

INVESTIGATING THE COMPUTABLE FRIEDMAN–STANLEY JUMP

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Abstract. The Friedman–Stanley jump, extensively studied by descriptive set theorists, is a fundamental tool for gauging the complexity of Borel isomorphism relations. This paper focuses on a natural computable analog of this jump operator for equivalence relations on ω , written $\dot{+}$, recently introduced by Clemens, Coskey, and Krakoff. We offer a thorough analysis of the computable Friedman–Stanley jump and its connections with the hierarchy of countable equivalence relations under the computable reducibility \leq_c . In particular, we show that this jump gives benchmark equivalence relations going up the hyperarithmetical hierarchy and we unveil the complicated highness hierarchy that arises from $\dot{+}$.

§1. Introduction. The classification of analytic equivalence relations up to Borel reducibility, written \leq_B , is a prominent area of research in contemporary descriptive set theory and it serves as a general framework for assessing the complexity of isomorphism relations on classes of countable structures \mathbb{K} . Such a study is intertwined with that of the orbit equivalence relations induced by the action of countable groups, as there is a full duality between isomorphism relations on countable models of $\mathcal{L}_{\omega_1\omega}$ -formulas and S_∞ -equivalence relations (i.e., orbit equivalence relations induced by a continuous action of the infinite permutation group).

Say that a class \mathbb{K} of countable structures is *on top for* \leq_B , if every S_∞ -equivalence relation reduces to the isomorphism relation on \mathbb{K} , written $\cong_{\mathbb{K}}$. If so, $\cong_{\mathbb{K}}$ is analytic but not Borel. It turns out that several familiar classes of structures are on top, including: undirected graphs, trees, linear orders, nilpotent groups, fields of any fixed characteristic [20]; Boolean algebras [12]; and torsion-free abelian groups [28, 29]. This may be regarded as strong evidence that these classes are simply too rich to be simplified in terms of a reasonable system of invariants.

Clearly, many other classes of mathematical interest are *not* on top: e.g., the Borel complexity of torsion-free abelian groups of finite rank n strictly increases as n grows [33]. A fundamental yardstick for gauging the complexity of isomorphism relations for classes not on top is given by the Friedman–Stanley jump operator, which dates back to the very beginning of the theory of Borel equivalence relations.

DEFINITION 1.1 (Friedman and Stanley [20]). For E an equivalence relation on a standard Borel space X , the *Friedman–Stanley jump* E^+ of E is the equivalence

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relation on X^ω given by

$$f E^+ g \leftrightarrow [\text{ran}(f)]_E = [\text{ran}(g)]_E.$$

Friedman and Stanley [20] showed that this jump operator is proper: that is, $E^+ >_B E$ for any Borel equivalence relation E with more than one equivalence class. It immediately follows that the Borel hierarchy is unbounded. The *Friedman–Stanley tower* is then obtained by starting with the identity on X and then iterating the Friedman–Stanley jump transfinitely, along the countable ordinals. Friedman and Stanley [20] proved that this tower forms a cofinal family of Borel isomorphism relations.

Parallel to the theory of Borel reductions on equivalence relations, computability theorists have studied a computability theoretic way of characterizing some countable equivalence relations as more complex than others.

DEFINITION 1.2. For equivalence relations E and R on the natural numbers,

- E is *computably reducible* to R , written $E \leq_c R$, if there is a computable function f so that

$$x E y \leftrightarrow f(x) R f(y).$$

- An equivalence relation E is *universal* with respect to a complexity class \mathcal{C} , if every member of \mathcal{C} computably reduces to E .

Computable reducibility was first introduced by Ershov [16] and has seen a recent resurgence of interest with special attention paid to local structures of equivalence relations of a given complexity class such as the c.e. equivalence relations (*ceers*) [1, 2, 4–10] and how they naturally arise from algebra [15, 19, 22, 25, 27], or other levels of the arithmetical/Ershov hierarchy [11, 23, 26].

For a long time, the study of Borel and computable reducibility were conducted independently, despite the clear analogy between the two notions. Yet, there is rapidly emerging a theory of computable reductions which blends ideas from both computability theory and descriptive set theory [13, 14, 17, 21]. In particular, computable reductions are well-suited for assessing the complexity of isomorphism relations on classes of *computable* structures, as one can encode the atomic diagram of a computable structure by a single natural number. In [18], the authors prove that, for several classes \mathbb{K} , the isomorphism relation on the computable members of \mathbb{K} is a universal Σ_1^1 equivalence relation: examples include both classes which are also on top for \leq_B and classes that are not (e.g., torsion abelian groups). Moreover, since being *on top for* \leq_c coincides with being universal Σ_1^1 , every hyperarithmetical equivalence relation (on ω) is computably reducible to, e.g., the isomorphism of computable graphs, while there are Borel equivalence relations which are not Borel reducible to the isomorphism of countable graphs [24].

Less is known about the complexity of classes that are not on top for \leq_c . To deepen the natural connection between the Borel and the computable setting, the following computable analog of the Friedman–Stanley jump has been introduced (a finitary analog of this jump appeared in [21]):

DEFINITION 1.3 (Clemens, Coskey, and Krakoff [13]). For E an equivalence relation on ω , E^+ is given by $x E^+ y$ if and only if $[W_x]_E = [W_y]_E$, where W_i is the i th c.e. set.

Clemens, Coskey, and Krakoff [13] showed that $E^+ > E$ holds for any hyperarithmetic (HYP) equivalence relation. They also asked several carefully chosen questions (presented in Section 1.2) regarding features of the jump operator $\dot{+}$. In this paper, we answer several of these questions. By doing so, we advance understanding of both the computable Friedman–Stanley jump and the global hierarchy of countable equivalence relations explored in [3]. We will prove, in particular, that the computable Friedman–Stanley tower is a cofinal family of HYP equivalence relations (Corollary 6.13). This result provides natural benchmark equivalence relations, which may pave the way to a future classification of all sorts of isomorphism relations on computable structures, in analogy with the successful line of research pursued by descriptive set theorists. Interestingly, such benchmark relations lie at the Π levels of the hyperarithmetical hierarchy, while it is known that there is no universal Π_a^0 equivalence relation for $a \geq 2$ [23]. Finally, our investigation of the computable Friedman–Stanley jump will unveil a complicated highness hierarchy (see Sections 2 and 3) that give natural subclasses of the ceers.

Throughout the rest of this paper, the “jump” of an equivalence relation will always refer to the computable Friedman–Stanley jump operator $\dot{+}$.

1.1. Preliminaries. We assume that the reader is familiar with the fundamental notions and techniques of computability theory.

All our equivalence relations have domain the set ω of the natural numbers. Equivalence relations are *infinite* if they have infinitely many equivalence classes; otherwise, they are *finite*. For a c.e. set A , the equivalence relation E_A is given by $x E_A y$ if and only if $x = y$ or $x, y \in A$. A ceer of the form E_A is called *1-dimensional*.

The equality relation on ω is denoted by Id . Note that Id^+ is equivalent to $=^{ce}$, where $x =^{ce} y$ if and only if $W_x = W_y$. Following [3, 7], we say that:

- E is *light* if $\text{Id} \leq_c E$.
- E is *dark* if E is infinite and not light.
- E is *dark minimal* if it is dark and all equivalence relations $<_c E$ are finite.

The next lemma will be used a few times.

LEMMA 1.4 [3, Lemma 1.13], [7, Lemma 3.4]. *Let R be a dark minimal equivalence relation. If W_e intersects infinitely many R -classes, then W_e must intersect every R -class.*

Let R be an infinite ceer with the property that whenever W_e intersects infinitely many R -classes, then W_e intersects every R -class. Then R is a dark minimal equivalence relation.

For two equivalence relations E, R ,

- the *uniform join* $E \oplus R$ is the equivalence relation defined by $x E \oplus R y$ if and only if $x = 2k, y = 2l$ and $k E l$ or $x = 2k + 1, y = 2l + 1$ and $k R l$;
- the *cross product* $E \times R$ is the equivalence relation defined by

$$\langle x, y \rangle (E \times R) \langle u, w \rangle \Leftrightarrow (x E u \wedge y R w).$$

For a countable sequence $(E_i)_{i \in \omega}$, $\oplus_i E_i$ is given by $\langle x, y \rangle \oplus_i E_i \langle v, w \rangle$ if and only if $x = v$ and $y E_x w$.

The following definition gives a convenient notation for working with $\dot{+}$.

DEFINITION 1.5. For sets X, Y and an equivalence relation E , we write $X \subseteq_E Y$ to mean $[X]_E \subseteq [Y]_E$. Similarly, we write $X =_E Y$ to mean $[X]_E = [Y]_E$ and $X \subsetneq_E Y$ to mean $[X]_E \subsetneq [Y]_E$.

For any set X and equivalence relation E , we write X/E for the set of E -equivalence classes of members of X .

In answering [13, Question 1] and the discussion around it, we mention the following standard definitions from computability theory:

DEFINITION 1.6. A c.e. set A is hyperhypersimple if there is no infinite c.e. set $\{i_0, i_1, \dots\}$ so that the sets W_{i_j} are disjoint, each W_{i_j} is finite, and each W_{i_j} intersects $\omega \setminus A$.

A co-infinite c.e. set B is maximal if whenever $Y \supseteq B$ is c.e., either $Y \setminus B$ is finite or $\omega \setminus Y$ is finite.

A c.e. set C is quasimaximal if it is the intersection of finitely many maximal sets.

1.2. Questions of Clemens, Coskey, and Krakoff. For every ceer E , we have $E^+ \leq_c \text{Id}^+$ [13, Proposition 4.1]. This motivates the following definition:

DEFINITION 1.7. A ceer E is high for the jump if $\text{Id}^+ \leq_c E^+$.

We note that this is the notion of highness for ceers using this jump operator. Note that any light ceer satisfies $\text{Id}^+ \leq_c E^+$, so every light ceer is high for the jump.

Clemens, Coskey, and Krakoff [13, Question 1] ask for a characterization of the c.e. sets A so that E_A is high for the jump. In Section 2, we give the following solution:

THEOREM 2.8. For a c.e. set A , $E_A^+ \equiv \text{Id}^+$ if and only if A is not hyperhypersimple. Thus, the property of being high for the jump is Σ_4^0 -complete.

This line of inquiry led us to wonder what the picture looks like for the double-jump. That is, which sets A have the property that $\text{Id}^{++} \leq_c E_A^{++}$, i.e., E_A is high₂ for the computable FS-jump. And we also ask whether there are any ceers E so that $\text{Id}^{++} \not\leq_c E^{++}$. Similarly, we want to know whether every ceer is high _{n} for some $n \in \omega$. We use the following notation, where \hat{n} is used to avoid conflicting with notation for infinite iterates of the jump given below in Definition 1.9.

DEFINITION 1.8. For $n \in \omega$ and E an equivalence relation, we let $E^{\hat{n}}$ be the n th iterate of the jump over E .

We answer these questions in Section 3.

THEOREMS 3.1, 3.6, 3.7. For every co-infinite c.e. set A , $\text{Id}^{\hat{+}} \leq_c E_A^{\hat{+}}$. Yet there are infinite ceers E so that $\text{Id}^{\hat{+}} \not\leq_c E^{\hat{+}}$.

In fact, if E is a low _{m} dark minimal ceer for any m , then $\text{Id}^{\hat{k}} \not\leq_c E^{\hat{k}}$ for every k . Yet there are dark minimal ceers E so that $\text{Id}^{\hat{+}} \leq_c E^{\hat{+}}$.

Next, every infinite ceer E has the property that $\text{Id} \leq_c E^+$ [13, Theorem 4.2], but there are infinite Δ_4^0 equivalence relations E so that $\text{Id} \not\leq_c E^+$ [13, Theorem 4.4].

Clemens, Coskey, and Krakoff ask [13, Question 6] what is the least complexity of an infinite equivalence relation E so that $\text{Id} \not\leq_c E^+$. In Section 4, we answer this with the following theorems:

THEOREM 4.1. *If $E \in \Pi_2^0$ is infinite, then $\text{Id} \leq_c E^+$.*

THEOREM 4.4. *There exists an infinite Σ_2^0 equivalence relation E so that $\text{Id} \not\leq_c E^+$.*

Clemens, Coskey, and Krakoff [13] also examine the transfinite jump hierarchy, which they defined as follows:

DEFINITION 1.9. For $a \in \mathcal{O}$ and E an equivalence relation, E^{+a} is defined by induction as follows:

If $a = 1$ (the notation for 0), then $E^{+a} = E$.

If $a = 2^b$ then $E^{+a} = (E^{+b})^+$.

If $a = 3 \cdot 5^e$, then $E^{+a} = \oplus_i E^{+\varphi_e(i)}$

Clemens, Coskey, and Krakoff show [13, Theorem 3.1] that no jump fixed-point can be hyperarithmetical (HYP). In fact, they show that if E is a jump fixed point and X is a HYP set, then $X \leq_m E$ [13, Theorem 3.2]. They ask if notations matter in the definition of the jump [13, Question 2]. They also ask whether every HYP equivalence relation E reduces to Id^{+a} for some $a \in \mathcal{O}$ [13, Question 3]. In particular, this would imply that every jump fixed point must be an upper bound under computable reduction (not just m -reduction) for all HYP equivalence relations. We answer both in the affirmative in Sections 5 and 6.

