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UNIQUENESS OF TRACES ON LOG-POLYHOMOGENEOUS PSEUDODIFFERENTIAL OPERATORS

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Abstract

We show how to derive the uniqueness of graded or ordinary traces on some algebras of log-polyhomogeneous pseudodifferential operators from the uniqueness of their restriction to classical pseudodifferential ones.

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1. Introduction

We consider a closed connected Riemannian manifold M of finite dimension n and a finite rank hermitian vector bundle E over M. A pseudodifferential operator (PDO) acting on smooth sections of E is called classical (or polyhomogeneous) (see [13]) if, locally, its symbol is classical, that is, it admits an asymptotic expansion in positively homogeneous components.

A pseudodifferential operator L acting on smooth sections of E is called log-polyhomogeneous if, locally, its symbol has the form

$$a_k(x, \xi) \log^k |\xi| + a_{k-1}(x, \xi) \log^{k-1} |\xi| + \cdots + a_0(x, \xi), \tag{1.1}$$

where $k \in \mathbb{N}$ and $a_0, \ldots, a_k$ are classical symbols. We call the integer k the *log-degree* of L. We denote by:

- $\mathcal{L}$ the algebra of log-polyhomogeneous PDOs acting on smooth sections of E;
- $C\ell$ the subalgebra of classical PDOs in $\mathcal{L}$;
- Q an admissible classical PDO of positive order such that $\operatorname{Log} Q$ exists;

- $\mathcal{A}_{C\ell}$ a subalgebra of $C\ell$ such that $[\mathcal{A}_{C\ell}, \operatorname{Log} Q] \subset \mathcal{A}_{C\ell}$ (in general, we only know that the commutator of a classical PDO and a logarithm is a classical PDO; hence, we only have $[\mathcal{A}_{C\ell}, \operatorname{Log} Q] \subset C\ell$);
- $\mathcal{A}$ the subalgebra of $\mathcal{L}$ generated by $\operatorname{Log} Q$ and $\mathcal{A}_{C\ell}$.

The assumption on $\mathcal{A}_{C\ell}$ implies the following fundamental decomposition of $\mathcal{A}$ (see Lemma 2.2).

$$\mathcal{A} = \bigoplus_{k=0}^{+\infty} \mathcal{A}_{C\ell} \operatorname{Log}^k Q. \tag{1.2}$$

We assume that $\mathcal{A}_{C\ell}$ does not consist only of smoothing operators. Otherwise, $\mathcal{A}_{C\ell}$ and $\mathcal{A}$ are algebras of smoothing operators. Guillemin [4] has shown that the L^2 trace is the unique trace on such algebras.

We say that a linear form τ on an algebra is a trace if, for any operators A and B in the algebra, we have $\tau(AB) = \tau(BA)$. That is, τ vanishes on commutators. On a graded algebra $\mathcal{B} = \bigoplus_{k\geq 0} \mathcal{B}^k$, a graded trace is a sequence $(\tau_k)_{k\in\mathbb{N}}$ of linear forms τ_k on $\bigoplus_{0\leq l\leq k} \mathcal{B}^l$ which vanishes on $\bigoplus_{0\leq l\leq k-1} \mathcal{B}^l$ and satisfies $\tau_{k+m}(AB) = \tau_{k+m}(BA)$ for $A \in \bigoplus_{0\leq l\leq k} \mathcal{B}^l$ and $B \in \bigoplus_{0\leq l\leq m} \mathcal{B}^l$.

Under the assumption of the uniqueness of a trace τ_0 on $\mathcal{A}_{C\ell}$, we show that there exists a unique graded trace $(\overline{\tau_0}^{\,k})_{k\in\mathbb{N}}$ on the whole algebra $\mathcal{A}$ which extends τ_0 (see Theorem 3.3). We also prove that, if there exists a trace on $\mathcal{A}$ extending τ_0, then this extension is unique (see Theorem 3.5).

Our first result applies to the Wodzicki–Guillemin residue (also called the noncommutative residue) Res on the algebra $C\ell$ provided that M is of dimension $n \geq 2$. It is well known that the Wodzicki–Guillemin residue Res is the unique trace on $C\ell$. Setting $\operatorname{Res}_0 = \operatorname{Res}$, the extension $\overline{\operatorname{Res}_0}^{\,k}$ coincides, up to a multiplicative factor, with the higher noncommutative residue Res_k introduced by Lesch.

