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EXTENSIONS OF AUTOCORRELATION INEQUALITIES WITH APPLICATIONS TO ADDITIVE COMBINATORICS

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Abstract

Barnard and Steinerberger ['Three convolution inequalities on the real line with connections to additive combinatorics', Preprint, 2019, arXiv:1903.08731] established the autocorrelation inequality

$$\min_{0\le t\le 1}\int_{\mathbb{R}} f(x)f(x+t)\,dx \le 0.411\|f\|_{L^1}^2 \quad \text{for } f\in L^1(\mathbf{R}),$$

where the constant 0.411 cannot be replaced by 0.37. In addition to being interesting and important in their own right, inequalities such as these have applications in additive combinatorics. We show that for f to be extremal for this inequality, we must have

$$\max_{x_1\in\mathbb{R}}\min_{0\le t\le 1}[f(x_1-t)+f(x_1+t)] \le \min_{x_2\in\mathbb{R}}\max_{0\le t\le 1}[f(x_2-t)+f(x_2+t)].$$

Our central technique for deriving this result is local perturbation of f to increase the value of the autocorrelation, while leaving $\|f\|_{L^1}$ unchanged. These perturbation methods can be extended to examine a more general notion of autocorrelation. Let $d,n\in\mathbb{Z}^+$, $f\in L^1$, A be a $d\times n$ matrix with real entries and columns a_i for $1\le i\le n$ and C be a constant. For a broad class of matrices A, we prove necessary conditions for f to extremise autocorrelation inequalities of the form

$$\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx \le C\|f\|_{L^1}^n.$$

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1. Introduction

1.1. An autocorrelation inequality. A recent paper of Barnard and Steinerberger [1] asks the following: what is the smallest c so that

$$\min_{t\in[0,1]}\int_{\mathbb{R}} f(x)f(x+t)\,dx \le c\|f\|_{L^1}^2 \tag{1.1}$$

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holds for any $f \in L^1$? A first attempt at providing such a c uses Fubini's theorem to show that

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(x+t)\,dx\,dt = \frac{1}{2}\|f\|_{L^1}^2. \tag{1.2}$$

This shows that $c = 1/2$ satisfies (1.1), although it is not necessarily the smallest c to do so. This is a crude approximation, since to derive (1.2) we replace a minimum with an averaging integral. This bound ($c = 1/2$) was improved in [1], where the authors showed that

$$\min_{t\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \leq 0.411\|f\|_{L^1}^2. \tag{1.3}$$

This result was obtained using techniques from Fourier analysis, in particular the Wiener–Khintchine theorem. Barnard and Steinerberger also gave explicit examples of functions for which the left-hand side of (1.3) is large, showing that 0.411 cannot be reduced to 0.37.

1.2. A problem in continuous Ramsey theory. This problem can be viewed as a problem in continuous Ramsey theory.

In discrete mathematics, Ramsey theory is the study of questions of the form: *How large must a global structure be, in order to guarantee that a smaller substructure appears?* A classical problem in Ramsey theory is the study of Ramsey numbers. Given $r, b \in \mathbb{N}$, let $R(r, b) =: n$ be the smallest natural number such that a 2-colouring of the edges of K_n (with colours red and blue) must contain a red copy of K_r or a blue copy of K_b. Ramsey's theorem asserts that $R(r, b)$ exists for all $r, b \in \mathbb{N}$. *Continuous Ramsey theory* is the study of Ramsey-type questions in the continuous setting. An example of such a question is the problem of symmetric subsets studied by Martin and O'Bryant [10]. A subset of $[0, 1]$ is called *symmetric* if it is invariant under some reflection. Let λ denote the one-dimensional Lebesgue measure and define

$$D(x) := \sup\{r \in \mathbb{R}^+ : \forall A \subset [0,1] \text{ with } \lambda(A) = x,\ \exists\, S \subset A \text{ symmetric with } \lambda(S) \geq r\}.$$

By placing bounds on $D(x)$, Martin and O'Bryant analysed the size of symmetric sets S found within larger sets A.