THEOREM 5.4. *For any notation b for ω^2 there exists another notation a for ω^2 so that $\text{Id}^{+a} \not\leq_c \text{Id}^{+b}$.*

There are two notations a, b for ω^2 so that Id^{+a} and Id^{+b} are incomparable.

On the other hand, if $|a| = |b|$, then Id^{+a} and Id^{+b} are somewhat related as follows:

THEOREM 5.8. *For every computable ordinal α , there is an equivalence relation E which is $\Pi_{2,\alpha+1}^0$ so that whenever $a \in \mathcal{O}$ is a notation for α , we have $\text{Id}^{+a} \leq_c E$.*

THEOREM 6.13. *Every HYP equivalence relation reduces to Id^{+a} for some $a \in \mathcal{O}$.*

COROLLARY 6.15. *If E is a fixed point of the jump, i.e., $E \equiv E^+$ then E is an upper bound for every HYP equivalence relation.*

§2. Ceers which are high for the jump. In this section, we examine which ceers E are high for jump, i.e., $\text{Id}^+ \leq_c E^+$. We begin by introducing a combinatorial notion which will capture the notion of a ceer being high for the jump.

DEFINITION 2.1. A ceer E is *singly high for the jump* if there is a uniformly c.e. sequence $(V_i)_{i \in \omega}$ so that, $V_i \not\subseteq_E \bigcup_{j \neq i} V_j$ for every $i \in \omega$. That is, there is an $x \in V_i$ so that $[x]_E \cap V_j = \emptyset$ for every $j \neq i$.

This definition naturally captures a ceer being high for the jump in a way given by a map from ω into c.e. sets.

LEMMA 2.2. *Fix a ceer E . Then, E is singly high for the jump if and only if there exists a function f so that the map g which sends i to an index for $\bigcup_{j \in W_i} W_{f(j)}$ gives a reduction of Id^+ to E^+ .*

PROOF. Suppose first that E is singly high for the jump. We let $W_{f(j)} = V_j$. Since for each $a \in \omega$, V_a contains an element whose E -class is not intersected by any V_j with $j \neq a$, the set of E -classes in the image $\bigcup_{j \in W_i} W_{f(j)}$ of a c.e. set W_i determines whether $a \in W_i$. Thus, this gives a reduction of Id^+ to E^+ .

Next, suppose that there is a function f as given. If every element of $W_{f(j)}$ were to be E -equivalent to a member of $W_{f(k)}$ for some $k \neq j$, then the g -image of ω and $\omega \setminus \{j\}$ would be the same, so g would not be a reduction of Id^+ to E^+ . Thus the family $V_j = W_{f(j)}$ shows that E is singly high for the jump. \dashv

More surprisingly, we show that any ceer which is high for the jump is singly high for the jump. Before this, let us establish a useful lemma that constrains the behavior of any reduction from Id^+ to some E^+ .

LEMMA 2.3. *Let $h : \text{Id}^+ \leq_c E^+$, for a ceer E . The following hold:*

- (1) *If $W_i \subseteq W_j$, then $W_{h(i)} \subseteq_E W_{h(j)}$.*
- (2) *If W_i is infinite, then $W_{h(i)} =_E \bigcup_{W_a \subset_{\text{fin}} W_i} W_{h(a)}$.*

PROOF. (1): Suppose towards a contradiction that $x \in [W_{h(i)}]_E \setminus [W_{h(j)}]_E$. Then, we use an index e we control by the recursion theorem and we let $W_e = W_i$ unless we see $x \in [W_{h(e)}]_E$, in which case we make $W_e = W_j$. This will yield a contradiction in either case. If we never see $x \in [W_{h(e)}]_E$, then $W_e = W_i$, so we must have $x \notin [W_{h(i)}]_E$, which is a contradiction. If we do see $x \in [W_{h(e)}]_E$, then we make $W_e = W_j$, so $x \in [W_{h(j)}]_E$, also a contradiction.

(2): We already have $\bigcup_{W_a \subset_{\text{fin}} W_i} W_{h(a)} \subseteq_E W_{h(i)}$ by the first item. Suppose that $y \in [W_{h(i)}]_E$. Then, we use an index e we control by the recursion theorem and we begin enumerating $W_{h(i)}$ into W_e until we see $y \in [W_{h(e)}]_E$. At this point, we stop enumerating any new elements into W_e . We thus get a finite set W_e and $y \in [W_{h(e)}]_E$. \dashv

THEOREM 2.4. *A ceer is high for the jump if and only if it is singly high for the jump.*

PROOF. If E is singly high for the jump, then it is high for the jump by Lemma 2.2.

Let E be a ceer which is high for the jump and fix h to be a reduction of Id^+ to E^+ . We will construct a sequence $(V_i)_{i \in \omega}$ of c.e. sets witnessing that E is singly high for the jump.

We define a function from c.e. sets F to c.e. sets $W(F)$ by taking an index e we control by the recursion theorem and enumerating F into W_e . Then we let $W(F) = W_{h(e)}$. At a given stage s , we let $W(F)_s = W_{h(e),s}$. We observe that for any index i of F , we have $W(F) = W_{h(i)}$. Moreover, by Lemma 2.3, we may assume that for every s we have $W(F)_s \subseteq W(G)_s$ for any finite sets $F \subseteq G$.

We fix a sequence of equivalence relations E_s which limit to E and we assume that at most one pair of classes collapses at any given stage s . Our construction is designed to meet the following requirements:

- $\mathcal{P}_i: (\exists x \in V_i)(x \text{ is not } E\text{-equivalent to any } y \in V_j, \text{ for } j \neq i).$ \dashv

2.1. Strategy. Intuitively, the strategy to satisfy \mathcal{P}_i acts as follows: We choose a number a_i and begin with a set $B_i = \emptyset$. We want to exploit the fact that $W(B_i) \subsetneq_E W(B_i \cup \{a_i\})$. So, we choose a number z which we believe is in $W(B_i \cup \{a_i\}) \setminus [W(B_i)]_E$ and we put this z into V_i . If we see z become E -equivalent to a member of set V_j with $j > i$, then this is because some set $B_j \cup \{a_j\}$ which does not contain a_i has $z \in [W(B_j \cup \{a_j\})]_E$. We now give up on z and update our parameter B_i to contain $B_j \cup \{a_j\}$ and try to use the fact that $W(B_i) \subsetneq_E W(B_i \cup \{a_i\})$ for this new larger set B_i , and we choose a new number z . If this happens infinitely often, and each choice of z ends up in $\bigcup_{j>i} [V_j]_E$, then we will have built a set B_i not containing a_i so that $W(B_i) = W(B_i \cup \{a_i\})$ contradicting the fact that h is a reduction of Id^+ to E^+ .

If we see z go into $\bigcup_{j<i} [V_j]_E$, it is possible that this E -class is the only one distinguishing between $[W(B_i)]_E$ and $[W(B_i \cup \{a_i\})]_E$. So, we put a_i into B_i and choose a new parameter a_i . Now this class is already in $W(B_i)$, and since $\bigcup_{j<i} V_j$ will be finite, we will have to do this only finitely often, so the above strategy will eventually find us a $z \in V_i \setminus \bigcup_{j \neq i} [V_j]_E$.

2.2. Construction. The strategy for the \mathcal{P}_i requirement will have parameters a_i , B_i , and z_i . These should be understood as follows: B_i is a finite set which does not contain a_i . We want to use the fact that $W(B_i) \neq W(B_i \cup \{a_i\})$ to find an E -class which “represents” a_i . The parameter z_i defines an element which is in $V_i \setminus [\bigcup_{j \neq i} V_j]_E$ at the current stage. To refer to the value of a parameter at the end of stage s , we give it a superscript s .

The strategy for \mathcal{P}_i requires attention at stage $s + 1$ if its parameter z_i is undefined or is contained in $[\bigcup_{j \neq i} V_{j,s}]_{E_s}$ or if it has been injured since it last acted. At any given stage, the highest priority strategy which requires attention acts. All lower-priority strategies are injured. For bookkeeping reasons, if a strategy is injured, it keeps its parameters but just knows that it is injured. The strategy acts as follows when acting at stage $s + 1$:

2.2.1. Step (I). If the strategy has been injured since it last acted or if it has never acted before, then it chooses new parameters as follows: If it currently has parameters a_i^s and B_i^s defined, then it lets the parameter B_i have value $B_i^s \cup \{a_i^s\}$. Otherwise, it lets the parameter B_i have value \emptyset . It also chooses a new parameter a_i to be a fresh number which has never before been considered.

2.2.2. Step (II) If z_i is currently defined we run the module $\text{TryTheNumber}(z_i)$. Otherwise, we run the PickANumber module.

We now describe the module $\text{TryTheNumber}(c)$:

- (1) If $c \not E_s w$ for every $w \in \bigcup_{j \neq i} V_{j,s}$, then we let $z_i^{s+1} = c$ and enumerate c into V_i .
- (2) If $c E_s w$ for some $w \in V_{j,s}$ with $j < i$, then we let $D = B_i \cup \{a_i\}$ and we pick a new number b . We then reset the parameters $B_i = D$ and $a_i = b$. We then call the PickANumber module with these new parameters.
- (3) If $c E_s w$ for some $w \in V_j$ with $j > i$, then we let $D = B_i \cup \{a_j\} \cup B_j$. We reset B_i to be D and we call the PickANumber module with the new parameters (note that a_i has not changed).

We now describe the `PickANumber` module:

Find the first $t > s$ so that $[W(B_i \cup \{a_i\})_t]_{E_s} \neq [W(B_i)_t]_{E_s}$ and let c be the least element of $W(B_i \cup \{a_i\})_t \setminus [W(B_i)_t]_{E_s}$. We then call the module `TryTheNumber(c)`.

2.3. Verification. Note that we only ever enumerate a number into V_i if it is already in $W(B_i \cup \{a_i\})$ and we only ever grow the set $B_i \cup \{a_i\}$ (either by putting a_i into B_i in case the strategy is injured or in case 2 of the `TryTheNumber` module, or by keeping a_i the same and growing B_i in case 3 of the `TryTheNumber` module), so we always have $V_{i,s} \subseteq W(B_i^s \cup \{a_i^s\})_s$.

LEMMA 2.5. *If the strategy \mathcal{P}_i begins the `PickANumber` module, it eventually terminates in case (1).*

PROOF. It suffices to see that the strategy cannot take outcome (2) or (3) of the `TryTheNumber` module infinitely many times. Every time it takes outcome (2) or (3), we have a new element $w \in \bigcup_{j \neq i} V_{j,s}$ so that $w \in [W(B_i)]_{E_s}$. Note that c was in $W(B_i \cup \{a_i\})_t \setminus [W(B_i)_t]_{E_s}$ before the change of parameters, but $c \in [W(B_i)_t]_{E_s}$ after the change of parameters. Since $c E_s w$, we also see w has entered the set $[W(B_i)_t]_{E_s}$. Note that since we only ever grow B_i , once something is seen to be in $W(B_i)$, it remains there. Since $\bigcup_{j \neq i} V_{j,s}$ is finite at a given stage of the construction, this process must eventually stop. \dashv

LEMMA 2.6. *At every stage s , if $i < j$ and a_i is defined, then $a_i \notin B_j \cup \{a_j\}$.*

PROOF. This is by induction on stages. When a_i is chosen, it is chosen new so this holds at that stage. Similarly, a_j is chosen new so $a_i \neq a_j$. At later stages, elements can enter B_j by either adding a_j to B_j in outcome (2) of the `TryTheNumber` module or by adding $\{a_k\} \cup B_k$ for some $k > j$ into B_j . But $a_i \notin \{a_k\} \cup B_k$ by the inductive hypothesis. \dashv

LEMMA 2.7. *Every strategy eventually settles with a parameter $z_i \notin [\bigcup_{j \neq i} V_j]_E$. Thus, every V_i is finite and contains an element which is not E -equivalent to a member of any other V_j .*

PROOF. We proceed by induction. We may assume that every strategy \mathcal{P}_j for $j < i$ has found such parameters z_i by stage s . Since these parameters never change after stage s , those strategies never act after stage s and the \mathcal{P}_i -strategy is never injured after stage s . The \mathcal{P}_i -strategy can then only take outcome (2) of the `TryTheNumber` module finitely often as there are only finitely many members of V_j for $j < i$.

Let $t > s$ be a stage late enough that the \mathcal{P}_i -strategy never takes outcome (2) of the `TryTheNumber` module after stage t . Then the parameter a_i at stage t is permanent. Further, note that a_i never enters B_i after stage t . This cannot happen via outcome (2), since outcome (2) never happens after stage t and a_i never enters B_i via outcome (3) by Lemma 2.6.