The kth residue of an operator L in $\mathcal{L}$ of log-degree k with local symbol

$$\sigma(L) = a_k(x,\xi)\log^k|\xi| + a_{k-1}(x,\xi)\log^{k-1}|\xi| + \cdots + a_0(x,\xi)$$

is defined by

$$\operatorname{Res}_k(L) = (k+1)! \int_M \int_{S^*M} \operatorname{tr}((a_k)_{-n}(x,\xi))\, d\xi\, dx.$$

When $\mathcal{L}$ is seen as a graded algebra (the grading is given by log-degrees), Lesch has shown in [8] that the sequence $(\operatorname{Res}_k)_{k\in\mathbb{N}}$ is the unique graded trace. We recover the same result by an alternative approach.

Our second result applies to the canonical trace on the algebra of odd-class log-polyhomogeneous PDOs when the manifold M is odd dimensional. According to Kontsevitch and Vishik [6, 7], a classical operator A of order $m \in \mathbb{Z}$ is of *odd class* if, locally, the positively homogeneous components of its symbol $\{a_{m-j} \mid j \in \mathbb{Z}\}$ are simply homogeneous, that is, they have the property that

$$a_{m-j}(x,-\xi) = (-1)^{m-j} a_{m-j}(x,\xi). \tag{1.3}$$

The odd-class classical PDOs form an algebra. Following [10], we say that a log-polyhomogeneous PDO L is of odd class if, locally, all the classical symbols $a_0, \ldots, a_k$ arising as coefficients of powers of $\log|\xi|$ in its symbol (see (1.1)) have the above property (1.3). In a similar manner to that for odd-class classical PDOs, one can easily check that odd-class log-polyhomogeneous PDOs form an algebra.

When M is odd dimensional, the canonical trace TR of [6] is well defined on the algebra of odd-class classical PDOs. Recently, Maniccia *et al.* [9] proved that TR is the unique trace on this algebra. The canonical trace was first extended by Lesch [8] to log-polyhomogeneous PDOs of noninteger orders (this means that the orders of $a_0, \ldots, a_k$ are not integers). It has further been extended by Paycha and Scott [10] to odd-class log-polyhomogeneous PDOs when M is odd dimensional. We refer the reader to Section 4 for more details. To the best of our knowledge, our proof of the uniqueness of TR on odd-class log-polyhomogeneous PDOs is new.

2. Preliminaries

Following [6, 12, 13], we begin by briefly reviewing the definitions of complex powers and then those of logarithms of an operator of positive order.

Let Q be an invertible classical PDO of positive order q. We say that Q is *admissible* if there exists a closed angle of vertex zero which does not intersect the spectrum of the leading symbol of Q. If Q is admissible, then there is a half-line

$$L_\theta = \{z \in \mathbb{C} \mid \arg z = \theta\}$$

which does not meet the spectrum of Q. We call such a half-line a *spectral cut*.

Now let Q be an admissible PDO with spectral cut L_θ. Then complex powers Q_θ^s are well defined for all $s \in \mathbb{C}$ and the logarithm $\mathrm{Log}_\theta Q$ is obtained as the derivative at zero $\mathrm{Log}_\theta Q = D_s Q_\theta^s|_{s=0}$. The complex power Q_θ^s is still an invertible classical PDO of order qs, whereas $\mathrm{Log}_\theta Q$ is no longer classical. Locally, its symbol reads

$$q \log|\xi|\mathrm{id} + \sigma_0(x, \xi),$$

where σ_0 is a classical symbol of order zero. Thus, $\mathrm{Log}_\theta Q$ is a PDO of positive order ε for all $\varepsilon > 0$. Since the choice of a spectral cut will not be important when taking an admissible operator, we will omit the mention of θ.

From now on, let Q be an admissible operator and let $\mathcal{A}_{C\ell}$ be an algebra of classical PDOs which are not reduced to smoothing operators and such that $[\mathcal{A}_{C\ell}, \mathrm{Log}\, Q]$ lies in $\mathcal{A}_{C\ell}$. Let $\mathcal{A}$ be the algebra generated by $\mathrm{Log}\, Q$ and $\mathcal{A}_{C\ell}$.

The following are some elementary, yet fundamental, results that we shall require in the sequel.

