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# ON A QUESTION OF MORETÓ

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**Abstract**

We present a family of counterexamples to a question proposed recently by Moretó concerning the character codegrees and the element orders of a finite solvable group.

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## 1. Introduction

Let $G$ be a finite group, and write $\mathrm{Irr}(G)$ to denote the set of irreducible complex characters of $G$. The concept of character codegree, originally defined as $|G|/\chi(1)$ for any nonlinear irreducible character $\chi$ of $G$, was introduced in [1] to characterise the structure of finite groups. However, a nonlinear character $\chi \in \mathrm{Irr}(G/N)$, where $N$ is a nontrivial normal subgroup of $G$, will have two different codegrees when it is considered as a character of $G$ and of $G/N$. To eliminate this inconvenience, Qian *et al.* in [9] redefined the codegree of an arbitrary character $\chi$ of $G$ as

$$\chi^c(1) = |G : \mathrm{Ker}\,\chi|/\chi(1).$$

Many properties of codegrees have been studied, including variations on Huppert's $\rho$-$\sigma$ conjecture, the relationship between codegrees and element orders, groups with few codegrees, and recognising simple groups using the codegree set.

The authors believe that among the above-mentioned results, the most interesting is the relation between codegrees and element orders (see [4, 7, 8]). Here we mention a result of Qian [7, Theorem 1.1] which says that if a finite solvable group $G$ has an element $g$ of square-free order, then $G$ must have an irreducible character of codegree divisible by the order $o(g)$ of $g$. Isaacs [4] established the same result for an arbitrary finite group. Recently, Qian [8] strengthened his earlier result, showing that for every

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element $g$ of a finite solvable group $G$, there necessarily exists some $\chi \in \mathrm{Irr}(G)$ such that $o(g)$ divides $\chi^c(1)$.

Motivated by the results in [4, 7, 8], Moretó considered the converse relation of codegrees and element orders and proposed an interesting question [6, Question B]. He also mentioned that counterexamples, if they exist, seem to be rare.

QUESTION 1.1. Let $G$ be a finite solvable group and let $\chi \in \mathrm{Irr}(G)$. Does there exist $g \in G$ such that $\pi(\chi^c(1)) \subseteq \pi(o(g))$? Here, $\pi(n)$ denotes the set of prime divisors of a positive integer $n$.

In this note, we will construct a family of examples to show that this question has a negative answer in general. For notation and terminology of character theory, we refer to [3].

## 2. Counterexamples

We begin with some facts about automorphisms of extra-special $p$-groups, which are more or less well-known but we give a complete proof for the reader's convenience.

THEOREM 2.1. *For any distinct primes $p, r$ with $r > 3$, choose an extra-special $p$-group $P$ of order $p^{2r+1}$, such that $P$ has exponent $p$ if $p > 2$, and $P$ is the central product of $r - 1$ dihedral groups of order 8 and a quaternion group if $p = 2$.*

(1) *There exists a prime $q$ dividing $p^r + 1$ but not $p^2 - 1$. In particular, $r$ divides $q - 1$, so that the semi-direct product $C_q \rtimes C_r$ makes sense.*
(2) $\mathrm{Aut}(P)$ *contains a subgroup $A$, which acts trivially on $Z(P)$ and is isomorphic to $C_q \rtimes C_r$.*

PROOF (1) By Zsigmondy's prime theorem (see [2, Theorem IX.8.3]) and the condition that $r > 3$, there exists a prime $q$ dividing $p^{2r} - 1$ but not $p^i - 1$ for all $i = 1, \ldots, 2r - 1$. It follows that $2r$ is the order of $p$ modulo $q$, establishing (1).

(2) Write $S = \mathrm{Sp}(2r, p)$ for $p > 2$ and $S = \mathrm{O}_-(2r, 2)$ for $p = 2$. Let $\Lambda$ denote the subgroup of $\mathrm{Aut}(P)$ consisting of those automorphisms of $P$ acting trivially on $Z(P)$. By the construction of $P$, it is well known that $\Lambda/\mathrm{Inn}(P)$ is isomorphic to $S$ (see, for example, [10, Theorem 1]). Since the orders of $C_q \rtimes C_r$ and $\mathrm{Inn}(P)$ are relatively prime (as $p$ does not divide $qr$), it suffices to show that $S$ contains a subgroup isomorphic to $C_q \rtimes C_r$ by the Schur–Zassenhaus theorem.

To do this, we consider the finite field $\mathbb{F}_{p^{2r}}$ of $p^{2r}$ elements and let $V = \mathbb{F}_{p^{2r}}$ be the vector space over $\mathbb{F}_p$ of dimension $2r$. We need to construct a nonsingular symplectic form $\langle\ ,\ \rangle$ on $V$ for $p > 2$ and a nonsingular quadratic form $Q$ on $V$ for $p = 2$.

Fix an element $a \in \mathbb{F}_{p^2} - \mathbb{F}_p$. Then $a \notin \mathbb{F}_{p^r}$ since $r$ is odd. Let $\mathrm{tr} : \mathbb{F}_{p^r} \to \mathbb{F}_p$ be the trace map. It is easy to see that

$$\langle v, w\rangle = \mathrm{tr}((a - a^{p^r})(vw^{p^r} - v^{p^r}w)), \quad v, w \in V,$$

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defines a nonsingular symplectic form on $V$ and the corresponding symplectic group is $S$ for $p > 2$ (see [5, Example 8.5]). When $p = 2$,

$$Q(v) = \mathrm{tr}(v^{1+p^r}), \quad v \in V,$$

defines a nonsingular quadratic form on $V$ and the corresponding orthogonal group is $S = \mathrm{O}_-(2r, 2)$.