Rephrased, (1.1) asks the following. Given a function $f \in L^1$, how does the function

$$g(t) = \int_{\mathbb{R}} f(x)f(x+t)\,dx$$

behave? If we replace f by $c \cdot f$ for some $c \in \mathbb{R}$, then $g(t) = c^2 g(t)$. Thus, we must take into account the size of f. The L^1 norm is a natural choice for measuring the size of a function $f : \mathbb{R} \to \mathbb{R}$. For our measurement of substructure, we follow Barnard and Steinerberger [1] and investigate, as a function of t,

$$\frac{1}{\|f\|_{L^1}^2} \int_{\mathbb{R}} f(x)f(x+t)\,dx, \tag{1.4}$$

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where we normalise by $\|f\|_{L^1}^2$ in order to provide invariance under the scaling discussed above.

The central question behind (1.4) is Ramsey in the following sense. In the classical discrete setting, we are given an edge colouring of some graph of known size and asked to find which monochromatic subgraphs must appear. Just as there are some colourings which have huge amounts of structure, there are some functions with trivial behaviour under $\min_{t\in[0,1]} g(t)$. For example, if $\operatorname{supp}(t) \subseteq [0, 1]$, then $\min_{t\in[0,1]} g(t) = 0$. However, the value

$$\sup_{t\in L^1} \min_{t\in[0,1]} g(t)$$

(corresponding to those graph edge colourings which are least structured) is not well understood.

1.3. A problem in additive combinatorics. In addition to being a Ramsey-type problem, this problem is also fundamentally connected to a problem in additive combinatorics.

Given some $n \in \mathbb{N}$, a set of integers $A \subset \mathbb{Z}$ is called a *difference basis* with respect to n if, for the difference set

$$A - A := \{a_1 - a_2 \mid a_1, a_2 \in A\},$$

we have $\{1, \dots, n\} \subset A - A$. The value

$$H(n) := \min\{|A| : A \subset \mathbb{Z} \text{ and } \{1, \dots, n\} \subset A - A\}$$

has been studied extensively. The connection to (1.4) is through probability: if $f(n)$ is a probability distribution on $n \in \mathbb{Z}$, then $g(t)$ is the probability distribution given by taking the difference $f - f$. The function $H(n)$ was proposed and studied in [4, 6, 8, 11]. Lower bounds on $H(n)$ as $n \to \infty$ were proved in [9] and later improved in [3], while upper bounds were shown in [7]. Since $|A - A|$ is at most quadratic in $|A|$, it is immediate that $H(n) \geq \sqrt{2n}$. In fact, these are the correct asymptotics; the best known results are

$$\sqrt{2.435n} \leq H(n) \leq \sqrt{2.645n}.$$

This connection to additive combinatorics motivates our investigation. It is possible that the discrete and continuous problems could inform one another.

1.4. Our results. We provide necessary conditions for the existence of a function f maximising (1.4). This is a question which applies only to continuous Ramsey theory, as opposed to discrete Ramsey theory. In a discrete problem, such as the study of Ramsey numbers, extremal structures trivially exist. For example, given that $R(r, b) = n$, it is clear that there must exist some colouring of K_{n-1} which contains no red copy of K_r or blue copy of K_b. Furthermore, since there are but a finite number of such colourings, we know there is only a finite number of such extremal graphs, none 'more extreme' than any other. In the continuous case, it is not clear if there exist function(s) maximising (1.4).

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Our methods are based on perturbation theory. Given a candidate extremal function f, we attempt to increase its value under (1.4) by adding a function g which is small in L^1 norm. In fact, our perturbation techniques can be extended to prove results on generic convolution-type integrals. If d, n are positive integers, A a $d \times n$ matrix with columns a_i for $1 \leq i \leq n$ and $f \in L^1$, then we study

$$\frac{1}{\|f\|_{L^1}^n} \min_{\mathbf{t} \in [0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + \mathbf{t} \cdot a_i)\, dx. \tag{1.5}$$

In Section 2 we state Theorem 2.1, our main result, as well as Corollaries 2.4, 2.5 and 2.6. We prove Theorem 2.1 in Section 3 and conclude in Section 4 with a discussion of potential directions in which our work might be extended.