LEMMA 2.1. *If A is in $\mathcal{A}_{C\ell}$ and if $k \geq 1$, then $[A, \mathrm{Log}^k Q]$ is in $\mathcal{A}$ and is of log-degree $k - 1$.*

PROOF. The fact that $[A, \mathrm{Log}\, Q]$ is classical is stated in [3]. For the sake of completeness, here are the detailed calculations that prove this fact. We recall the

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composition law of symbols corresponding to the composition of PDOs (see [13]):

$$\sigma \star \sigma'(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(x, \xi) \partial_x^\alpha \sigma'(x, \xi).$$

Locally, let σ be a symbol of A and let

$$q \log |\xi| \mathrm{id} + \sigma_0(x, \xi)$$

be a symbol of $\mathrm{Log}\, Q$ (as above, q denotes the order of Q). Then a symbol of $A \,\mathrm{Log}\, Q$ is

$$\sigma(A \,\mathrm{Log}\, Q)(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(x, \xi) \partial_x^\alpha (q \log |\xi| \mathrm{id} + \sigma_0(x, \xi)),$$
$$\sigma(A \,\mathrm{Log}\, Q)(x, \xi) = (\sigma \star \sigma_0)(x, \xi) + q \log |\xi| \sigma(x, \xi)$$

and a symbol of $\mathrm{Log}\, QA$ is

$$\sigma(\mathrm{Log}\, QA)(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha (q \log |\xi| \mathrm{id} + \sigma_0(x, \xi)) \partial_x^\alpha \sigma(x, \xi),$$
$$\sigma(\mathrm{Log}\, QA)(x, \xi) = (\sigma_0 \star \sigma)(x, \xi) + q \log |\xi| \sigma(x, \xi) + q \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \log |\xi| \partial_x^\alpha \sigma(x, \xi).$$

The derivative $\partial_\xi^\alpha \log |\xi|$ is homogeneous of degree $-|\alpha|$. Thus, a symbol of $[A, \mathrm{Log}\, Q] = A \,\mathrm{Log}\, Q - \mathrm{Log}\, QA$ is classical. By our assumption on $\mathcal{A}_{C\ell}$, we have $[A, \mathrm{Log}\, Q] \in 0\mathcal{A}_{C\ell}$.

The general result can be proved by induction on k, since we have

$$A \,\mathrm{Log}^k Q - \mathrm{Log}^k QA = (A \,\mathrm{Log}^{k-1} Q - \mathrm{Log}^{k-1} QA) \,\mathrm{Log}\, Q + \mathrm{Log}^{k-1} QA \,\mathrm{Log}\, Q - \mathrm{Log}^k QA,$$
$$[A, \mathrm{Log}^k Q] = [A, \mathrm{Log}^{k-1} Q] \,\mathrm{Log}\, Q + \mathrm{Log}^{k-1} Q [A, \mathrm{Log}\, Q].$$

LEMMA 2.2. *If $A \in \mathcal{A}$ has log-degree k, then there exist classical PDOs $A_0, A_1, \dots, A_k$ in $\mathcal{A}_{C\ell}$ such that*

$$A = A_0 + A_1 \,\mathrm{Log}\, Q + \cdots + A_k \,\mathrm{Log}^k Q.$$

That is,

$$\mathcal{A} = \bigoplus_{k=0}^{+\infty} \mathcal{A}_{C\ell} \,\mathrm{Log}^k Q.$$

PROOF. By definition of $\mathcal{A}$, A is a linear combination of terms of the form

$$C_1 \,\mathrm{Log}^{\alpha_1} Q C_2 \,\mathrm{Log}^{\alpha_2} Q \cdots C_m \,\mathrm{Log}^{\alpha_m} Q C_{m+1},$$

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where the C_i are in $\mathcal{A}_{C\ell}$ and the α_i are nonnegative integers which sum up to k. We notice that

$$\operatorname{Log}^{\alpha_m} Q C_{m+1} = C_{m+1} \operatorname{Log}^{\alpha_m} Q + [\operatorname{Log}^{\alpha_m} Q, C_{m+1}].$$