Considering the limiting value of B_i , since $a_i \notin B_i$, we see that $[W(B_i)]_E \subsetneq [W(B_i \cup \{a_i\})]_E$. Let c be the least element of $W(B_i \cup \{a_i\}) \setminus [W(B_i)]_E$ and let $u > t$ be a stage large enough that $[W(B_i^u)]_{E_u} \cap [0, c] = [W(B_i)]_E \cap [0, c]$ and $[W(B_i^u \cup \{a_i^u\})]_{E_u} \cap [0, c] = [W(B_i \cup \{a_i\})]_E \cap [0, c]$. Then when we next run the `PickANumber` module after stage u , we pick this value of c and we cannot take

outcome (2) of $\text{TryTheNumber}(c)$ because $u > t$ and we cannot take outcome (3) as this would put c into $[W(B_i)]_E$. Thus we must take outcome 1 so $c \in V_i$.

Now we argue that $c \notin [\bigcup_{j \neq i} V_j]_E$. Suppose towards a contradiction that $c \in E w$ for $w \in V_j$ with $j \neq i$. Then the \mathcal{P}_i strategy requires attention and since every higher priority strategy has settled, it gets to act. It then runs the $\text{TryTheNumber}(c)$ module and must take outcome (2) or (3) depending on whether $j < i$ or $j > i$. This cannot take outcome (2) as $u > t$. If it takes outcome (3), then we see $c \in [W(B_i)]_E$ contradicting the choice of c . \dashv

This concludes the proof that the property of being high for the jump coincides with the property of being singly high for the jump. \dashv

We now shift the focus to the case of 1-dimensional ceers. Indeed, it is natural to ask for which c.e. sets A is E_A high for the jump [13, Question 1]. Clemens, Coskey, and Krakoff proved the following: on the one hand, if A is not hyperhypersimple then E_A is high for the jump [13, Theorem 4.8]. In particular, there are dark ceers which are high for the jump: Any E_A for A simple but not hyperhypersimple. On the other hand, if A is quasimaximal, then E_A is not high for the jump [13, Theorem 4.17]. This is not a characterization, as there are sets which are hyperhypersimple yet are not quasimaximal [30]. Recall Definition 1.6 for these terms. The next theorem settles the problem.

THEOREM 2.8. *For a c.e. set A , $E_A^+ \equiv Id^+$ if and only if A is not hyperhypersimple. Thus, the property of being high for the jump is Σ_4^0 -complete.*

PROOF. If A is nonhyperhypersimple, then E_A is high for the jump by [13, Theorem 4.8].

Suppose E_A is high for the jump. Then E_A is singly high for the jump. Let $\mathcal{V} = (V_i)_{i \in \omega}$ witness this. We may assume that every V_i has an element z_i which is not in $\bigcup_{j \neq i} V_j$ and $z_i \notin A$. This is because all of A constitutes a single class in E_A , so omitting one set from the sequence of V_i suffices to guarantee this. We may also assume that at every stage at most one number is enumerated into at most one set V_i .

We now define the sets X_i defined as follows: $z \in X_i$ if

- (1) $z \in V_i$ and V_i is the first set in \mathcal{V} which z enters.
- (2) $(\exists s > z \forall w < z)(w \in V_{i,s} \rightarrow w \in A_s \cup \bigcup_{j \neq i} V_{j,s})$.

The first condition implies that $\mathcal{X} = (X_i)_{i \in \omega}$ is a uniformly c.e. array of disjoint sets. Since every V_i contains a member which is not contained in $A \cup \bigcup_{j \neq i} V_j$, the second condition ensures that each X_i is finite. Finally, for each V_i , let z be the least member of $V_i \setminus (A \cup \bigcup_{j \neq i} V_j)$. Then $z \in X_i$. Thus \mathcal{X} witnesses that A is not hyperhypersimple.

Next we consider the index set of being high for the jump. It is easy to calculate that being high for the jump is a Σ_4^0 problem. To conclude, it is sufficient to recall that the index set of nonhyperhypersimple c.e. sets is Σ_4^0 -complete (see [34, 35], where the result is announced, and [32, Theorem XII 4.13] where it is proved). \dashv

We finish our discussion of which ceers are high for the jump by focusing on a special class of ceers which will also be considered in the next section: dark minimal ceers, i.e., dark ceers E so that $R <_c E$ implies that R is finite. Dark minimal ceers are of special interest for the theory of ceers. For example, we code graphs onto the

dark minimal ceers to show that the theory of the partial order of ceers is as rich as possible, being computably isomorphic with first-order arithmetic [6].

PROPOSITION 2.9. *No dark minimal ceer is high for the jump.*

PROOF. Towards a contradiction, suppose that there exists $h : \text{Id}^+ \leq_c E^+$, for a dark minimal ceer E . Lemma 2.3(2) guarantees that, if W_i is infinite, then $W_{h(i)}$ must intersect infinitely many E -classes, as otherwise there would be finite c.e. sets $W_a \subset W_b \subset W_i$ so that $[W_{h(a)}]_E = [W_{h(b)}]_E$, a contradiction. So, let W_{e_0} and W_{e_1} be the evens and the odds, respectively. Since E is dark minimal, by Lemma 1.4, we obtain that $[W_{h(e_0)}]_E = [W_{h(e_1)}]_E = \omega$, a contradiction. \dashv

§3. The higher jump hierarchy of ceers. We now turn our attention to higher jumps applied to ceers. We first consider the 1-dimensional case where, contrary to the picture for the single jump, every co-infinite c.e. set A has the property that E_A has the highest possible double-jump. Of course, we focus on the co-infinite c.e. sets because, if A is co-finite, then E_A has only finitely many classes.

THEOREM 3.1. *If A is a co-infinite c.e. set, then $\text{Id}^{++} \leq_c E_A^{++}$.*

PROOF. We describe an algorithm h for reducing Id^{++} to E_A^{++} . Let $F : \omega \rightarrow \omega$ be so $F(n)$ is the n th element of $\omega \setminus A$. Note that F is Δ_2^0 , so we fix also F_s a uniformly computable sequence of functions limiting to F .

We arrange it so that for any index e , $h(e)$ is an index for a uniformly c.e. family consisting of ω , all finite sets, and $\omega \setminus \{F(k)\}$ for each k so that $W_k = W_i$ for some $i \in W_e$. We observe that this is a reduction from Id^{++} into E^{++} : If e, e' are indices for the same family of c.e. sets, then $h(e)$ and $h(e')$ are indices for the same family of c.e. sets. If $h(e)$ and $h(e')$ are so they describe the same family of c.e. sets up to E_A -equivalence, then since each $F(k)$ is a singleton class in E_A , they must be the same family of c.e. sets. In particular, $W_j = W_i$ for some $i \in W_e$ if and only if $\omega \setminus F(j) = W_m$ for some $m \in W_{h(e)}$ if and only if $\omega \setminus F(j) = W_m$ for some $m \in W_{h(e')}$ if and only if $W_j = W_i$ for some $i \in W_{e'}$.

Given an index e , we must uniformly produce the uniform family which is to be its image under h . Begin with a uniform enumeration of ω and all finite sets. We add to this a sequence of sets $V_{i,k}^m$. If i enters W_e , then make $V_{i,k}^0$ active. If $V_{i,k}^j$ is active for some j and $F_{s+1}(k) \neq F_s(k)$, then we deactivate $V_{i,k}^j$, make $V_{i,k}^j = \omega$ and we activate $V_{i,k}^{j+1}$. If $V_{i,k}^j$ is active at stage s and both s and the length of agreement between W_i and W_k at stage s are $\geq \ell$, then we enumerate $[0, \ell] \setminus \{F_s(k)\}$ into $V_{i,k}^j$.

It is straightforward to check that this gives a uniform enumeration of the described family. \dashv

Next we see that, unlike the 1-dimensional case, there are ceers which are not high_n for the computable Friedman–Stanley jump for any n . That is, $\text{Id}^{+\hat{n}} \not\leq_c E^{+\hat{n}}$. We do this by considering the low dark minimal ceers. Dark minimal ceers have been used heavily in the literature, and we now note that there are dark minimal ceers which are also low.¹

¹We emphasize that we are using lowness in the sense of the Turing jump on sets, not any of the equivalence relation jumps from Definition 1.3.

LEMMA 3.2. *There are low dark minimal ceers.*

PROOF SKETCH. The construction of a dark minimal ceer E has requirements of two types, which suffice by Lemma 1.4:

$\mathcal{R}_{e,n}$: If W_e intersects infinitely many E -classes, then it intersects $[n]_E$.

\mathcal{I}_m : E has $\geq m$ classes.

To these, we can add the lowness requirement:

\mathcal{L}_e : If for infinitely many stages s we have $\varphi_e^{E^s}(e) \downarrow$, then $\varphi_e^E(e)$ converges.

\mathcal{L} -requirements only place restraint on some finite collection of E -classes preventing collapse. This fits in the finite injury construction of a dark minimal ceer, as given in [7, Theorem 3.3] (i.e., to a lower-priority requirement, this restraint is no different than the restraints placed by higher-priority \mathcal{I} -requirements). \dashv

Recall that all dark minimal ceers E have the property that if W_e intersects infinitely many E -classes, then W_e must intersect every E -class. The following few lemmas use this property to bound the complexity of the jumps of dark minimal ceers.

LEMMA 3.3. *If E is a dark minimal ceer, then for each $k \in \omega$, the set of i so that W_i/E has size $\geq k$ is a $\Delta_2^0(E)$ set.*

Further, the set of triples (i, j, k) so that $|W_i/E| = k$ and $W_i =_E W_j$ is $\Delta_2^0(E)$.

In particular, if E is a low dark minimal ceer then these sets are both Δ_2^0 .

PROOF. The quotient W_i/E has size at least k if and only if $\exists x_1 \dots x_k \in W_i (\bigwedge_{k \neq j} x_k \not\equiv x_j)$. This is $\Sigma_1^0(E)$.

To check if (i, j, k) is so that $|W_i/E| = k$ and $W_i =_E W_j$, we can in a $\Delta_2^0(E)$ way check that $|W_i/E| = k$ and $|W_j/E| = k$ by the above. Then, if this is the case, we can in an E -computable way find elements $x_1 \dots x_k \in W_i$ so that $\bigwedge x_i \not\equiv x_j$ and $y_1 \dots y_k \in W_j$ so that $\bigwedge y_i \equiv y_j$. Then we need only check in an E -computable way that $\bigwedge_{i \leq k} x_i \equiv y_{\sigma(i)}$ for some permutation σ . \dashv

LEMMA 3.4. *If E is a dark minimal ceer, then $E^{\hat{+}\hat{+}}$ is $\Delta_4^0(E)$.*

In particular, if E is a low dark minimal ceer then $E^{\hat{+}\hat{+}}$ is Δ_4^0 .

PROOF. Let $\mathcal{V}_i, \mathcal{V}_j$ be two uniformly c.e. families of c.e. sets (given by appropriate indices, i.e., $\mathcal{V}_i = \{W_m : m \in W_i\}$). Then $W_i \subset_{E^+} W_j$ if and only if the following hold:

$$(\forall S \in \mathcal{V}_i)(\forall k \in \omega) [|S/E| = k \rightarrow (\exists F \in \mathcal{V}_j)(F =_E S)]. \tag{1}$$

$$(\exists S \in \mathcal{V}_i)(\forall k \in \omega) [|S/E| > k \rightarrow (\exists S \in \mathcal{V}_j)(\forall k \in \omega)(|S/E| > k)]. \tag{2}$$

The conditions $|S/E| = k$ and $F =_E S$ in (1) are $\Delta_2^0(E)$ by Lemma 3.3. Thus, the condition (1) is $\Pi_3^0(E)$. Similarly, using Lemma 3.3, (2) is $\Delta_4^0(E)$. Thus, $W_i =_{E^+} W_j$, or $i E^{\hat{+}\hat{+}} j$ is a $\Delta_4^0(E)$ condition. \dashv

COROLLARY 3.5. *If E is a dark minimal ceer, then for any $k > 2$, the equivalence relation $E^{\hat{+}\hat{k}}$ is $\Pi_{2k-1}^0(E)$. In particular, if E is a low $_{2k-2}$ dark minimal ceer then $E^{\hat{+}\hat{k}}$ is Π_{2k-1}^0 .*

PROOF. This is by induction with base case $k = 3$: E^{+++} is Π_2^0 over E^{++} , which is $\Delta_4^0(E)$, so is $\Pi_5^0(E)$. Then $E^{+\widehat{(k+1)}} = (E^{+\hat{k}})^+$ is Π_2^0 over $E^{+\hat{k}}$ which is $\Pi_{2k-1}^0(E)$ by induction, so $E^{+\widehat{(k+1)}}$ is $\Pi_{2(k+1)-1}^0(E)$. ⊣

Below, in Corollary 6.7, we will show that $\text{Id}^{+\hat{n}}$ is not Π_{2n-1}^0 . It follows from this that if E is a low $_{2k-2}$ dark minimal ceer and $k > 2$, then $\text{Id}^{+\hat{k}} \not\leq_c E^{+\hat{k}}$. Since being low $_m$ implies being low $_{m+1}$, thus shows that E cannot be high $_n$ for any n . Thus we will have the following theorem.

THEOREM 3.6. *If E is a low $_m$ dark minimal ceer for any $m \in \omega$, then E is not high $_n$ for the computable Friedman–Stanley jump for any $n \in \omega$.*

We now see that the assumption of lowness is necessary here, since there are dark minimal ceers so that $\text{Id}^{+++} \leq_c E^{+++}$.