2. Main results

In Section 2.1 we state our main result, Theorem 2.1. The statement of the theorem relies on a technical result, Lemma 2.2, which we state and prove in Section 2.2. In Section 2.3 we state Corollary 2.4, Corollary 2.5 and Corollary 2.6, which are special cases of Theorem 2.1. Finally, in Section 2.5, we state conditions under which the continuity hypothesis of Theorem 2.1 can be relaxed.

2.1. Statement of Theorem 2.1. We now present our main results on the existence of functions f maximising (1.5). First we present the theorem in its full generality.

THEOREM 2.1. *Let $d, n \in \mathbb{N}$ and let A be a $d \times n$ matrix with columns a_i for $1 \leq i \leq n$ satisfying Lemma 2.2. Then a continuous function f maximising (1.5) must satisfy both*

$$\max_{x_1 \in \mathbb{R}} \min_{\mathbf{t} \in [0,1]^d} \sum_{i=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_1 + \mathbf{t} \cdot (a_i - a_j)) \leq \frac{n}{\|f\|_{L^1}} \min_{\mathbf{t} \in [0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + \mathbf{t} \cdot a_i)\, dx \tag{2.1}$$

and

$$\max_{x_1 \in \mathbb{R}} \min_{\mathbf{t} \in [0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_1 + \mathbf{t} \cdot (a_i - a_j)) \leq \min_{x_2 \in \operatorname{supp}(f)} \max_{\mathbf{t} \in [0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_2 + \mathbf{t} \cdot (a_i - a_j)). \tag{2.2}$$

This theorem is most easily interpreted in the one-dimensional case, when $d = 1$. In this scenario, we find that a function f maximising (1.5) must satisfy both

$$\max_{x_1 \in \mathbb{R}} \min_{t \in [0,1]} \sum_{i=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_1 + t \cdot (a_i - a_j)) \leq \frac{n}{\|f\|_{L^1}} \min_{t \in [0,1]} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + t \cdot a_i)\, dx$$

and

$$\max_{x_1 \in \mathbb{R}} \min_{t \in [0,1]} \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_1 + t \cdot (a_i - a_j)) \leq \min_{x_2 \in \operatorname{supp}(f)} \max_{t \in [0,1]} \sum_{j=1}^{n} \prod_{i=1, i \neq j}^{n} f(x_2 + t \cdot (a_i - a_j)).$$

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2.2. Technical lemmas. Our goal is to study the existence of functions f maximising (1.5). However, there exist choices of A for which (1.5) is unbounded. For example, if $A = \mathbf{0}_{d\times n}$, then (1.5) is not necessarily even finite for individual f. In the specific case $n = d = 2$, $\int_{\mathbb{R}} |f(x)|\,dx < \infty$ does not imply that $\int_{\mathbb{R}} f(x)^2\,dx < \infty$.

Analogous to the reasoning in (1.2), we can use Fubini's theorem to give a sufficient condition on A for which (2.1) is bounded from above. This includes the choice of A studied in [1].

Lemma 2.2. *If the* $(d+1)\times n$ *matrix*

$$B = \left[\frac{1\cdots 1}{A}\right]$$

has rank at least n, then (1.5) is finite for all choices of $f \in L^1$.

Proof. First we see that $d \geq n-1$ is implied by the rank criteria on A. Then we observe the right-multiplication

$$\begin{bmatrix} x & t_1 & \dots & t_d \end{bmatrix} \cdot B = \begin{bmatrix} x + \mathbf{t}\cdot a_1 & x + \mathbf{t}\cdot a_2 & \dots & x + \mathbf{t}\cdot a_n \end{bmatrix}.$$

Since B has rank at least n, there exists an invertible linear transformation $C : \mathbf{R}^d \to \mathbf{R}^d$ so that

$$\begin{bmatrix} x & \mathbf{t}\cdot C \end{bmatrix} \cdot B = \begin{bmatrix} x & x + t_1 & \dots & x + t_{n-1} \end{bmatrix}.$$

Returning to our problem, we use the sequence of upper bounds

$$\begin{aligned} \min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx &\leq \int_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx\,d\mathbf{t} \\ &\leq \int_{\mathbf{t}\in\mathbb{R}^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx\,d\mathbf{t}. \end{aligned}$$