Hence, by induction on α_m and Lemma 2.1, there exist classical PDOs $A_{m+1,0}$, $A_{m+1,1}, \ldots, A_{m+1,\alpha_{m-1}}$ in $\mathcal{A}_{C\ell}$ such that

$$\begin{aligned}\operatorname{Log}^{\alpha_m} Q C_{m+1} = C_{m+1} \operatorname{Log}^{\alpha_m} Q + A_{m+1,0} \\ + A_{m+1,1} \operatorname{Log} Q + \cdots + A_{m+1,\alpha_{m-1}} \operatorname{Log}^{\alpha_{m-1}} Q.\end{aligned} \tag{2.1}$$

The result follows by induction on m. □

PROPOSITION 2.3. *Let*

$$A = A_0 + A_1 \operatorname{Log} Q + \cdots + A_k \operatorname{Log}^k Q \in \mathcal{A}$$

be an operator of log-degree k. Then:

(1) *the classical PDO A_k is unique up to a smoothing operator;*
(2) *if there exists a trace τ_0 on $\mathcal{A}_{C\ell}$ which vanishes on smoothing operators, then the linear form $\overline{\tau_0}^k : A \mapsto \tau_0(A_k)$ defines a graded trace on $\mathcal{A}$.*

PROOF. For part (1), let us consider an alternative decomposition of A, namely

$$A = A'_0 + A'_1 \operatorname{Log} Q + \cdots + A'_k \operatorname{Log}^k Q.$$

The two decompositions lead to two descriptions of the local symbol of A:

$$\sigma(A) = q^k \sigma(A_k) \log^k |\xi| + \sigma$$

and

$$\sigma(A) = q^k \sigma(A'_k) \log^k |\xi| + \sigma',$$

where $\sigma(A_k)$ and $\sigma(A'_k)$ are symbols of A_k and A'_k, respectively, σ and σ' are log-polyhomogeneous symbols of log-degree $k-1$ and q is the order of Q. It follows that the classical symbols $\sigma(A_k)$ and $\sigma(A'_k)$ differ from smoothing symbols. This implies that the difference $A_k - A'_k$ is a smoothing operator.

In part (2), if τ_0 vanishes on smoothing operators, then, since A_k is unique modulo a smoothing operator, the linear form $\overline{\tau_0}^k$ is well defined. Let A and B be two log-polyhomogeneous PDOs written as

$$A = A_0 + A_1 \operatorname{Log} Q + \cdots + A_k \operatorname{Log}^k Q$$

and

$$B = B_0 + B_1 \operatorname{Log} Q + \cdots + B_m \operatorname{Log}^m Q.$$

We have

$$\overline{\tau_0}^{k+m}(AB) = \overline{\tau_0}^{k+m}(A_k \operatorname{Log}^k Q B_m \operatorname{Log}^m Q).$$

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Using (2.1), we permute $\mathrm{Log}^k Q$ and B_m to obtain

$$\overline{\tau_0}^{\,k+m}(AB) = \overline{\tau_0}^{\,k+m}(A_k B_m \,\mathrm{Log}^{k+m} Q).$$

This implies that $\overline{\tau_0}^{\,k+m}(AB) = \tau_0(A_k B_m)$. A similar argument allows us to prove that $\overline{\tau_0}^{\,k+m}(BA) = \tau_0(B_m A_k)$. The result follows. □

3. Results and proofs

For $k \geq 0$, we let $\mathcal{A}^k$ denote the vector space of operators in $\mathcal{A}$ of log-degree k.

LEMMA 3.1. *If there exists a unique nontrivial trace τ_0 on $\mathcal{A}_{C\ell}$ and if $A \in \mathcal{A}^k$, then, for any operator P in $\mathcal{A}_{C\ell}$ such that $\tau_0(P) = 1$, there exists a finite number of operators P_i in $\mathcal{A}_{C\ell}$, Q_i in $\mathcal{A}^k$ and a finite number of complex scalars α_i such that*

$$A = \sum_{i=1}^{M} [P_i, Q_i] + P(\alpha_0 + \alpha_1 \,\mathrm{Log}\, Q + \cdots + \alpha_k \,\mathrm{Log}^k Q). \tag{3.1}$$

In particular, $\alpha_k = \tau_0(A_k)$ when A is written as

$$A = A_0 + A_1 \,\mathrm{Log}\, Q + \cdots + A_k \,\mathrm{Log}^k Q.$$

If $\mathcal{A}_{C\ell}$ does not admit any nontrivial trace, then every $A \in \mathcal{A}^k$ can be written in the form

$$A = \sum_{i=1}^{M} [P_i, Q_i] \tag{3.2}$$

with P_i in $\mathcal{A}_{C\ell}$ and Q_i in $\mathcal{A}^k$.