THEOREM 3.7. *There is a dark minimal ceer E so that $\text{Id}^{+++} \leq_c E^{+++}$.*

PROOF. We begin with a description of how we will give the reduction witnessing $\text{Id}^{+++} \leq_c E^{+++}$. Along with the ceer E , we will construct uniformly in each $j, k, \bar{x} \in \omega$, a finite sequence of c.e. sets $U_{j,k,\bar{x}}^n$ for $n \leq N(j, k, \bar{x})$ satisfying the following Informal Requirement:

IR $_{j,k,\bar{x}}$: For every $n < N(j, k, \bar{x})$, $U_{j,k,\bar{x}}^n = \omega$. Regarding $U_{j,k,\bar{x}}^{N(j,k,\bar{x})}$:

- If \bar{x} is a $2k$ -tuple which is E -distinct and $W_j = W_k$, then

$$U_{j,k,\bar{x}}^{N(j,k,\bar{x})} = [\bar{x}]_E.$$

- Otherwise, $|U_{j,k,\bar{x}}^{N(j,k,\bar{x})}/E|$ is odd or $U_{j,k,\bar{x}}^{N(j,k,\bar{x})} = \omega$.

From the success of these requirements, we give a reduction of Id^{+++} to E^{+++} . Given a uniformly c.e. family $\mathcal{V}_i = \{W_j : j \in W_i\}$, we map this to a family \mathcal{F}_i which contains each set $U_{j,k,\bar{x}}^n$ for each $j \in W_i, k \in \omega$ and $\bar{x} \in \omega^{2k}$, and $n \leq N(j, k, \bar{x})$. We also include an enumeration of ω and sets $X_{\bar{x}}$ for every \bar{x} of odd size where $X_{\bar{x}}$ enumerates $[\bar{x}]_E$ unless we see that \bar{x} is not E -distinct, in which case $X_{\bar{x}}$ enumerates ω . It is easy to check that the sets enumerated as $X_{\bar{x}}$ are exactly ω and every E -closed set Y so that $|Y/E|$ is odd. Further, if W_k is represented in \mathcal{V}_i , then there is some $j \in W_i$ so that $W_j = W_k$. In this case, $U_{j,k,\bar{x}}^{N(j,k,\bar{x})}$ for various \bar{x} will enumerate every E -closed set Y so that $|Y/E|$ has size $2k$. So, this gives the necessary reduction to witness that $\text{Id}^{+++} \leq_c E^{+++}$. ⊣

3.1. Formal construction. We now verify that we can construct a dark minimal ceer E along with the uniform sequence of sets $U_{j,k,\bar{x}}^n$ satisfying the IR-requirements.

We have the full set of requirements for $m, n, o, j, k \in \omega$ and $\bar{x} \in \omega^{2k}$.

\mathcal{I}_m : E has at least m equivalence classes.

$\mathcal{P}_{n,o}$: If W_n intersects infinitely many E -classes, then W_n intersects $[o]_E$.

$\mathcal{Q}_{j,k,\bar{x}}$: Enumerate a c.e. set U so that:

- If \bar{x} is not E -distinct, then $U = \omega$.
- If \bar{x} is E -distinct and $W_j = W_k$, then $U = [\bar{x}]_E$.
- If \bar{x} is E -distinct and $W_j \neq W_k$, then $|U/E|$ is odd.

We note that whenever a $\mathcal{Q}_{j,k,\bar{x}}$ -requirement is reinitialized, we will let the constructed set be ω and have the strategy begin constructing a new set U . This explains the finite sequence of sets $U_{j,k,\bar{x}}^n$ described in $\text{IR}_{j,k,\bar{x}}$ and $N(j, k, \bar{x})$ will be the number of times this strategy is reinitialized. So the success of the strategy which satisfies $\mathcal{Q}_{j,k,\bar{x}}$, after finitely many reinitializations gives us success of $\text{IR}_{j,k,\bar{x}}$. Thus, we need only give a finite injury construction satisfying each of these requirements, then we know that E is a dark minimal ceer and $\text{Id}^{++} \leq_c E^{++}$.

We enumerate the strategies in order type ω . Whenever a \mathcal{P} -strategy causes collapse, all lower-priority strategies are reinitialized.

The strategies for \mathcal{I} and \mathcal{P} -requirements are familiar from the usual construction of a dark minimal ceer in [7, Theorem 3.3]: \mathcal{I} -requirements simply choose a new tuple and place restraint.

$\mathcal{P}_{n,o}$ -strategies seek to find an element of W_n which is not (currently) E -equivalent to any restrained number. Then it E -collapses this number with o .

$\mathcal{Q}_{j,k,\bar{x}}$ strategies act as follows: If it ever sees some $x_i \ E \ x_j$, then it just stops and makes $U = \omega$, and the requirement is satisfied. Nonetheless, the strategy restrains the tuple \bar{x} . We begin by enumerating $[\bar{x}]_E$ into U . We use the Π_2^0 approximation to the statement $W_j = W_k$. That is, at every stage, we have a computable guess as to whether or not $W_j = W_k$. If we infinitely often guess that $W_j = W_k$, then they are equal. When our guess switches from saying $W_j = W_k$ to saying that they are not equal, we take a new number y , and we add y to U . Further, we place restraint on the number y so that lower priority requirements will not collapse y with any element of \bar{x} . If we later guess that $W_j = W_k$, then we collapse y with x_0 . We then undefine the parameter y and unrestrain it (it is restrained automatically anyway by our restraint on x_0).

The construction is put together via standard finite injury machinery. At every stage s , the first s strategies get to act in order.

3.2. Verification.

LEMMA 3.8. *At every moment of the construction, the set of parameters of y for various \mathcal{Q} -requirements and the set of restrained elements for \mathcal{I} -requirements are all E -distinct.*

At every moment of the construction, if $\mathcal{Q}_{i,j,\bar{x}}$ is higher priority than $\mathcal{Q}_{i',j',\bar{x}'}$, then the latter's parameter y' (if defined) is not E -equivalent to any $x \in \bar{x}$.

PROOF. These statements are preserved by the choice of parameters, since they are chosen new. Collapse occurs only via action from \mathcal{P} or \mathcal{Q} -requirements. In the former case, $\mathcal{P}_{n,o}$ collapses some member z of W_n to o . This z was not equivalent to any element restrained by a higher-priority requirement, and since all lower-priority requirements are reinitialized, we have added no restrained number to the class of o . Next we consider collapse caused by a $\mathcal{Q}_{j,k,\bar{x}}$ -strategy. Since $\mathcal{Q}_{j,k,\bar{x}}$ previously restrained y , the inductive hypothesis shows that no other parameter y' for a \mathcal{Q}' -requirement or an element restrained by an \mathcal{I} -requirement was equivalent to y . Since after the collapse of y with x_0 , this y is no longer the parameter for $\mathcal{Q}_{j,k,\bar{x}}$, we have added no such element to the class of x_0 . Thus the first statement is proved.

It remains to see that a collapse caused by a $\mathcal{Q}_{i^0,k^0,\bar{x}^0}$ -strategy does not cause a violation of the second statement. By the first statement, no two y -parameters

could have been equivalent. So, the only way this could have caused the violation is if $x_0^0 E_s y'$ and $y^0 E_s x_0$. But by inductive hypothesis, the former implies Q_{i^0, k^0, \bar{x}^0} is lower priority than $Q_{i', j', \bar{x}'}$ and the latter implies $Q_{i, j, \bar{x}}$ is priority than Q_{i^0, j^0, \bar{x}^0} . Thus we would have $Q_{i, j, \bar{x}}$ being lower priority than $Q_{i', j', \bar{x}'}$, so this is not a violation of the second statement after all. \dashv

LEMMA 3.9. *Every strategy succeeds.*

PROOF. Since only \mathcal{P} -requirements reinitialized lower priority requirements, and each can act at most once, every requirement is reinitialized only finitely often.

We first see that every \mathcal{I}_m -strategy succeeds. Take a stage after which the strategy is not reinitialized and consider the tuple restrained by the strategy. By the previous Lemma, each of its restrained elements are E -distinct, so the strategy succeeds.

Next, consider a $\mathcal{P}_{n,o}$ -strategy. Let s be a stage large enough that the strategy is not reinitialized after stage s . Let \bar{a} be the full tuple of elements restrained by higher-priority \mathcal{I} -strategies (which has settled by stage s). Let Q_{i_q, j_q, \bar{x}_q} for $q < K$ be the collection of higher-priority \mathcal{Q} -strategies. Suppose that W_n/E is infinite, and let $t > s$ be a stage after which W_n contains at least $|\bar{a} \cup \bigcup_{q < K} \bar{x}_q| + K + 1E$ -distinct elements. At any such stage, at most KE_t -classes are restrained as parameters y by higher priority \mathcal{Q} -strategies, so there must be an unrestrained member of $W_{n,t}$ which the strategy will collapse with o and thus be permanently satisfied.

Finally, we consider a $Q_{i, j, \bar{x}}$ -strategy. We consider the three cases: If \bar{x} is not E -distinct, then this is seen at some point and we set $U = \omega$. If \bar{x} is E -distinct and $W_j = W_k$, then infinitely often, we add some n to U , but then we collapse this n in with x_0 . So, $U = [\bar{x}]_E$. If $W_j \neq W_k$, then let s be the least stage so that the strategy is not reinitialized after stage s and the approximation says $W_j \neq W_k$ for all $t > s$. Let y be the parameter chosen at stage s . Then we need only see that $y \notin [\bar{x}]_E$. We consider what strategy might cause this collapse. It cannot be a higher priority \mathcal{P} -requirement, since the strategy is not reinitialized after stage s . It cannot be lower priority $\mathcal{P}_{n,o}$ -requirements since both \bar{x} and y are restrained by $Q_{i, j, \bar{x}}$, so neither can be E_t -equivalent to the chosen element $z \in W_n$. It cannot be a \mathcal{Q} -requirement, since the lower-priority strategy's parameter y cannot be equivalent to either the higher-priority strategy's y or x , by the previous lemma. \dashv

This concludes the proof of Theorem 3.7. \dashv

We have showed that there are properly high₂ dark minimal ceers. We ask the next natural question:

QUESTION 1. *For each $n > 2$, does there exist a dark minimal ceer which is properly high_n for the computable Friedman–Stanley jump?*

§4. Dark jumps. In the remaining three sections, we move out from the realm of ceers and consider equivalence relations of higher complexity. In particular, we now ask how complex an infinite equivalence relation E must be for its jump to be dark. Clemens, Coskey, and Krakoff [13, Theorems 4.2 and 4.4] show that E^+ is light for every infinite ceer E and there are infinite Δ_4^0 equivalence relations E so that E^+ is dark. Here we prove that Σ_2^0 is the lowest arithmetical complexity of an equivalence relation E such that E^+ is dark (thus answering [13, Question 6]).

First, we show that the jump of every infinite Π_2^0 equivalence relation is light.

THEOREM 4.1. *If $E \in \Pi_2^0$ is infinite, then $\text{Id} \leq_c E^\dagger$.*

PROOF. We let E_s be computable approximations to E so that $x E y$ if and only if there are infinitely many stages s so that $x E_s y$. We construct a uniform sequence of c.e. sets W_{i_j} , for $j \in \omega$, so that $W_{i_j} \subsetneq_E W_{i_{j+1}}$, for each $j \in \omega$.

We let $W_{i_0} = \{0\}$. We define $W_{i_{j+1}}$ as follows:

$$x \in W_{i_{j+1}} \text{ if and only if } (\forall y < x)(\exists s \geq x)(\exists z \in W_{i_j})(y E_s z).$$

LEMMA 4.2. *If $[0, x) \subseteq_E W_{i_j}$, then $x \in W_{i_{j+1}}$.*

PROOF. For each $y < x$, there is a $z \in W_{i_j}$ so that $y E z$. Thus for infinitely many s we have $y E_s z$, witnessing $x \in W_{i_{j+1}}$. ⊢

LEMMA 4.3. *Each W_{i_j} is a finite initial segment of ω .*

PROOF. We prove this by induction. This is true for $j = 0$.

Fix an element $y \notin [W_{i_j}]_E$. This exists because W_{i_j} is finite and E has infinitely many classes. Then let s be a stage large enough that $W_{i_j} = W_{i_{j,s}}$ and every $z \in W_{i_j}$ and $t > s$ we have $y \not E_t z$. Then no $x > s$ can ever enter $W_{i_{j+1}}$. ⊢

It follows that $W_{i_j} \subsetneq_E W_{i_{j+1}}$ for each j . To see this, note that $W_{i_j} \subseteq W_{i_{j+1}}$ by Lemma 4.3 and the definition of $W_{i_{j+1}}$. Then consider the least $x \notin [W_{i_j}]_E$ which exists since E is infinite and W_{i_j} is finite. Then $[0, x) \subseteq_E W_{i_j}$, so $x \in W_{i_{j+1}} \setminus [W_{i_j}]_E$ by Lemma 4.2. Thus $j \mapsto i_j$ is a reduction of Id to E^\dagger . ⊢

On the other hand, there are Σ_2^0 sets whose jumps are dark.