We exchange the $\mathbf{t}\cdot a_i$ for t_i by applying C,

$$\begin{aligned} \int_{\mathbf{t}\in\mathbf{R}^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx\,d\mathbf{t} &\leq \int_{\mathbf{t}\in C^{-1}\mathbb{R}^d} \int_{\mathbb{R}} f(x) \prod_{i=1}^{n-1} f(x+t_i)\,dx\,d\mathbf{t} \\ &\leq \|f\|_{L^1}^n. \end{aligned}$$

Thus, (1.5) is necessarily finite. □

A generalisation of Lemma 2.2 holds when $d+1 = n$.

Corollary 2.3. *Let* $d+1 = n$ *and let A be an* $n\times n$ *matrix with columns* a_i *such that each* a_i *contains at least one nonzero entry. Then (1.5) is finite for all choices of* $f \in L^1$. *In fact, we have the bound*

$$\frac{1}{\|f\|_{L^1}^n} \min_{\mathbf{t}\in[0,1]^d} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx \leq \frac{1}{\sqrt{D}},$$

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where D is given by

$$D = \inf\left\{\det\left(\sum_{i=1}^{n} \lambda_i a_i' \cdot a_i\right)\Big/\prod_{i=1}^{n} \lambda_i \;\Big|\; \lambda_i \in \mathbb{R}^{>0}\right\}$$

and a_i' denotes the transpose of a_i.

Corollary 2.3 follows immediately from the Brascamp–Lieb inequality (see [2]).

2.3. Theorem 2.1 for specific d, n and A. Corollary 2.4 addresses a question asked in [1] about the existence of f extremising (1.1). It is obtained by setting $n = 2$, $d = 1$ and $A = [\,0 \quad 1\,]$ in Theorem 2.1.

Corollary 2.4. *A continuous function f maximising*

$$\frac{1}{\|f\|_{L^1}^2} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx$$

must satisfy both

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]} [f(x_1 - t) + f(x_1 + t)] \le \frac{2}{\|f\|_{L^1}} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx$$

and

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]} [f(x_1 - t) + f(x_1 + t)] \le \min_{x_2\in\mathbb{R}} \max_{\mathbf{t}\in[0,1]} [f(x_2 - t) + f(x_2 + t)].$$

Setting $d = n$ and $A = I$ in Theorem 2.1 gives the following corollary.

Corollary 2.5. *Let n be a positive integer. A continuous function f maximising*

$$\frac{1}{\|f\|_{L^1}^n} \min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+t_i)\,dx$$

must satisfy both

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^n} \sum_{i=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1 + t_i - t_j) \le \frac{n}{\|f\|_{L^1}} \min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x+t_i)\,dx$$

and

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^n} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1 + t_i - t_j) \le \min_{x_2\in\operatorname{supp}(f)} \max_{t\in[0,1]^n} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_2 + t_i - t_j).$$

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2.4. Theorem 2.1 for convex or concave functions. The value

$$\max_{x_1\in\mathbb{R}} \min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1 + \mathbf{t}\cdot(a_i - a_j)) \tag{2.3}$$

found in Theorem 2.1 engenders some discussion on how it is connected to the structure of f. Consider the function $r : \mathbb{R} \to \mathbb{R}$ given by

$$r(x_1) = \min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} f(x_1 + \mathbf{t}\cdot(a_i - a_j)).$$

In other words, (2.3) asks for $\max_x r(x)$. The value r takes on at a given point x_1 is not truly global; we may alter f outside of an interval around x_1 without altering the value taken there. Nor is it truly local; no amount of information locally around x_1 can provide enough information to determine this value, because we allow t to extend to 1.

In the case $d = 1$, (2.3) reduces to

$$\max_{x_1\in\mathbb{R}} \min_{t\in[0,1]} [f(x-t) + f(x+t)].$$

When f is convex, the minimum is always obtained for $t = 0$, since increasing t increases the symmetric sum about the point x. Conversely, for f concave the minimum occurs when $t = 1$. This provides the next corollary, which follows from simplifying the result of Theorem 2.1 under the additional assumptions that $d = 1$ and f is concave or convex.