PROOF. We proceed by induction on k.

Let $k = 0$. In this case, A belongs to $\mathcal{A}_{C\ell}$. Let us assume that there exists a unique nontrivial trace τ_0 on $\mathcal{A}_{C\ell}$. Let $P \in \mathcal{A}_{C\ell}$ be such that $\tau_0(P) = 1$.

Let $\mathcal{D}$ be the vector subspace of $\mathcal{A}_{C\ell}$ generated by the commutators of $\mathcal{A}_{C\ell}$ and let $\mathcal{D}^{\perp}$ be the orthogonal of $\mathcal{D}$ in the algebraic dual space of $\mathcal{A}_{C\ell}$. By definition, $\mathcal{D}^{\perp}$ is the set of linear forms which vanish on $\mathcal{D}$. Hence, a trace on $\mathcal{A}_{C\ell}$ is an element of the subspace $\mathcal{T} = \mathcal{D}^{\perp}$ and the assumption of the uniqueness of τ_0 implies that $\mathcal{T}$ is generated by τ_0. Now it is a general fact that any vector subspace F, of finite or infinite dimension, of a vector space $\mathcal{E}$ satisfies $(F^{\perp})^{\perp} = F$. Here $(F^{\perp})^{\perp}$ is the vector subspace of $\mathcal{E}$ orthogonal to $F^{\perp}$ for the pairing between $\mathcal{E}$ and its dual (see, for example, [1, Section 7, no. 5, Theorem 7]). Hence, we have $(\mathcal{D}^{\perp})^{\perp} = \mathcal{D}$. That is, $\mathcal{T}^{\perp} = \mathcal{D}$. But, by definition of

$$\mathcal{T}^{\perp} = \{A \in \mathcal{A}_{C\ell} \mid \tau_0(A) = 0\},$$

we have $\mathcal{T}^{\perp} = \mathrm{Ker}\, \tau_0$. Hence, $\mathcal{D} = \mathrm{Ker}\, \tau_0$ is of codimension one. This leads to the following decomposition of A:

$$A = \sum_{i=1}^{M} [P_i, Q_i] + \tau_0(A) P$$

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with P_i and Q_i in $\mathcal{A}_{C\ell}$ for all i. Otherwise, if $\tau_0 = 0$, we simply have that $A = \sum_{i=1}^{M} [P_i, Q_i]$.

Suppose that property (3.1) holds for some $k \geq 0$. Let $A \in \mathcal{A}^{k+1}$ and let the elements $A_0, \ldots, A_{k+1}$ of $\mathcal{A}_{C\ell}$ be such that

$$A = A_0 + A_1 \operatorname{Log} Q + \cdots + A_{k+1} \operatorname{Log}^{k+1} Q.$$

Since A_{k+1} lies in $\mathcal{A}_{C\ell}$, the case for $k = 0$ gives us the decomposition

$$A_{k+1} = \sum_{i=1}^{N} [P_{k+1,i}, Q_{k+1,i}] + \tau_0(A_{k+1}) P$$

with $P_{k+1,i}$ and $Q_{k+1,i}$ in $\mathcal{A}_{C\ell}$ for all i.

In any algebra, we have that

$$[X, Y]Z = [X, YZ] + Y[Z, X].$$

Thus,

$$\begin{aligned} A_{k+1} \operatorname{Log}^{k+1} Q &= \left(\sum_{i=1}^{N} [P_{k+1,i}, Q_{k+1,i}] + \tau_0(A_{k+1}) P \right) \operatorname{Log}^{k+1} Q \\ &= \sum_{i=1}^{N} [P_{k+1,i}, Q_{k+1,i} \operatorname{Log}^{k+1} Q] + \sum_{i=1}^{N} Q_{k+1,i} [\operatorname{Log}^{k+1} Q, P_{k+1,i}] \\ &\quad + \tau_0(A_{k+1}) P \operatorname{Log}^{k+1} Q. \end{aligned}$$

Now $[P_{k+1,i}, Q_{k+1,i} \operatorname{Log}^{k+1} Q]$ is a commutator with $P_{k+1,i}$ in $\mathcal{A}_{C\ell}$ and $Q_{k+1,i} \operatorname{Log}^{k+1} Q$ in $\mathcal{A}^{k+1}$. By Lemma 2.1, $Q_{k+1,i}[\operatorname{Log}^{k+1} Q, P_{k+1,i}]$ is of log-degree k so that we can apply the inductive hypothesis to $\sum_{i=1}^{N} Q_{k+1,i}[\operatorname{Log}^{k+1} Q, P_{k+1,i}]$. We may also apply the inductive hypothesis to

$$A_0 + A_1 \operatorname{Log} Q + \cdots + A_k \operatorname{Log}^k Q.$$

Property (3.1) follows.