THEOREM 4.4. *There exists an infinite Σ_2^0 equivalence relation E so that $\text{Id} \not\leq_c E^\dagger$.*

PROOF. We construct E as a c.e. set via a finite injury argument over $\mathbf{0}'$. We have requirements:

\mathcal{R}_i : If W_i is infinite, then it contains two elements which are E^\dagger -equivalent.

\mathcal{Q}_j : There are x_1, \dots, x_j which are E -inequivalent.

If φ were a reduction of Id to E^\dagger , then the image of φ would be a c.e. set no two elements of which are E^\dagger equivalent. Thus, the \mathcal{R} -requirements ensure that there is no reduction from Id to E^\dagger , while the \mathcal{Q} -requirements ensure that E is infinite. We place these requirements in order-type ω . A \mathcal{Q} -requirement acts by placing a restraint. At every stage s , we allow the first s requirements to act in turn. In fact, \mathcal{R} -requirements may act at infinitely many stages and cause infinitely many E -collapses.

The strategy for an \mathcal{R}_n -requirement is as follows: Let \bar{x} be the tuple of elements restrained by higher-priority \mathcal{Q} -requirements. Using $\mathbf{0}'$, we seek a set \mathcal{I} of $3 \cdot 2^{|\bar{x}|} + 1$ numbers in W_n . If there are not this many, then W_n is not infinite and the requirement is satisfied. From these numbers, we use $\mathbf{0}'$ to find four that agree on the (current) classes of \bar{x} . That is, for each of these $3 \cdot 2^{|\bar{x}|} + 1$ indices $j \in W_n$ and $x \in \bar{x}$, we use $\mathbf{0}'$ to ask if any member (there will be only finitely many) of $[x]_{E_s}$ is in W_j . Then by the pigeon-hole principle, there are four that give the same answer for every $x \in \bar{x}$. Fix these indices: j, k, l, m . If there are two indices $i, i' \in \{j, k, l, m\}$ so that W_i and $W_{i'}$ are contained in $[\bar{x}]_{E_s}$, then $\mathbf{0}'$ sees this and the requirement will be automatically

satisfied, so no further action is taken. So, we may suppose W_j, W_k, W_l are each not contained in $[\bar{x}]_{E_s}$. Note that the family $\{W_j, W_k, W_l\}$ must contain two finite sets or two infinite sets. We begin with working with the pair j, k and, until proven otherwise, we guess that W_j and W_k are both infinite.

Then, we perform the following Collapse(j, k) module:

At each stage s greater than every $x \in \bar{x}$, we ask $\mathbf{0}'$ if there is a $y \geq s$ which is in W_j and we ask if there is a $y \geq s$ which is in W_k . We distinguish two cases.

- (1) If the answer is no to either, then we stop this module and we call the FoundFiniteSet module instead.
- (2) Assuming case (1) didn't happen, we now act to ensure that every $z < s$ is either in both or neither of $[W_j]_{E_s}$ and $[W_k]_{E_s}$. We act successively for each $z \in (\max(\bar{x}), s)$. If z is not least in its E_s -equivalence class, then we have already ensured this when previously considering a number $y < s$ which is E_s -equivalent to z , so we do nothing. Otherwise, we ask $\mathbf{0}'$ if $z \in [W_j]_{E_s}$ and if $z \in [W_k]_{E_s}$.
 - (a) If it is in neither or both, we do no action.
 - (b) If it is in one and not the other, then we find the least $n > s$ in the other set and we E -collapse the interval $[z, n]$.

We now describe the FoundFiniteSet module:

- (1) If this is the first time we call this procedure, say having found that W_j is finite, then we simply return to the Collapse(k, l) module (we just assume W_k and W_l are infinite until we see otherwise).
- (2) If this is the second time we call this procedure, say having found that W_j and W_k are finite, then we simply collapse $[\max(\bar{x}) + 1, \max(W_j, W_k)]$ to a single E -class.

Note that since every collapse involves an interval, the classes of E are intervals as well.

A \mathcal{Q}_j strategy acts as follows: Let \bar{x} be the tuple restrained by \mathcal{Q}_{j-1} (or $\bar{x} = \emptyset$ if $j = 0$). Wait to find a stage s and a number $y < s$ so that y is the greatest element of $[\max(\bar{x} + 1)]_{E_s}$ and $[y]_{E_s} = [y]_{E_{s-1}}$. Once such a y is found, the strategy places a restraint on the tuple $\bar{x}y$.

The strategies are interwoven in priority order: $\mathcal{R}_0 < \mathcal{Q}_0 < \mathcal{R}_1 < \mathcal{Q}_1 < \dots$. Whenever an \mathcal{R} -strategy runs a FoundFiniteSet module, all lower priority strategies are reinitialized. This is the only source of injury. At each stage s , we allow the requirements to act in order until one of them ends the stage. A \mathcal{Q}_j -strategy which is still waiting to find a y or which acts by declaring its restraint $\bar{x}y$ ends the stage, and a \mathcal{R}_n -strategy which runs a FoundFiniteSet module ends the stage.

LEMMA 4.5. *Suppose that a \mathcal{Q}_k strategy restrains a tuple $\bar{x}y$ at stage s , and $t > s$. Then either the strategy has been reinitialized between stages s and t or $\bar{x}y$ are the largest members of the first $|\bar{x}y|_{E_t}$ -equivalence classes. In particular, $[z]_{E_s} = [z]_{E_t}$ for every $z \in \bar{x}y$.*

PROOF. The result holds by induction for every $x_i \in \bar{x}$. Namely, \bar{x} is restrained by the strategy \mathcal{Q}_{j-1} at a stage $r < s$. By inductive hypothesis applying the claim to the \mathcal{Q}_{j-1} -strategy, x_i is the greatest number in the $i + 1$ th E_t -class as needed. We must consider the E -class of y . Since $y \in E_s \max(\bar{x}) + 1$, we need only show that as

long as the \mathcal{Q}_k has not been reinitialized, no number $> y$ ever becomes equivalent to y .

Since $[y]_{E_{s-1}} = [y]_{E_s}$, each higher priority \mathcal{R} -strategy (without loss of generality, suppose it is running the $\text{Collapse}(j, k)$ module) has considered the class $[y]_{E_s}$ on its previous pass and found that it intersected either both or neither of W_j and W_k . Thus, at any future stage $t > s$ where $[y]_{E_t} = [y]_{E_s}$, as long as the strategy remains in the $\text{Collapse}(j, k)$ module, this strategy will never have a need to collapse any element with y . If the strategy takes the FoundFiniteSet module, then the \mathcal{Q}_k -strategy is reinitialized and the desired result holds. Thus, no higher priority strategy can ever cause the E -class of y to grow.

Consider the collapses caused by lower-priority \mathcal{R} -strategies at a stage $t > s$ and suppose that we have $[y]_{E_t} = [y]_{E_s}$. The strategy collapses finite intervals of numbers $[z, n]$ which are greater than the largest element in the restrained tuple. Since y is the largest number in its E_t -equivalence class, no number in this finite interval can be equivalent to y , so this collapse does not add any elements to y 's E -class. \dashv

LEMMA 4.6. *Each strategy is satisfied.*

PROOF. Each strategy may injure lower priority requirements at most twice (each time it runs the FoundFiniteSet module), so every strategy is reinitialized only finitely often.

Suppose towards a contradiction that the first strategy that fails is a \mathcal{R}_n -strategy. Fix \bar{x} to be the numbers restrained by higher-priority \mathcal{Q} -strategies. Then \mathcal{R}_n begins by choosing indices j, k, l . Note that for any $x \in \bar{x}$, we have $[x]_{E_s} \cap W_j = \emptyset \leftrightarrow [x]_{E_s} \cap W_k = \emptyset \leftrightarrow [x]_{E_s} \cap W_l = \emptyset$ where s is the stage when j, k, l were chosen after the last time the \mathcal{R}_n -strategy is reinitialized. But by the previous claim, $[x]_{E_s} = [x]_E$, so

$$[x]_E \cap W_j = \emptyset \Leftrightarrow [x]_E \cap W_k = \emptyset \Leftrightarrow [x]_E \cap W_l = \emptyset.$$

So, on these classes, the three sets agree.

First suppose that both of W_j and W_k are infinite. We now check that the $\text{Collapse}(j, k)$ module ensures that $W_j =_E W_k$. Fix $z > \max(\bar{x})$ (i.e., a class distinct from the ones considered above) and suppose that $z \in [W_j]_E$. Then at some stage $s > z$ we have $z \in [W_j]_{E_s}$. Then at this stage, we ensure that $z \in [W_k]_{E_s}$. This covers every class by the previous claim, so $j E^+ k$ satisfying the \mathcal{R}_n requirement.

Similarly, if exactly one of W_j or W_k is finite (without loss of generality, assume it is W_j), and W_l is infinite then the $\text{Collapse}(k, l)$ module ensures that $W_k =_E W_l$. If two of the sets, say W_j and W_k are finite, then the FoundFiniteSet module ensures that $W_j =_E W_k$ since they must both intersect the class of $\max(\bar{x}) + 1$ (since they were chosen to not be contained in $[\bar{x}]_{E_s} = [\bar{x}]_E$) and no larger class. Thus, the strategy succeeds after all.

Next, suppose towards a contradiction that \mathcal{Q}_j is the first strategy that fails. From the above lemma, we need only show that the wait to find a y as needed must end. At each stage t , let $y_t = \max([\max(\bar{x}) + 1]_{E_t})$. This would work for our choice of y unless $[\max(\bar{x}) + 1]_{E_t} \neq [\max(\bar{x}) + 1]_{E_{t-1}}$. This can only happen due to the action of a higher priority \mathcal{R} -requirement, since \mathcal{Q}_j ends the stage since it is waiting to find its y . We can suppose, without loss of generality, that the higher priority strategy is in a $\text{Collapse}(j, k)$ module, since the $\text{Collapse}(k, l)$ module is

symmetric and it can run the FoundFiniteSet module at most twice. Then growing the E -class of $\max(\bar{x}) + 1$ must be because $\max(\bar{x}) + 1$ was seen to be in exactly one of $[W_j]_{E_{t-1}}$ or $[W_k]_{E_{t-1}}$. But this can happen only once in the Collapse(j, k) module, since after stage t it is in both. Thus, after finitely many stages, we must have $[\max(\bar{x}) + 1]_{E_t} = [\max(\bar{x}) + 1]_{E_{t-1}}$ and \mathcal{Q}_j can choose its element y . \dashv

This concludes the proof of Theorem 4.4. \dashv

§5. Jumps depend on notations. We now consider the transfinite jump hierarchy. Clemens, Coskey, and Krakoff [13, Question 2] ask whether the degree of E^{+a} depends on the notation $a \in \mathcal{O}$ or only the ordinal $|a|$. We show that it does indeed depend on the notation, but we give a bound on how much it can depend on the notation.

NOTATION. *To avoid having towers of exponentials to represent successor ordinals, we introduce the function $P(x) = 2^x$ and we write $P^{(k)}(x)$ for the k th iterate of the function P on x . Note that if n is a notation for the ordinal α , then $P^{(k)}(n)$ is a notation for the ordinal $\alpha + k$.*

The following observation follows directly from the definitions.

OBSERVATION 5.1. *For any notations $a <_{\mathcal{O}} b$, there is a computable function $f_{a,b}$ so that $f_{a,b}$ witnesses $E^{+a} \leq_c E^{+b}$ for any equivalence relation E . Further, $f_{a,b}$ can be uniformly found from the notations a and b .*

PROOF. This is seen by induction on the notation b with base case $b = a$. In this case, the reduction is the identity. There is a uniform reduction f from E to E^+ for any E via the map sending x to an index for the c.e. set $\{x\}$ as in [13, Proposition 2.2(a)]. Using this, when $b = 2^c$, we can take a reduction g which reduces E^{+a} to E^{+c} for any E and compose with f to get a reduction to E^{+b} . This is uniform from the reduction g .

Similarly, if $b = 3 \cdot 5^e$ with $a <_{\mathcal{O}} b$, then there is some k so that $a <_{\mathcal{O}} \varphi_e(k)$. Then by inductive hypothesis, we have a reduction g of E^{+a} to $E^{+\varphi_e(k)}$. But the k th column of E^{+b} is exactly $E^{+\varphi_e(k)}$, so composing g with the function $x \mapsto \langle k, x \rangle$ gives the uniform reduction to E^{+b} . Finally, note that k , and thus the reduction, can be found uniformly from a, b . \dashv

The following lemma will be used to manage possible reductions into E^{+a} where $|a|$ is a limit ordinal.

LEMMA 5.2. *For any equivalence relation E , the classes of E^+ are computably inseparable.*

PROOF. Suppose towards a contradiction that $[i]_{E^+}$ and $[j]_{E^+}$ are separated by the computable set A . That is, $[i]_{E^+} \subseteq A$ and $[j]_{E^+} \cap A = \emptyset$. By the recursion theorem, we can take an index e so that $W_e = W_i$ if $e \notin A$ and $W_e = W_j$ if $e \in A$. In either case, this gives a contradiction. \dashv

We first consider ordinals $< \omega^2$, and show that the notation does not matter in this case.