Corollary 2.6. *A continuous convex function f maximising*

$$\frac{1}{\|f\|_{L^1}^2} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx \tag{2.4}$$

must satisfy both

$$\max_{x_1\in\mathbb{R}} f(x_1) \leq \frac{1}{\|f\|_{L^1}} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx$$

and

$$2\max_{x_1\in\mathbb{R}} f(x_1) \leq \min_{x_2\in\mathbb{R}}[f(x_2-1) + f(x_2+1)].$$

Similarly, a concave function maximising (2.4) must satisfy both

$$\max_{x_1\in\mathbb{R}} f(x_1-1) + f(x_1+1) \leq \frac{2}{\|f\|_{L^1}} \min_{\mathbf{t}\in[0,1]} \int_{\mathbb{R}} f(x)f(x+t)\,dx$$

and

$$\max_{x_1\in\mathbb{R}}[f(x_1-1) + f(x_1+1)] \leq 2\min_{x_2\in\mathbb{R}} f(x_2).$$

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2.5. Theorem 2.1 for discontinuous functions. We may relax the hypothesis on the continuity of f if, in the conditions given in the theorem, we replace $f(x_0)$ with

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{x_0-\varepsilon/2}^{x_0+\varepsilon/2} f(x)\,dx$$

whenever we evaluate f at a point x_0. When f is continuous, this limit is $f(x_0)$. If f has a removable or jump discontinuity (such as those found in a construction in [1]), the limit is no longer identical to function evaluation, but still may be calculated. By standard results from measure theory, the limit is $f(x_0)$ almost everywhere on $\mathbb{R}$ for any choice of f.

3. Proof of Theorem 2.1

Before proving Theorem 2.1, we recall the theorem statement.

THEOREM 2.1. *Let $d, n \in \mathbb{N}$ and let A be a $d \times n$ matrix with columns a_i for $1 \le i \le n$ satisfying Lemma 2.2. Then a continuous function f maximising (1.5) must satisfy both*

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{i=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \le \frac{n}{\|f\|_{L^1}}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx \tag{2.1}$$

and

$$\max_{x_1\in\mathbb{R}}\min_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) \le \min_{x_2\in\mathrm{supp}(f)}\max_{\mathbf{t}\in[0,1]^d}\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_2+\mathbf{t}\cdot(a_i-a_j)). \tag{2.2}$$

PROOF OF THEOREM 2.1. For $\varepsilon > 0$ and $x_1 \in \mathbb{R}$, set $g(x) := \varepsilon\chi_{[x_1-\varepsilon/2,x_1+\varepsilon/2]}$. We show that if the given conditions fail, $f+g$ is an improvement over f. That is, we wish to show that

$$\frac{1}{\|f+g\|_{L^1}^n}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\,dx > \frac{1}{\|f\|_{L^1}^n}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx.$$

By the triangle inequality, it suffices to show that

$$\frac{1}{(\|f\|_{L^1}+\|g\|_{L^1})^n}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\,dx > \frac{1}{\|f\|_{L^1}^n}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx.$$

Now the simple form of g allows us to compute $\|g\|_{L^1} = \varepsilon^2$. By letting $\varepsilon \to 0$, we find it is enough to show that

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g)(x+\mathbf{t}\cdot a_i)\,dx}{\|f\|_{L^1}^n + n\varepsilon^2\|f\|_{L^1}^{n-1} + O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx}{\|f\|_{L^1}^n}.$$

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Since g is $O(\varepsilon)$, the product on the left-hand side will be dominated by those products containing only one g. Breaking up the minimum, we find the sufficient condition

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\sum_{i=1}^{n} g(x+\mathbf{t}\cdot a_j)\prod_{i=1,i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\,dx + O(\varepsilon^3)}{n\varepsilon^2\|f\|_{L^1}^{n-1}+O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx}{\|f\|_{L^1}^{n}}.$$

With f continuous, as we integrate over x, $g(x+\mathbf{t}\cdot a_j)$ approximates $f(x_1-\mathbf{t}\cdot a_j)$, so that again by letting $\varepsilon\to 0$, we need

$$\frac{\varepsilon^2\min_{\mathbf{t}\in[0,1]^d}\sum_{i=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) + O(\varepsilon^3)}{n\varepsilon^2\|f\|_{L^1}^{n-1}+O(\varepsilon^4)} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx}{\|f\|_{L^1}^{n}}.$$

Therefore, as higher order terms are eliminated, we find that

$$\min_{\mathbf{t}\in[0,1]^d}\sum_{i=1}^{n}\prod_{i=1,i\neq j}^{n} f(x_1+\mathbf{t}\cdot(a_i-a_j)) > \frac{n}{\|f\|_{L^1}}\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx$$

and taking the maximum over x_1 yields the first half of Theorem 2.1.