Suppose that property (3.2) holds and $\tau_0 = 0$. We may deduce from our calculations that, if A is in $\mathcal{A}^{k+1}$, then A is a sum of commutators of operators in $\mathcal{A}_{C\ell}$ and $\mathcal{A}^{k+1}$. □

REMARK 3.2. Lemma 3.1 extends to several traces on $\mathcal{A}_{C\ell}$. If there exist m linearly independent traces $\tau_0, \ldots, \tau_m$ on $\mathcal{A}_{C\ell}$, then there exist m operators $\widetilde{P}_0, \ldots, \widetilde{P}_m$ in $\mathcal{A}_{C\ell}$ and scalars $\alpha_{l,j}$, $0 \leq l \leq k$, $0 \leq j \leq m$ such that $\tau_j(\widetilde{P}_i) = \delta_{ij}$ and

$$A = \sum_{i=1}^{M} [P_i, Q_i] + \sum_{j=1}^{m} \widetilde{P}_j (\alpha_{0,j} + \alpha_{1,j} \operatorname{Log} Q + \cdots + \alpha_{k,j} \operatorname{Log}^k Q).$$

PROOF. In this case, the vector subspace $\mathcal{D}$ of $\mathcal{A}_{C\ell}$ generated by commutators satisfies $\mathcal{D} = \bigcap_{i=1}^{m} \operatorname{Ker} \tau_i$. Hence, $\mathcal{D}$ is of codimension m in $\mathcal{A}_{C\ell}$. The proof is similar to that for Lemma 3.1. □

THEOREM 3.3. *Suppose that there exists a unique nontrivial trace τ_0 on $\mathcal{A}_{C\ell}$. If τ_0 vanishes on smoothing operators, then the graded trace $(\overline{\tau_0}^k)_{k\in\mathbb{N}}$ on $\mathcal{A}$ extending τ_0 is unique up to a multiplicative factor (depending on k). In contrast, there is no nontrivial trace on $\mathcal{A}$.*

PROOF. Let $A \in \mathcal{A}$ be of log-degree $k \geq 1$. By Lemma 2.2, A can be written as

$$A = A_0 + A_1 \operatorname{Log} Q + \cdots + A_k \operatorname{Log}^k Q$$

with the A_i in $\mathcal{A}_{C\ell}$. Let us assume that τ_0 is nontrivial.

Applying Lemma 3.1, for any P in $\mathcal{A}_{C\ell}$ such that $\tau_0(P) = 1$ we have

$$A = \sum_{i=1}^{M} [P_i, Q_i] + L_k \tag{3.3}$$

with

$$L_k = P(\alpha_0 + \alpha_1 \operatorname{Log} Q + \cdots + \alpha_{k-1} \operatorname{Log}^{k-1} Q + \tau_0(A_k) \operatorname{Log}^k Q),$$

where the P_i are in $\mathcal{A}_{C\ell}$, the Q_i are in $\mathcal{A}^k$ and the α_i are complex numbers.

Suppose that τ_0 vanishes on smoothing operators. We can choose P to be nonsmoothing. Let $(\tau_k)_{k\in\mathbb{N}}$ be a graded trace on $\mathcal{A}$. Then we have

$$\tau_k(A) = \tau_k(L_k) = \tau_k(P \operatorname{Log}^k Q)\tau_0(A_k).$$

Since P is independent of A, $\tau_k(A)$ is equal to $\tau_0(A_k)$ up to a multiplicative factor which is independent of k. Using the notation of Proposition 2.3, we have $\tau_0(A_k) = \overline{\tau_0}^k(A)$. The uniqueness of $(\overline{\tau_0}^k)_{k\in\mathbb{N}}$ follows.