Now, let $\Gamma_0(V) = \{v \mapsto av \mid 0 \neq a \in V\}$, consisting of multiplications, and let $\sigma : v \mapsto v^p$ be the Frobenius field automorphism of $\mathbb{F}_{p^{2r}}$. Then $\sigma$ acts naturally on $\Gamma_0(V)$, and we consider the corresponding semi-direct product $\Gamma_0(V) \rtimes \langle\sigma\rangle$. Observe that $\Gamma_0(V)$ is cyclic of order $p^{2r} - 1$ and the order of $\sigma$ is $2r$. Recall that the prime $q$ divides $p^r + 1$ and let $C$ be the unique subgroup of order $q$ in $\Gamma_0(V)$. It is clear that $C$ is invariant under $\sigma$, and furthermore, by elementary Galois theory, the fixed point of $\sigma^2$ in $C$ is trivial since $q$ does not divide $p^2 - 1$. So we can form the semi-direct product $C \rtimes \langle\sigma^2\rangle$, which is clearly isomorphic to $C_q \rtimes C_r$. What remains is to show that $C \rtimes \langle\sigma^2\rangle \leq S$.

A simple calculation shows that both the symplectic form $\langle\ ,\ \rangle$ and the quadratic form $Q$ defined above are preserved by the map on $V$ induced by multiplication by an element of order $p^r + 1$, and thus the unique cyclic subgroup of $\Gamma_0(V)$ of order $p^r + 1$ must be contained in $S$. In particular, we have $C \leq S$. To prove $\sigma^2 \in S$, we distinguish two cases. For $p = 2$, since the Galois group of $\mathbb{F}_{p^r}/\mathbb{F}_p$ can be identified with $\langle\sigma^2\rangle$ (as $r$ is odd), we conclude that $\sigma^2$ must preserve the trace map from $\mathbb{F}_{p^r}$ to $\mathbb{F}_p$ and hence lies in $S$. For $p > 2$, we need to establish that $\langle v^{\sigma^2}, w^{\sigma^2}\rangle = \langle v, w\rangle$ for all $v, w \in V$. Let $b = a - a^{p^r}$ and $x = vw^{p^r} - v^{p^r}w$. It suffices to prove $\mathrm{tr}(bx^{\sigma^2}) = \mathrm{tr}(bx)$. Since $(bx)^{p^r} = (-b)(-x) = bx$, we have $bx \in \mathbb{F}_{p^r}$. It follows that $\mathrm{tr}((bx)^{\sigma^2}) = \mathrm{tr}(bx)$. By the choice of $a$, we know that $b \in \mathbb{F}_{p^2}$ and hence $b$ is fixed by $\sigma^2$. Thus, $\mathrm{tr}(bx^{\sigma^2}) = \mathrm{tr}(bx)$ and $\sigma^2 \in S$, as required. □

As an application of Theorem 2.1, we can now construct a family of counterexamples to Moretó's question.

EXAMPLE 2.2. In the notation of Theorem 2.1, let $G = P \rtimes A$ be the corresponding semi-direct product, so that $G$ is solvable. Then there exists an irreducible character $\chi$ of $G$ such that $\chi^c(1) = p^{r+1}qr$ but $G$ contains no element of order divisible by $pqr$.

PROOF. Note that $Z(P)$ is cyclic of order $p$, and thus we can choose a faithful linear character $\lambda$ of $Z(P)$. Then it is well known that $\lambda^P = p^r\theta$ for some $\theta \in \mathrm{Irr}(P)$. Since $A$ acts trivially on $Z(P)$, it fixes $\lambda$ and hence $\theta$ is $A$-invariant. It follows that $\theta$ extends to some $\chi \in \mathrm{Irr}(G)$ by Gallagher's theorem (see [3, Corollary 6.28]), so that $\chi(1) = \theta(1) = p^r$. Also, let $B$ be the unique subgroup of $A$ of order $q$. Then $P/Z(P)$ is irreducible as an $\mathbb{F}_p[B]$-module because $|P/Z(P)| = p^{2r}$ and the order of $p$ modulo $q$ is exactly $2r$. From this, we conclude that $Z(P)$ is the unique minimal normal subgroup of $G$, and since $\theta$ is faithful, we have $\mathrm{Ker}\,\chi \cap P = \mathrm{Ker}\,\theta = 1$. Obviously, $P$ is the Fitting subgroup of the solvable group $G$, and thus $\mathrm{Ker}\,\chi$ contains no minimal normal subgroup of $G$, which forces $\mathrm{Ker}\,\chi = 1$. Therefore, we have $\chi^c(1) = |G|/\chi(1) = p^{r+1}qr$.

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Finally, if $G$ has an element of order divisible by all the primes $p, q, r$, then $A$ contains an element of order $qr$ and thus $A$ is cyclic, which is not the case by the choice of primes $q, r$. The proof is now complete. □

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