LEMMA 5.3. *Let α be an ordinal $< \omega^2$ and $a, b \in \mathcal{O}$ have $|a| = |b| = \alpha$. Then for any E , we have $E^{+a} \equiv E^{+b}$.*

PROOF. The proof is by induction on α . We note that if the result is shown for α , then $E^{+a} \leq_c E^{+b}$ for any notation a for α and b with $|b| > \alpha$. To see this, take the notation c with $c < \ominus b$ and $|c| = \alpha$. Then $E^{+a} \equiv E^{+c} \leq_c E^{+b}$. We call this the “reduction form” of the inductive hypothesis.

The lemma clearly holds for all finite α . The set of α for which this is true is also clearly closed under successor. It suffices to show the result for limit ordinals $\alpha < \omega^2$.

Let $a = 3 \cdot 5^i$ and $b = 3 \cdot 5^j$ be notations for $\omega \cdot n$. Let c be least so that $|\varphi_i(c)| \geq \omega \cdot (n - 1)$ and d be least so that $|\varphi_j(d)| \geq |\varphi_i(c)|$. For every $k > c$, $\varphi_i(k) = P^{(z)}(\varphi_i(c))$ for some z . Similarly, for every $k > d$, $\varphi_j(k) = P^{(z)}(\varphi_j(d))$ for some z .

We build a reduction of E^{+a} to E^{+b} as follows: We send the first c columns of E^{+a} to the columns d through $d + c - 1$ of E^{+b} . This can be done by the reduction form of the inductive hypothesis since the first c columns of E^{+a} are all E^{+g} for some g with $|g| < \omega \cdot (n - 1)$ and the images are of the form E^{+h} where $|h| \geq \omega \cdot (n - 1)$.

Next, we send the c th column of E^{+a} to the $(d + c)$ th column of E^{+b} which again we can do by the reduction form of the inductive hypothesis. To figure out how to send the $c + 1$ th column, we find the number k so that $\varphi_i(c + 1) = P^{(k)}(\varphi_i(c))$. Then we find the first unused column e in E^{+b} so that $\varphi_j(e) = P^{(l)}(d)$ with $l > k$. We can then use the reduction from E^{+c} to E^{+d} to uniformly find a reduction from $E^{P^{(k)}(c)}$ to $E^{P^{(l)}(d)}$. Repeating as such, we uniformly send every column of E^{+a} into E^{+b} giving the needed reduction. \dashv

Next we see that notation does matter at ω^2 .

THEOREM 5.4. *For any notation b for ω^2 there exists another notation a for ω^2 so that $\text{Id}^{+a} \not\leq_c \text{Id}^{+b}$.*

There are two notations a, b for ω^2 so that Id^{+a} and Id^{+b} are incomparable.

PROOF. Let $b = 3 \cdot 5^j$ be a given notation for ω^2 .

We take $a = 3 \cdot 5^e$ for an index e which we control by the recursion theorem. For each x , we let $\varphi_e(x) = P(3 \cdot 5^{i_x})$ for an infinite sequence of indices i_x which we control by the recursion theorem. Until we determine otherwise, we define, stage by stage that $\varphi_{i_0}(0) = 1$, $\varphi_{i_x}(s + 1) = P(\varphi_{i_x}(s))$, and $\varphi_{i_{x+1}}(0) = P(3 \cdot 5^{i_x})$.

We perform the following actions for the sake of diagonalization. To ensure that φ_k is not a reduction of Id^{+a} to Id^{+b} , we wait for $\varphi_k(\langle k, 0 \rangle)$ to converge, say to $\langle m, n \rangle$. Since $|P(3 \cdot 5^{i_k})|$ is a successor ordinal, Lemma 5.2 shows that the classes of $\text{Id}^{+P(3 \cdot 5^{i_k})}$ are computably inseparable. Thus we know that if φ_k is a reduction, then it must send the entire k th column into the m th column of Id^{+b} . But the m th column of Id^{+b} is equivalent to $\text{Id}^{+\varphi_j(m)}$. So, at the stage s when we see that $\varphi_k(\langle k, 0 \rangle) \downarrow = \langle m, n \rangle$, we make $\varphi_{i_k}(s + 1) = \varphi_{i_k}(s) \oplus \varphi_j(m) \oplus 1$. This ensures that $|P(3 \cdot 5^{i_x})| > |\varphi_j(m)|$.

For each column, we will only perform this operation once (for all $t > s$, we set $\varphi_{i_k}(t + 1) = P(\varphi_{i_k}(t))$). Thus, if $3 \cdot 5^{i_x}$ is a notation for some limit ordinal less than ω^2 , then $3 \cdot 5^{i_{x+1}}$ is also a notation for a limit ordinal less than ω^2 . Thus, this is true for all x by induction and thus a is a notation for ω^2 .

Suppose towards a contradiction that φ_k is a reduction of Id^{+a} to Id^{+b} . Then on the k th column, φ_k gives a reduction of $\text{Id}^{+P(3 \cdot 5^{i_k})}$ to $\text{Id}^{+\varphi_j(k)}$. Let c be so

$c <_{\mathcal{O}} P(3 \cdot 5^{ix})$ and $|c| = |\varphi_j(k)|$. Then $\text{Id}^{\dagger c} \equiv \text{Id}^{\dagger \varphi_j(k)}$ by Lemma 5.3. But then $(\text{Id}^{\dagger c})^{\dagger} \leq_c \text{Id}^{\dagger P(3 \cdot 5^{ix})} \leq_c \text{Id}^{\dagger \varphi_j(k)} \equiv \text{Id}^{\dagger c}$. But then $\text{Id}^{\dagger c}$ m -bounds every HYP set [13, Theorem 3.2], but this is a contradiction since $\text{Id}^{\dagger c}$ is itself HYP.

Using the symmetric strategy to ensure $\text{Id}^{\dagger a} \not\leq_c \text{Id}^{\dagger b}$, and interleaving the requirements, we can construct a and b so that $\text{Id}^{\dagger a}$ and $\text{Id}^{\dagger b}$ are incomparable. \dashv

We next see that for any computable ordinal α , the equivalence relations $\text{Id}^{\dagger a}$ for a with $|a| = \alpha$ form a reasonably well bounded collection of equivalence relations. We will need the following observation:

OBSERVATION 5.5. *There is a computable function $x \mapsto 2 \cdot_{\mathcal{O}} x$ which sends a notation a for α to a notation for $2 \cdot \alpha$. Further, $x <_{\mathcal{O}} 2 \cdot_{\mathcal{O}} x$ for every $x \in \mathcal{O}$.*

PROOF. This is done via transfinite recursion and the recursion theorem. We define $2 \cdot_{\mathcal{O}} P(a)$ to be $P^{(2)}(2 \cdot_{\mathcal{O}} a)$ and we define $2 \cdot_{\mathcal{O}} (3 \cdot 5^e)$ as $3 \cdot 5^i$ where $\varphi_i(x) = 2 \cdot_{\mathcal{O}} \varphi_e(x)$. \dashv

THEOREM 5.6. *For any computable ordinal a , $\text{Id}^{\dagger a} \leq_c =_{\Sigma^0_{2 \cdot_{\mathcal{O}} a}}$ where $=_{\Sigma^0_c}$ is the equivalence relation of equality of Σ^0_c sets (given by a notation $c \in \mathcal{O}$).*

Further, this is uniform in the notation a .

PROOF. We prove this by induction on the notation a . For the base of the induction, let $a = 1$, i.e., the notation for the ordinal 0. Then $\text{Id}^{\dagger a} = \text{Id}$ and $\Sigma^0_{2 \cdot_{\mathcal{O}} a} = \Sigma^0_0$. We can send n to an index for the Σ^0_0 set $\{n\}$.

Next suppose that $a = P(b)$. Then we assume $\text{Id}^{\dagger b}$ reduces to $=_{\Sigma^0_{2 \cdot_{\mathcal{O}} b}}$ sets. Then $\text{Id}^{\dagger a}$ reduces to $(=_{\Sigma^0_{2 \cdot_{\mathcal{O}} b}})^{\dagger}$. Thus it suffices to show the following claim:

CLAIM 5.7. *For any $c \in \mathcal{O}$, $(=_{\Sigma^0_c})^{\dagger} \leq_c =_{\Sigma^0_{P(2)(c)}}$.*

PROOF. Let $(S_m)_{m \in \omega}$ be a natural indexing of all Σ^0_c sets. Let F be a function which sends i to a $\Sigma^0_{P(2)(c)}$ -index for the set $\{m : \exists k (S_m = S_k \wedge k \in W_i)\}$, and observe that F is a reduction. \dashv

Finally, suppose that $a = 3 \cdot 5^i$. Then by the assumed uniformity for all ordinal notations $<_{\mathcal{O}} a$, we have uniform reductions of each $\text{Id}^{\dagger \varphi_i(k)}$ to $=_{\Sigma^0_{2 \cdot_{\mathcal{O}} \varphi_i(k)}}$. Since we can uniformly turn $\Sigma^0_{2 \cdot_{\mathcal{O}} \varphi_i(k)}$ -indices for a set into a $\Sigma^0_{2 \cdot_{\mathcal{O}} a}$ -index for the same set, we see that each $\text{Id}^{\dagger \varphi_i(k)}$ reduces to $=_{\Sigma^0_{2 \cdot_{\mathcal{O}} a}}$. By coding on distinct columns, i.e., using the fact that $=_{\Sigma^0_{2 \cdot_{\mathcal{O}} a}} \times \text{Id} \leq_c =_{\Sigma^0_{2 \cdot_{\mathcal{O}} a}}$, we see that $\text{Id}^{\dagger a} \leq_c =_{\Sigma^0_{2 \cdot_{\mathcal{O}} a}}$. And again this is uniform. \dashv

THEOREM 5.8. *For every computable ordinal α , there is an equivalence relation E which is $\Pi^0_{2 \cdot \alpha + 1}$ so that whenever $a \in \mathcal{O}$ is a notation for α , we have $\text{Id}^{\dagger a} \leq_c E$.*

PROOF. By Spector’s uniqueness theorem [31, Theorem 4.5], if $|a| = |b|$, then $H(a) \equiv H(b)$. Further, this is uniform. Thus for any b with $|b| = |a|$, we can uniformly turn a $\Sigma^0_{2 \cdot_{\mathcal{O}} b}$ -index for a set into a $\Sigma^0_{2 \cdot_{\mathcal{O}} a}$ -index for the same set. Thus

fixing any chosen notation e for α , for any notation a for α , $\text{Id}^{+a} \leq_c \equiv_{\Sigma^0_2, \mathcal{O}^e}$ and $\equiv_{\Sigma^0_2, \mathcal{O}^e} \in \Pi^0_{2\alpha+1}$. +

OBSERVATION 5.9. *When $\alpha = \omega^2$, there is no single notation $b \in \mathcal{O}$ so $|b| = \alpha$ and $\text{Id}^{+b} \equiv_{\Sigma^0_2, \mathcal{O}^\alpha}$. This follows immediately from the first statement of Theorem 5.4.*

§6. Every HYP equivalence relation reduces to some Id^{+a} . Friedman and Stanley [20] proved that the collection of transfinite jumps of the identity relation on reals form a cofinal family in the Borel hierarchy of all Borel isomorphism relations. In this final section, we offer an effective analogue of this result. Namely, we will prove that any HYP equivalence relation is bounded by some Id^{+a} .

As for many other places of this paper, our starting point is [13]. We give a definition of a strong way to reduce a set $A \subseteq \omega$ to an equivalence relation E . This is similar to and inspired by [13, Definition 3.4]; whereas they aren't concerned with the image $h(x)$ if $x \notin A$ (so long as it is E -contained in the image of the reduction for an $x \in A$), we demand only two possible images depending on whether or not $x \in A$.

Observe that the cross product $E \times \text{Id}$ (as defined in the preliminaries) is equivalent to a uniform join of E with itself countably many times.

DEFINITION 6.1. *A set A strong subset reduces to E^+ if there is a computable function h and a pair i, j so that $W_i \subseteq_E W_j$, $h(x) E^+ j$ for every $x \in A$, and $h(x) E^+ i$ for every $x \notin A$.*

This form of reduction is strong enough to give us a way to transfer set reductions to Id^{+a} into equivalence relation reductions to Id^{+a} . In the following lemma and throughout this section, we focus on equivalence relations E so that $E \times \text{Id} \leq_c E$. This is a reasonable assumption since we are trying to build reductions into equivalence relations of the form Id^{+a} and all such equivalence relations satisfy $E \times \text{Id} \leq_c E$ [13, Corollary 2.9].

LEMMA 6.2. *Suppose that R is an equivalence relation and let $A := \{\langle x, y \rangle : x R y\}$. Suppose that either A or the complement of A strong subset reduces to E^+ . Suppose further that $E \times \text{Id} \leq_c E$. Then $R \leq_c E^+$.*

PROOF. Let (h, i, j) witness that A or its complement strong subset reduces to E^+ .