Now suppose that there exists a trace τ extending τ_0 to $\mathcal{A}$. To conclude that τ is trivial, we use an argument of Lesch (see [8]). For $A \in \mathcal{A}$ of log-degree k, we have $\overline{\tau_0}^{k+1}(A) = 0$. By the uniqueness of $(\overline{\tau_0}^k)_{k\in\mathbb{N}}$, A is a sum of commutators $[P_i, Q_i]$ with P_i in $\mathcal{A}_{C\ell}$ and Q_i in $\mathcal{A}^{k+1}$. Hence, A is a sum of commutators and $\tau(A) = 0$. □

REMARK 3.4. If there exist m linearly independent traces $\tau_0, \ldots, \tau_m$ on $\mathcal{A}_{C\ell}$ which all vanish on smoothing operators, then any graded trace on $\mathcal{A}$ is a linear combination of the m extensions $(\overline{\tau_0}^k)_{k\in\mathbb{N}}, \ldots, (\overline{\tau_m}^k)_{k\in\mathbb{N}}$.

PROOF. The result is a straightforward application of Remark 3.2 to the proof of Theorem 3.3. □

THEOREM 3.5. *Suppose that there exists a unique nontrivial trace τ_0 on $\mathcal{A}_{C\ell}$. If τ_0 does not vanish on smoothing operators and if there exists a trace τ extending τ_0 to $\mathcal{A}$, then τ is unique.*

PROOF. Suppose that τ_0 is nontrivial, but does not vanish on smoothing operators. We begin as in the proof of Theorem 3.3. In formula (3.3), we can choose P to be smoothing and satisfying $\tau_0(P) = 1$. Since P is smoothing, L_k is also smoothing. Now let Tr be the unique trace on smoothing operators. It follows that there exists β in $\mathbb{C}^*$, independent of A and such that $\tau(A) = \tau(L_k) = \beta \operatorname{Tr}(L_k)$. This implies the uniqueness of the trace τ on $\mathcal{A}$. □

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REMARK 3.6. If $\tau_0 = 0$, then any graded trace or ordinary trace extending τ_0 on $\mathcal{A}$ vanishes. Notice also that, if τ_0 does not vanish on smoothing operators, then any graded trace on $\mathcal{A}$ extending τ_0 vanishes on $\mathcal{A} \setminus \mathcal{A}_{C\ell}$.

PROOF. If $\tau_0 = 0$, then by Lemma 3.1 A is a sum of commutators of operators. The first result follows.

Now let us assume that τ_0 does not vanish on smoothing operators, and let $(\tau_k)_{k\in\mathbb{N}}$ be a graded trace extending τ_0. Following the proof of Theorem 3.5, τ_k should be proportional to the trace of a smoothing operator for any $k \geq 0$. Hence, $\tau_k = 0$ for all $k \geq 1$. □

4. Applications

We let $\mathcal{L}_{\mathrm{odd}}$ denote the algebra of odd-class log-polyhomogeneous PDOs and let $C\ell_{\mathrm{odd}}$ denote the algebra of odd-class classical PDOs. The following proposition says that both $\mathcal{L}$ and $\mathcal{L}_{\mathrm{odd}}$ satisfy property (1.2) of the introduction. We begin with a lemma about the odd-parity class of a logarithm, which can be found in [6] or [2].

LEMMA 4.1. *Let Q be an admissible operator. If Q is of odd class and even order, then* Log Q *is of odd class.*

PROOF. Let Q be of odd class and of even order q. We recall that Q^s is a classical PDO of order qs. We will denote by σ_{q-j}, for $j \geq 0$, the homogeneous components of a symbol of Q. For $s \in \mathbb{C}$ of negative real part and for a suitable contour Γ (see [12]), the positively homogeneous components of Q^s of degree $qs - j$ are expressed by

$$\sigma_{qs-j}(x, \xi) = \frac{i}{2\pi} \int_\Gamma \lambda^s b_{-q-j}(x, \xi, \lambda)\, d\lambda$$

with $b_{-q} = (\sigma_q - \lambda \mathrm{Id})^{-1}$ and satisfying, for $j \geq 1$,

$$b_{-q-j} = -b_{-q} \sum_{k+l+|\alpha|=j, l<j} i^{-|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{q-k} \partial_x^\alpha b_{-q-l}.$$

We have

$$b_{-q}(x, -\xi, \lambda) = (\sigma_q(x, -\xi) - \lambda \,\mathrm{Id}(x, \xi))^{-1}.$$

