrals that Euler successfully evaluated: as Dunham puts it in his very readable chapter on Euler in [14], ‘Euler was one of history’s foremost integrators, and the more bizarre the integrand, the better.’ The following unusual proof that \( \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} \) is due to Lobachevsky ([15], generalised in [10, p. 515]) but uses some of Euler’s most beautiful formulae involving trigonometric functions from [E61, 1742]. On substituting the infinite products \( \sin x = \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2\pi^2} \right) \), \( \cos x = \prod_{n=0}^{\infty} \left( 1 - \frac{4x^2}{(2n+1)^2\pi^2} \right) \) into the identity \(-\ln \cot \frac{x}{2} = \ln \sin \frac{x}{2} - \ln \cos \frac{x}{2} \) and differentiating, we obtain the ‘partial fraction’ expansion \( \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2x}{x^2 - n^2\pi^2} \). Now let \( I = \int_{0}^{\infty} \frac{\sin x}{x} \, dx \). Then

\[
I = \int_{0}^{\pi} \frac{\sin x}{x} \, dx + \int_{\pi}^{2\pi} \frac{\sin x}{x} \, dx + \int_{2\pi}^{3\pi} \frac{\sin x}{x} \, dx + \ldots \\
= \int_{0}^{\pi} \left( \frac{\sin x}{x} - \frac{\sin x}{x + \pi} + \frac{\sin x}{x + 2\pi} - \ldots \right) \, dx.
\]

Substituting \( \pi - x \) for \( x \) then shows that

\[
I = \int_{0}^{\pi} \left( \frac{\sin x}{x - \pi} + \frac{\sin x}{x - 2\pi} - \ldots \right) \, dx
\]

and, on adding the two expressions for \( I \), that

\[
2I = \int_{0}^{\pi} \sin x \left( \frac{1}{x} - \frac{2x}{x^2 - \pi^2} + \frac{2x}{x^2 - 4\pi^2} - \ldots \right) \, dx = \int_{0}^{\pi} \sin x \csc x \, dx = \pi
\]

from which \( I = \frac{\pi}{2} \). Incidentally, it is worth noting in passing that the behaviour of the sequence of functions \( s_n(x) = \frac{\pi}{2} \int_{0}^{x} \frac{\sin t}{t} \, dt \) is the key to the explanation of Gibb’s phenomenon. For, by the above, on the one hand, as

\[
n \to \infty, \quad s_n(x) \to s(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

considering the area under the graph \( y = \frac{\sin t}{t} \), we see that, for fixed \( n \), \( s_n(x) \)
has a global maximum value of \( s_n \left( \frac{\pi}{n} \right) = \frac{2}{\pi} \int_0^{\pi/2} \sin t \, dt \approx 1.179 \) and a global minimum value of \( s_n \left( -\frac{\pi}{n} \right) \approx -1.179. \) Thus, if we plot the graphs of \( s_n(x) \) for increasing \( n \), pointwise they converge to \( s(x) \), but in all neighbourhoods of 0 the approximations \( s_n(x) \) stubbornly overshoot their targets of \( \pm 1 \) by some 17.9%! Nahin’s recent book, [16], contains an entertaining account of the tangled history of this spectacular manifestation of the non-uniformity of the convergence of \( (s_n(x)) \) to \( s(x) \) in the context of Fourier series. Readers may also like to explore the use to which the sinc integral is put in [17] and the references therein.

6. The probability integral, \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \)

We start by reminding ourselves of Timothy Jameson’s lovely evaluation of the probability integral from the pages of the Gazette, [18]. The ‘hill’ \( z = e^{-(x^2+y^2)} \) has volume \( \int_{-\infty}^{\infty} e^{-x^2} \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \, dy = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 \). But it also is the volume of revolution generated by rotating \( x = \sqrt{-\ln z} \) around the \( z \)-axis; thus \( \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \pi \int_0^1 x^2 \, dz = \pi \Gamma \left( \frac{1}{2} \right) \), on recalling from Box 1 that \( \int_0^1 (-\ln z)^n \, dz = n! \). It follows that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \).

We now use this result as a catalyst to find the volume, \( V_n(r) \), and the surface area, \( S_n(r) \), of the hypersphere \( x_1^2 + \ldots + x_n^2 \leq r^2 \); this neat argument is due to Weyl and is reproduced in [19]. For 

\[
\pi^{n/2} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-(x_1^2 + \ldots + x_n^2)} \, dx_1 \ldots \, dx_n \\
= \int_0^\infty \ldots \int_0^\infty e^{-r^2} S_n(r) \, dr, \text{ on integrating radially over ‘hypershells’} \\
= S_n(1) \int_0^\infty e^{-r^2} r^{n-1} \, dr, \text{ since } S_n(r) = r^{n-1} S_n(1), \text{ by scaling} \\
= \frac{1}{2} S_n(1) \int_0^\infty e^{-t} t^{n-1} \, dt, \text{ on substituting } t = r^2 \\
= \frac{1}{2} S_n(1) \Gamma \left( \frac{n}{2} \right). 
\]