For each $x \in \omega$, let (h_x, i_x, j_x) witness that A or its complement strong subset reduces to the x th column of $(E \times \text{Id})^+$. That is, $W_{h_x(a)} = \{\langle x, y \rangle \mid y \in W_{h(a)}\}$, $W_{i_x} = \{\langle x, y \rangle \mid y \in W_i\}$, and $W_{j_x} = \{\langle x, y \rangle \mid y \in W_j\}$.

For each $x \in \omega$, let e_x be a c.e. index for the set $\bigcup_{y \in \omega} W_{h_y(\langle x, y \rangle)}$. Each $W_{h_y(\langle x, y \rangle)}$ is contained in the y th column and either has the same $E \times \text{Id}$ -closure as W_{i_y} or W_{j_y} . We now check that $x \mapsto e_x$ is a reduction of R to $(E \times \text{Id})^+$.

If $a R b$, then $\{y \mid y R a\} = \{y \mid y R b\}$. Similarly, $\{y \mid y \not R a\} = \{y \mid y \not R b\}$. So, for every y , $W_{h_y(\langle a, y \rangle)}$ has the same $E \times \text{Id}$ -closure as $W_{h_y(\langle b, y \rangle)}$, so $e_a (E \times \text{Id})^+ e_b$. If $a \not R b$ then $W_{h_a(\langle a, a \rangle)}$ has the same $E \times \text{Id}$ -closure as W_{j_a} (or W_{i_a} if it is the complement of A which strong subset reduces to E^+), but $W_{h_a(\langle b, a \rangle)}$ has the same $E \times \text{Id}$ -closure as W_{i_a} (or W_{j_a} if it is the complement of A which strong subset

reduces to E^+) showing that $e_a (E \times \text{Id})^+ e_b$. Thus $x \mapsto e_x$ is a reduction of R to $(E \times \text{Id})^+$, which is equivalent to E^+ . \dashv

We note the similarity between the above and the fact that every Σ_1^0 equivalence relation E reduces to Id^+ . That is proved by sending x to $[x]_E$. This is essentially what we do here, but instead of putting y into the set when y is equivalent to x , we put W_{j_y} into the set if y is equivalent to x .

Below, it will be convenient to reduce into $E \times \text{Id}$ instead of E . The following lemma shows how to return to E .

LEMMA 6.3. *Let $R \leq_c E$. Suppose that A strong subset reduces to R^+ , then A strong subset reduces to E^+ . Similarly, suppose that A strong subset reduces to R^{++} then A strong subset reduces to E^{++} .*

PROOF. Let g be a reduction of R to E . Take (h, i, j) witnessing that A strong subset reduces to R^+ . Then we define $f(n) = e_n$ so that $W_{e_n} = \{g(x) \mid x \in W_{h(n)}\}$. Let $W_a = \{g(x) \mid x \in W_i\}$ and $W_b = \{g(x) \mid x \in W_j\}$. Then (f, a, b) strong subset reduces A to E^+ .

Since we assumed that $R \leq_c E$, we also have $R^+ \leq_c E^+$, so the second case follows from the first. \dashv

In what follows, we will focus on the collection of sets which strong subset reduces to an equivalence relation Id^{+a} , since we now know that, by Lemma 6.2, we can transfer strong subset reductions to equivalence relation reductions. The following easy fact will serve as the base of our induction.

LEMMA 6.4. *Every Σ_1^0 set strong subset reduces to Id^+ .*

PROOF. Fix S a c.e. set. Let i be a c.e. index for the empty set and j be a c.e. index for ω . Let $h(x)$ be an index for an enumeration which either gives \emptyset or ω depending on whether or not we see $x \in S$. \dashv

Next we give an induction which covers every arithmetical equivalence relation.

LEMMA 6.5. *Suppose that A strong subset reduces to E^+ . Further suppose that for every n and p , the set $\{q \mid A(\langle n, p, q \rangle)\}$ is an initial subset of ω . Finally, suppose that $E \times \text{Id} \leq_c E$. Then $B(n) := \exists p \forall q A(n, p, q)$ strong subset reduces E^{++} .*

PROOF. Fix (h, i, j) witnessing A strong subset reduces to E^+ . This shows A strong subset reduces to every column of $(E \times \text{Id})^+$. That is, we have a uniformly computable sequence of functions h_x and indices i_x and j_x as above so that $W_{i_x}, W_{j_x} \subseteq \omega^{[x]}$, $W_{i_x} \subsetneq_{E \times \text{Id}} W_{j_x}$ and $h_x(y) (E \times \text{Id})^+ i_x$ if $y \notin A$ and $h_x(y) (E \times \text{Id})^+ j_x$ if $y \in A$.

For each n , we let W_{e_n} be a collection containing:

- (1) For every $y \in \omega$, a c.e. index for the set $\bigcup_{x < y} W_{j_x} \cup \bigcup_{x \geq y} W_{i_x}$.
- (2) For every $p \in \omega$, a c.e. index for the set $\bigcup_{x \in \omega} W_{h_x(\langle n, p, x \rangle)}$.

Since for every pair n, p , the set of x so that $h_x(\langle n, p, x \rangle) E^+ j_x$ is an initial segment of ω , the sets in the (2) are either already enumerated in (1) or are exactly equal to $\bigcup_{x \in \omega} W_{j_x}$.

Finally, take the map $g : n \mapsto e_n$, let a be a c.e. index for just the sets in (1), and let b be a c.e. index for the sets in (1) along with the set $\bigcup_{x \in \omega} W_{j_x}$. Then (g, a, b)

strong subset reduces B to $(E \times \text{Id})^{++}$. Thus, Lemma 6.3 shows that B strong subset reduces to E^{++} . –

THEOREM 6.6. *For every $n \in \omega$, every Σ_{2n-1}^0 and Π_{2n-1}^0 equivalence relation reduces to $\text{Id}^{+\hat{n}}$.*

PROOF. We first show that for every $n \in \omega$, every Σ_{2n-1}^0 set strong subset reduces to $\text{Id}^{+\hat{n}}$. We use Lemma 6.4 as the base of this induction.

Let X be a Σ_{2n+1}^0 set. Write $X(n) = \exists p \forall q A(\langle n, p, q \rangle)$. Rewrite this definition as: $X(n) = \exists p \forall q (\forall m < q A(\langle n, p, m \rangle))$. We observe that $\forall m < q A(\langle n, p, m \rangle)$ is a Σ_{2n-1}^0 set. Thus, it strong subset reduces to $\text{Id}^{+\widehat{n-1}}$ by inductive hypothesis and, by Lemma 6.5, X strong subset reduces to $\text{Id}^{+\hat{n}}$. Note that the hypotheses that $\text{Id}^{+\widehat{n-1}} \times \text{Id} \leq_c \text{Id}^{+\widehat{n-1}}$ holds by [13, Corollary 2.9].

Finally, applying Lemma 6.2 shows that if R is a Σ_{2n-1}^0 or Π_{2n-1}^0 equivalence relation, then $R \leq_c \text{Id}^{+\hat{n}}$. –

COROLLARY 6.7. *The equivalence relation $\text{Id}^{+\hat{n}}$ is not Π_{2n-1}^0 or Σ_{2n-1}^0 .*

PROOF. It is easy to see that there are equivalence relations which are Σ_{2n-1}^0 and not Π_{2n-1}^0 (consider 1-dimensional equivalence relations with a single class comprised of a Σ_{2n-1}^0 -complete set) and similarly equivalence relations which are Π_{2n-1}^0 and not Σ_{2n-1}^0 . If $\text{Id}^{+\hat{n}}$ were Σ_{2n-1}^0 , then every Π_{2n-1}^0 -equivalence relations would have to be Σ_{2n-1}^0 by virtue of reducing to $\text{Id}^{+\hat{n}}$. Similarly we get a contradiction if $\text{Id}^{+\hat{n}}$ were Π_{2n-1}^0 . –

We note that Theorem 6.6 is sharp on the scale of the arithmetical hierarchy since $\text{Id}^{+\hat{n}}$ is a Π_{2n}^0 equivalence relation and thus there is a Δ_{2n}^0 equivalence relation which does not reduce to $\text{Id}^{+\hat{n}}$ [23]. We can look closer using the Ershov hierarchy:

THEOREM 6.8. *There is a d-c.e. equivalence relation E so that $E \not\leq_c \text{Id}^+$.*

PROOF. We partition the odd numbers into countably many sets S_e for $e \in \omega$. Let $z_{\langle e,i \rangle}$ be the i th element of S_e . We construct a d-c.e. equivalence relation E by stages. We never make any pair of even numbers E -equivalent. We may make elements of S_e be E -equivalent to $4e$ or $4e + 2$ or neither.

We satisfy the following requirements:

\mathcal{R}_e : φ_e is not a reduction of E to Id^+ .

The strategy for meeting the \mathcal{R} -requirements is twofold. On the one hand, we ensure that $4e \not E 4e + 2$, for all e (in fact every pair of even numbers are E -inequivalent). This action forces $W_{\varphi_e(4e)} \neq W_{\varphi_e(4e+2)}$, otherwise φ_e would not be a reduction. But, on the other hand, we use the $z_{\langle e,i \rangle}$'s to gradually copy $W_{\varphi_e(4e)}$ into $W_{\varphi_e(4e+2)}$ and vice versa. Let's discuss in more detail the module for diagonalizing against a potential reduction φ_e :

Let $e_0 = 4e$ and $e_1 = 4e + 2$.

- (1) If at some stage s a number w appears in $W_{\varphi_e(e_k)}$, for $k \in \{0, 1\}$, we take the least unused $z_{\langle e,i \rangle}$ and we let $e_k E z_{\langle e,i \rangle}$.
- (2) We wait to see if w appears in $W_{\varphi_e(z_{\langle e,i \rangle})}$. If this happens, we declare $e_k \not E z_{\langle e,i \rangle}$ and we let $e_{1-k} E z_{\langle e,i \rangle}$ instead.

Now, towards a contradiction, suppose that there is a reduction φ_j from E to Id^+ . Since the construction ensures that $4j \not E 4j + 2$, it must be the case that $W_{\varphi_j(j_0)} \neq W_{\varphi_j(j_1)}$. Without loss of generality, let $v \in W_{\varphi_j(4j)} \setminus W_{\varphi_j(4j+2)}$. But then, by item (1) of the module, we have that, at some stage s , $4j$ is E -collapsed with some $z_{\langle j,i \rangle}$. Observe that, after this collapse, v must enter in $W_{\varphi_j(z_{\langle j,i \rangle})}$ (as otherwise, we would have that $W_{\varphi_j(z_{\langle j,i \rangle})} \neq W_{\varphi_j(4j)}$ but $4j E z_{\langle j,i \rangle}$, a contradiction). When this happens, by item (2), we make $4j \not E z_{\langle j,i \rangle}$ and we let $4j + 2 E z_{\langle j,i \rangle}$ instead. This action guarantees that there is a stage at which v appears in $W_{\varphi_j(4j+2)}$ (as otherwise, $W_{\varphi_j(z_{\langle j,i \rangle})} \neq W_{\varphi_j(4j+2)}$ but $4j + 2 E z_{\langle j,i \rangle}$), contradicting the assumption that $v \in W_{\varphi_j(4j)} \setminus W_{\varphi_j(4j+2)}$.

Finally, it immediately follows from the construction that E is d-c.e., since there is no pair of numbers on which E makes more than two mind changes. \dashv

Theorem 6.6 gives a nice way to represent the arithmetical equivalence relations in terms of FS-jumps, but it is not sharp at the even layers. For example, every Σ_2^0 and Π_2^0 equivalence relation reduces to Id^{+2} , but Id^{+2} is Π_4^0 and we should expect to find a Π_3^0 equivalence relation that is universal for all Σ_2^0 and Π_2^0 -equivalence relations. The next lemma gives us an analogous result at the even layers of the arithmetical hierarchy.

LEMMA 6.9. *Let Z be a universal Π_1^0 -equivalence relation (which exists by [23, Theorem 3.3]). Then every Σ_{2n}^0 and Π_{2n}^0 equivalence relation reduces to $Z^{+\hat{n}}$.*

PROOF. We first observe that since Z is Π_1^0 -universal and $Z \times \text{Id}$ is Π_1^0 , we have $Z \times \text{Id} \leq_c Z$. Thus $Z^{+a} \times \text{Id} \leq_c Z^{+a}$ for any $a \in \mathcal{O}$ [13, Proposition 2.8].

As above, we will first show by induction that every Σ_{2n}^0 set strong subset reduces to $Z^{+\hat{n}}$. As the base of our induction, we first show that every Σ_2^0 set strong subset reduces to Z^+ . To see this, we fix a Σ_2^0 set A and we construct a Π_1^0 -equivalence relation Y and show that A strong subset reduces to Y^+ . This suffices by Lemma 6.3.

We fix a computable approximation $(A_s)_{s \in \omega}$ to A so that $x \in A$ if and only if $x \in A_s$ for all sufficiently large s . We build a reduction by sending every x to an index e_x which we control by the recursion theorem. We enumerate the complement of $[0]_Y$ into each W_{e_x} . At stages s when $x \in A_s$, we take a fresh number z and enumerate z into W_{e_x} . If at a later stage $t > s$ we have $x \notin A_t$ then we make $z \not Y 0$. In fact, we make $[z]_Y = \{z\}$. If we never see such a t , we will maintain $z Y 0$. If $x \notin A$, then $[W_{e_x}]_Y = \omega \setminus [0]_Y$. If $x \in A$, then $[W_{e_x}]_Y = \omega$. This shows that every Σ_2^0 set strong subset reduces to Z^+ .