For the second half, we take a second point $x_2\in\operatorname{supp}(\mathbb{R})$ and in the spirit of g set $g_1 := \varepsilon\chi_{[x_1-\varepsilon/2,x_1+\varepsilon/2]}$ and $g_2 := \varepsilon\chi_{[x_2-\varepsilon/2,x_2+\varepsilon/2]}$. Then by taking ε small enough we know that $\|f+g_1-g_2\|_{L^1} = \|f\|_{L^1}$ and, so long as $x_1\neq x_2$, g_1 and g_2 have disjoint support. Then to prove that

$$\frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g_1-g_2)(x+\mathbf{t}\cdot a_i)\,dx}{\|f+g_1-g_2\|_{L^1}^{n}} > \frac{\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx}{\|f\|_{L^1}^{n}}$$

we show that

$$\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n}(f+g_1-g_2)(x+\mathbf{t}\cdot a_i)\,dx > \min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\prod_{i=1}^{n} f(x+\mathbf{t}\cdot a_i)\,dx.$$

By breaking open the minimum and expanding the product on the left-hand side, we find that

$$\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\sum_{j=1}^{n}(g_1-g_2)(x+\mathbf{t}\cdot a_j)\prod_{i=1,i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\,dx > 0.$$

Transferring those negative g_2 terms to the right,

$$\min_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\sum_{j=1}^{n} g_1(x+\mathbf{t}\cdot a_j)\prod_{i=1,i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\,dx$$
$$> \max_{\mathbf{t}\in[0,1]^d}\int_{\mathbb{R}}\sum_{j=1}^{n} g_2(x+\mathbf{t}\cdot a_j)\prod_{i=1,i\neq j}^{n} f(x+\mathbf{t}\cdot a_i)\,dx.$$

Once again, g_1 and g_2, when integrated against a product, return the value of that product evaluated at a specific value of x as $\varepsilon \to 0$, giving

$$\min_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_1 + \mathbf{t}\cdot(a_i - a_j)) > \max_{\mathbf{t}\in[0,1]^d} \sum_{j=1}^{n} \prod_{i=1,i\neq j}^{n} f(x_2 + \mathbf{t}\cdot(a_i - a_j)).$$

Taking the best possible x_1, x_2 gives the second half of Theorem 2.1. □

4. Future work

Theorem 2.1 gives conditions which must hold for any f maximising (1.5). Future work might be able to show whether or not such functions exist. We have defined a much broader class of convolution-type inequalities than the one studied in [1]. There, the focus is on placing upper and lower bounds on (1.4). Can we find (the best possible) constants C_1, C_2 and function f_0, all depending on d, n, A, such that for any choice of $f \in L^1$,

$$\frac{1}{\|f\|_{L^1}^n} \min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f(x + t_i)\, dx \leq C_1,$$

while

$$\frac{1}{\|f_0\|_{L^1}^n} \min_{\mathbf{t}\in[0,1]^n} \int_{\mathbb{R}} \prod_{i=1}^{n} f_0(x + t_i)\, dx \geq C_2?$$

While inspired by autocorrelations found in number theory, connections to the Brascamp–Lieb inequality suggest that further abstraction could prove fruitful. The standard formulation of Brascamp–Lieb considers products of functions f_i each operating on the domain $\mathbb{R}^{n_i}$, while here we limited ourselves to functions on $\mathbb{R}$. Our perturbation approach appears quite general; could this technique be effective in the more general setting? We can pose these additive problems using probability distributions on Banach spaces [5]; perhaps these methods might be useful there.

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