It follows that \( S_n(r) = r^{n-1} S_n(1) = \frac{2\pi^{n/2}}{\Gamma \left( \frac{n}{2} \right)} r^{n-1} \) and, since \( V_n(r) = \int_0^r S_n(x) \, dx = \int_0^r S_n(1) x^{n-1} \, dx = \frac{S_n(1)}{n} r^n \), \( V_n(r) = \frac{2\pi^{n/2}}{n\Gamma \left( \frac{n}{2} \right)} r^n = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)} r^n \). The more usual proofs of these formulae use a recursive argument. For example, by considering \( x_1^2 + \ldots + x_n^2 \leq r^2 \) as a volume of revolution,

\[
V_n(r) = \int_{-r}^{r} V_{n-1}(\sqrt{r^2-x^2}) \, dx = \int_{-r}^{r} \left( 1 - \frac{x^2}{r^2} \right)^{(n-1)/2} \, dx = 2r V_{n-1}(r) \int_0^{\pi/2} \sin^n t \, dt, 
\]
on setting \( x = r \cos t \). Substituting for \( \int_0^{\pi/2} \sin^n t \, dt \) then gives the recursive formula; here, we can reverse the argument to give \( \int_0^{\pi/2} \sin^n t \, dt = \frac{V_n(r)}{2r V_{n-1}(r)} \)

\[
= \frac{\sqrt{\pi} \Gamma \left( \frac{n+1}{2} \right)}{2 \Gamma \left( \frac{n}{2} + 1 \right)} \text{ and we have recovered a unified odd/even-case formula}
\]
for the ‘Wallis’s product’ integral with which we started this article. Having
gone full circle, ‘Our revels now are ended’! My role has largely been to
function (in the words of Montaigne) as the thread holding together a
bouquet of other people’s flowers, flowers which both trace their roots and
bloom in homage to ‘Analysis Incarnate’, the mathematical genius who was
Leonhard Euler.

Box 5: Euler and the probability integral

Did Euler know the probability integral? The first explicit evaluation
of it is usually attributed to Laplace in a 1774 paper on probability. But
there is no doubt that, well before this, Euler knew the equivalent result,
\[ \int_0^1 (\ln x)^{-1/2} dx = \sqrt{\pi}, \]

obtained by substituting for \( e^{-x^2} \) in the probability
integral. Indeed (Box 1), having derived \( \int_0^1 (\ln x)^{1/2} dx = \frac{\sqrt{\pi}}{2} \),
the value of \( \int_0^1 (\ln x)^{-1/2} dx = \Gamma (\frac{1}{2}) \) follows immediately either from
\( \Gamma (x) = (x - 1) \Gamma (x - 1) \) with \( x = \frac{3}{2} \), or from \( B(m, n) = \frac{\Gamma (m) \Gamma (n)}{\Gamma (m + n)} \)
with \( m = n = \frac{1}{2} \), or from the reflection formula \( \Gamma (x) \Gamma (1 - x) = \frac{\pi}{\sin \pi x} \)
with \( x = \frac{1}{2} \) — all three of which are due to Euler! Beautiful definite
integrals abound in Euler’s writings: spoilt for choice, one might cite
\[ \int_0^1 \frac{\ln x}{\sqrt{1 - x^2}} dx = \int_0^\frac{\pi}{2} \ln \sin x \, dx = -\frac{1}{2} \pi \ln 2 \text{ (from [E499, 1780])}, \]
\[ \int_0^1 (\ln x)^5 \, dx = \frac{-8 \pi^6}{63}, \int_0^1 (\ln x)^4 \, dx = \frac{31 \pi^6}{252}, \int_0^1 (\ln x)^3 \, dx = \frac{5 \pi^5}{64} \]
\text{(from [E463, 1775]),}

\[ \int_0^\infty \frac{x^{m-1}}{1 + x^n} \, dx = \int_0^1 \frac{x^{m-1}}{\sqrt{(1 - x^n)^m}} = \frac{\pi}{n \sin \frac{m \pi}{n}} \]

for \( n > m > 0 \) (from [E500, 1777/80], [E588, 1785], E[675, 1794]),

\[ \int_0^\infty e^{-px} \sin qx \, \frac{dx}{\sqrt{x}} = \sqrt{\frac{\pi (\sqrt{p^2 + q^2} - p)}{2 (p^2 + q^2)}}, \]
\[ \int_0^\infty e^{-px} \cos qx \, \frac{dx}{\sqrt{x}} = \sqrt{\frac{\pi (\sqrt{p^2 + q^2} + p)}{2 (p^2 + q^2)}} \]

for \( p, q > 0 \) (from [E675, 1794], with the Fresnel integrals
\( p = 0, q = 1 \) as limiting cases),

\[ \int_0^\infty e^{-px} \sin qx \, \frac{dx}{x} = \tan^{-1} \frac{q}{p} \]

for \( p > 0 \) (from [E675, 1794], with the sinc integral \( p = 0, q = 1 \)
as a limiting case).

The attribution of integrals to their original evaluators is a notoriously
tricky area: one inspirational recent source with useful historical
references and a fulsome bibliography is [20].

‘There will, I believe, always be something new to be learned in
reading Euler.’ (Nahin, [16, p.323])
References


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