As the step of our induction, we apply Lemma 6.5 as in the proof of Theorem 6.6 using the fact that $Z^{+\hat{n}} \times \text{Id} \leq_c Z^{+\hat{n}}$.

Finally, Lemma 6.2 shows that every Σ_{2n}^0 or Π_{2n}^0 equivalence relation reduces to $Z^{+\hat{n}}$. \dashv

To move to transfinite levels in the HYP hierarchy, we show that we can handle negations and effective unions.

LEMMA 6.10. *If A strong subset reduces to E^+ then the complement of A strong subset reduces to E^{++} .*

PROOF. Let (h, i, j) witness that A strong subset reduces to E^\dagger . Then let $W_{g(x)}$ enumerate the collection of all c.e. supersets of $W_{h(x)}$. Let W_a be an index for the collection of all c.e. supersets of W_i and W_b be an index for the collection of all c.e. supersets of W_j . Then (g, b, a) witnesses that the complement of A strong subset reduces to $E^{\dagger\dagger}$. \dashv

LEMMA 6.11. *Suppose that each member of $(A_k)_{k \in \omega}$ uniformly strong subset reduces to E^\dagger via (h_k, i_k, j_k) . Further suppose that $E \times \text{Id} \leq_c E$. Let $B(n)$ hold if and only if $\exists k A_k(n)$. Then B strong subset reduces $E^{\dagger\dagger}$.*

PROOF. As in Lemma 6.5, for each x , let g_x be the function showing that A_x strong subset reduces to $(E \times \text{Id})^\dagger$ using only the x th column. That is, $W_{g_x(n)} = \{\langle x, y \rangle \mid y \in W_{h_x(n)}\}$.

We first show that B strong subset reduces to $(E \times \text{Id})^{\dagger\dagger}$.

For each $x \in \omega$, let W_i^x be the set $\{\langle x, y \rangle \mid y \in W_{i_x}\}$. Similarly for W_j^x . Finally, let $V^x = W_i^x \cup \bigcup_{y \neq x} W_j^y$.

Let $f(n)$ be a c.e. index for a set which contains indices for every V^x and also contains indices for the sets $V^x \cup W_{g_x(n)}$. If $n \notin B$, then for every n , $g_x(n)$ is an index for W_i^x , so each set $V^x \cup W_{g_x(n)}$ is a copy of V^x . So, the family is exactly the collection of V^x 's. If $n \in B$, then for some n we have $W_{g_x(n)} = W_j^x$, so $V^x \cup W_{g_x(n)} = \bigcup_{z \in \omega} W_j^z$.

Finally, $E \times \text{Id} \leq_c E$ gives the result by Lemma 6.3. \dashv

At this point, we can take effective unions and we can take negations. That's all we need to induct up the HYP hierarchy:

LEMMA 6.12. *Every HYP set strong subset reduces to $\text{Id}^{\dagger a}$ for some $a \in \mathcal{O}$.*

PROOF. We proceed by induction on notations for computable ordinals with the base case done by Lemma 6.4.

Formally, we show that for every notation c for an ordinal α , there is some a so that every Σ_α^0 set uniformly strong subset reduces to $\text{Id}^{\dagger a}$ (i.e., we can find the index of the witness (h, i, j) uniformly from an index of A as a Σ_c^0 set). Further our construction will produce a computable function H going from c to the notation a . Further, whenever $c <_{\mathcal{O}} d$, we will have $H(c) <_{\mathcal{O}} H(d)$. \dashv

6.1. Successor step. Suppose every Σ_α^0 set uniformly strong subset reduces to $\text{Id}^{\dagger a}$. Then every Π_α^0 set uniformly strong subset reduces to $\text{Id}^{\dagger P(a)}$ by Lemma 6.10. Let A be a $\Sigma_{\alpha+1}^0$ set. Then A is an effective union of Π_α^0 sets. Thus A strong subset reduces to $\text{Id}^{\dagger P^{(2)}(a)}$ by Lemma 6.11, and this argument is uniform.

6.2. Limit step. Let $c = 3 \cdot 5^i$. Then we let $a = 3 \cdot 5^e$ where $\varphi_e(n) = H(\varphi_i(n))$. Since by the inductive hypothesis, we know that $H(\varphi_i(n)) <_{\mathcal{O}} H(\varphi_i(n+1))$ for every $n \in \omega$, we have $a \in \mathcal{O}$.

If A is a Σ_c^0 set, then it is an effective union of Σ_b^0 sets for $b <_{\mathcal{O}} c$. Each of these uniformly strong subset reduces to $\text{Id}^{\dagger P(a)}$ by the uniformity in Observation 5.1 and Lemma 6.3. So, the effective union strong subset reduces to $\text{Id}^{\dagger P^{(2)}(a)}$ by Lemma 6.11. This argument is uniform, and we can let $H(c) = P^{(2)}(a)$.

THEOREM 6.13. *Every HYP equivalence relation reduces to $\text{Id}^{\dagger a}$ for some $a \in \mathcal{O}$.*

PROOF. Combine the above with Lemma 6.2. ⊢

COROLLARY 6.14. *Every HYP equivalence relation reduces to $=_{\Sigma_a}$ for some $a \in \mathcal{O}$. The degree of this only depends on the ordinal $|a|$.*

PROOF. Combine the above with Lemma 5.8. ⊢

COROLLARY 6.15. *If E is a fixed point of the jump, i.e., $E \equiv E^{\dagger}$ then E is an upper bound for every HYP equivalence relation.*

PROOF. If $E^{\dagger} \leq_c E$, then E is above $\text{Id}^{\dagger a}$ for every $a \in \mathcal{O}$ by [13, Propositions 2.3 and 2.7]. So this follows immediately from Theorem 6.13. ⊢

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REFERENCES

- [1] U. ANDREWS and S. A. BADAEV, *On isomorphism classes of computably enumerable equivalence relations*, this JOURNAL, vol. 85 (2020), no. 1, pp. 61–86.
- [2] U. ANDREWS, S. BADAEV, and A. SORBI, *A survey on universal computably enumerable equivalence relations*, **Computability and Complexity**, Lecture Notes in Computer Science, vol. 10010, Springer, Cham, 2017, pp. 418–451.
- [3] U. ANDREWS, D. F. BELIN, and L. S. MAURO, *On the structure of computable reducibility on equivalence relations of natural numbers*, this JOURNAL, (2021), pp. 1–30.
- [4] U. ANDREWS, S. LEMPP, J. S. MILLER, K. M. NG, L. S. MAURO, and A. SORBI, *Universal computably enumerable equivalence relations*, this JOURNAL, vol. 79 (2014), no. 1, pp. 60–88.
- [5] U. ANDREWS, N. SCHWEBER, and A. SORBI, *Self-full ceers and the uniform join operator*. **Journal of Logic and Computation**, vol. 30 (2020), no. 3, pp. 765–783.
- [6] ———, *The theory of ceers computes true arithmetic*. **Annals of Pure and Applied Logic**, vol. 171 (2020), no. 8, p. 102811.
- [7] U. ANDREWS and A. SORBI, *Joins and meets in the structure of ceers*. **Computability**, vol. 8 (2019), nos. 3–4, pp. 193–241.
- [8] ———, *The complexity of index sets of classes of computably enumerable equivalence relations*, this JOURNAL, vol. 81 (2016), no. 4, pp. 1375–1395.
- [9] ———, *Jumps of computably enumerable equivalence relations*. **Annals of Pure and Applied Logic**, vol. 169 (2018), no. 3, pp. 243–259.
- [10] S. BADAEV and A. SORBI, *Weakly precomplete computably enumerable equivalence relations*. **Mathematical Logic Quarterly**, vol. 62 (2016), nos. 1–2, pp. 111–127.
- [11] N. BAZHENOV, M. MUSTAFA, L. S. MAURO, A. SORBI, and M. YAMALEEV, *Classifying equivalence relations in the Ershov hierarchy*. **Archive for Mathematical Logic**, vol. 59 (2020), nos. 7–8, pp. 835–864.
- [12] R. CAMERLO and S. GAO, *The completeness of the isomorphism relation for countable Boolean algebras*. **Transactions of the American Mathematical Society**, vol. 353 (2001), no. 2, pp. 491–518.
- [13] J. CLEMENS, S. COSKEY, and G. KRAKOFF, *Computable reducibility of equivalence relations and an effective jump operator*, this JOURNAL, vol. 88 (2023), no. 2, pp. 540–561. doi: [10.1017/jsl.2022.45](https://doi.org/10.1017/jsl.2022.45).
- [14] S. COSKEY, J. D. HAMKINS, and R. MILLER, *The hierarchy of equivalence relations on the natural numbers under computable reducibility*. **Computability**, vol. 1 (2012), no. 1, pp. 15–38.
- [15] V. DELLE ROSE, L. S. MAURO, and A. SORBI, *Word problems and ceers*. **Mathematical Logic Quarterly**, vol. 66 (2020), no. 3, pp. 341–354.
- [16] J. L. ERSHOV, *Positive equivalences*. **Algebra i Logika**, vol. 10 (1971), pp. 620–650.
- [17] E. B. FOKINA and S.-D. FRIEDMAN, *On Σ_1^1 equivalence relations over the natural numbers*. **Mathematical Logic Quarterly**, vol. 58 (2012), nos. 1–2, pp. 113–124.
- [18] E. B. FOKINA, S.-D. FRIEDMAN, V. HARIZANOV, J. F. KNIGHT, C. MCCOY, and A. MONTALBÁN, *Isomorphism relations on computable structures*, this JOURNAL, vol. 77 (2012), no. 1, pp. 122–132.

- [19] E. FOKINA, B. KHOUSSAINOV, P. SEMUKHIN, and D. TURETSKY, *Linear orders realized by c.e. equivalence relations*, this JOURNAL, vol. 81 (2016), no. 2, pp. 463–482.
- [20] H. FRIEDMAN and L. STANLEY, *A Borel reducibility theory for classes of countable structures*, this JOURNAL, vol. 54 (1989), no. 3, pp. 894–914.
- [21] S. GAO and P. GERDES, *Computationally enumerable equivalence relations*. *Studia Logica*, vol. 67 (2001), no. 1, pp. 27–59.
- [22] A. GAVRYUSHKIN, B. KHOUSSAINOV, and F. STEPHAN, *Reducibilities among equivalence relations induced by recursively enumerable structures*. *Theoretical Computer Science*, vol. 612 (2016), pp. 137–152.
- [23] E. IANOVSKI, R. MILLER, K. M. NG, and A. NIES, *Complexity of equivalence relations and preorders from computability theory*, this JOURNAL, vol. 79 (2014), no. 3, pp. 859–881.
- [24] A. KECHRIS and A. LOUVEAU, *The classification of hypersmooth Borel equivalence relations*. *Journal of the American Mathematical Society*, vol. 10 (1997), no. 1, pp. 215–242.
- [25] B. KHOUSSAINOV, *A journey to computably enumerable structures (tutorial lectures)*, *Sailing Routes in the World of Computation* (F. Manea, R. Miller, and D. Nowotka, editors), Springer, Cham, 2018, pp. 1–19.
- [26] K. M. NG and H. YU, *On the degree structure of equivalence relations under computable reducibility*. *Notre Dame Journal of Formal Logic*, vol. 60 (2019), no. 4, pp. 733–761.
- [27] A. NIES and A. SORBI, *Calibrating word problems of groups via the complexity of equivalence relations*. *Mathematical Structures in Computer Science*, vol. 28 (2018), no. 3, pp. 457–471.
- [28] G. PAOLINI and S. SHELAH, *Torsion-free abelian groups are Borel complete*, preprint, 2021. [arXiv:2102.12371](https://arxiv.org/abs/2102.12371).
- [29] M. C. LASKOWSKI and D. S. ULRICH, *A proof of the Borel completeness of torsion free abelian groups*, preprint. <https://arxiv.org/abs/2202.07452>.
- [30] R. W. ROBINSON, *Two theorems on hyperhypersimple sets*. *Transactions of the American Mathematical Society*, vol. 128 (1967), pp. 531–538.
- [31] G. E. SACKS, *Higher Recursion Theory*, Springer, Berlin, 1990.
- [32] R. I. SOARE, *Recursively Enumerable Sets and Degrees*, Springer, Berlin, 1987.
- [33] S. THOMAS, *The classification problem for torsion-free abelian groups of finite rank*. *Journal of the American Mathematical Society*, vol. 16 (2003), no. 1, pp. 233–258.
- [34] C. E. M. YATES, *On the degrees of index sets*. *Transactions of the American Mathematical Society*, vol. 121 (1966), no. 2, pp. 309–328.
- [35] ———, *On the degrees of index sets. II*. *Transactions of the American Mathematical Society*, vol. 135 (1969), pp. 249–266.

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