Since Q is of odd class and q is even, we have $b_{-q}(x, -\xi, \lambda) = b_{-q}(x, \xi, \lambda)$. By induction on j, we deduce that

$$b_{-q-j}(x, -\xi, \lambda) = (-1)^j b_{-q-j}(x, \xi, \lambda).$$

Thus, we have

$$\sigma_{qs-j}(x, -\xi) = (-1)^j \sigma_{qs-j}(x, \xi).$$

For $s = 0$, we have $A^0 = \mathrm{Id}$ which is odd and for $\mathrm{Re}(s) > 0$ we use $A^s A^{-s} = \mathrm{Id}$ to conclude that this equality still holds. Observing that

$$\sigma_{qs-j}(x, \xi) = |\xi|^{qs-j} \sigma_{qs-j}(x, \xi/|\xi|)$$

and differentiating at $s = 0$ gives us that Log Q is of odd class. □

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PROPOSITION 4.2. *Let Q be an admissible operator. Then:*

(1) $\mathcal{L} = \bigoplus_{k=0}^{+\infty} C\ell \operatorname{Log}^k Q$;

(2) *if Q is an admissible operator of odd class and of even order, then* $\operatorname{Log} Q$ *is of odd class and*

$$\mathcal{L}_{\mathrm{odd}} = \bigoplus_{k=0}^{+\infty} C\ell_{\mathrm{odd}} \operatorname{Log}^k Q.$$

PROOF. In part (1), the inclusion from right to left is straightforward. To show the inclusion from left to right, we proceed by induction on k. The inclusion certainly holds for $k = 0$. Suppose that it holds for some $k \geq 0$. Let $A \in \mathcal{A}$ be of log-degree $k + 1$ with symbol $a = \sum_{l=0}^{k+1} a_l \log^l |\xi|$. Using a partition of unity adapted to a finite trivializing covering of M for E, we associate to a_{k+1} a classical operator

$$A_{k+1} = Op(a_{k+1}).$$

Then the operator

$$A - \frac{1}{q^{k+1}} A_{k+1} (\operatorname{Log}^{k+1} Q)$$

lies in $\mathcal{A}$ and is of log-degree k.

The result follows by our inductive hypothesis.

The proof of part (2) proceeds like that of part (1) with A_{k+1} and $\operatorname{Log}^{k+1} Q$ of odd class. □

Now we recall the definition of the canonical trace on the algebra $\mathcal{L}_{\mathrm{odd}}$ when M is odd dimensional. For this definition, we follow [10] (see also [8]).

Let L be in $\mathcal{L}_{\mathrm{odd}}$ of log-degree k with local symbol $\sigma = \sum_{l=0}^{k} a_l \log^l |\xi|$. The canonical trace of L is defined by

$$\mathrm{TR}(L) = \int_M \mathrm{TR}_x(L)\, dx$$

via a well-defined global density on M

$$\mathrm{TR}_x(L)\, dx = \left(-\!\!\!\!\!\!\int_{T_x^* M} \mathrm{tr}_x(\sigma(x, \xi))\, d\xi \right) dx.$$

Here the finite part integral $-\!\!\!\!\!\!\int_{T_x^* M} \mathrm{tr}_x(\sigma(x, \xi))\, d\xi$ is the constant term in the asymptotic expansion of $\int_{|\xi|<R} \mathrm{tr}_x(\sigma(x, \xi))\, d\xi$ when $R \to +\infty$.

On smoothing operators, the canonical trace coincides with the L^2 trace, whereas the Wodzicki–Guillemin residue clearly vanishes.

APPLICATION 4.3. When the manifold is of dimension ≥ 2, we know from Wodzicki (see [5, 14]) and Guillemin (see [4]) that Res is the unique trace on $C\ell$ (see also [11] for detailed analysis on this subject). When combined with Proposition 4.2, Theorem 3.3 gives the existence and the uniqueness of $(\mathrm{Res}_k)_{k \in \mathbb{N}}$ on $\mathcal{L}$.

APPLICATION 4.4. When we combine Proposition 4.2 with the result of [9], the uniqueness of TR on odd-class classical PDOs for odd-dimensional manifolds and Theorem 3.5, we obtain a proof of the uniqueness of TR on $\mathcal{L}_{\rm odd}$ in odd dimensions.

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