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Lorentz Estimates for Weak Solutions of Quasi-linear Parabolic Equations with Singular Divergence-free Drifts

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Abstract. This paper investigates regularity in Lorentz spaces for weak solutions of a class of divergence form quasi-linear parabolic equations with singular divergence-free drifts. In this class of equations, the principal terms are vector field functions that are measurable in (x,t)-variable, and nonlinearly dependent on both unknown solutions and their gradients. Interior, local boundary, and global regularity estimates in Lorentz spaces for gradients of weak solutions are established assuming that the solutions are in BMO space, the John–Nirenberg space. The results are even new when the drifts are identically zero, because they do not require solutions to be bounded as in the available literature. In the linear setting, the results of the paper also improve the standard Calderón–Zygmund regularity theory to the critical borderline case. When the principal term in the equation does not depend on the solution as its variable, our results recover and sharpen known available results. The approach is based on the perturbation technique introduced by Caffarelli and Peral together with a "double-scaling parameter" technique and the maximal function free approach introduced by Acerbi and Mingione.

1 Introduction

This paper establishes local interior, local boundary, and global regularity estimates in Lorentz spaces for gradients of weak solutions of the following class of quasi-linear parabolic equations with singular divergence-free drifts, and with conormal boundary condition

(1.1)

$$\begin{aligned} u_t - \operatorname{div}\big[\mathbf{A}(x,t,u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)\big] &= f(x,t) \quad && (x,t)\in\Omega\times(0,T),\\ \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{v}\rangle &= 0 && (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) &= u_0(x), \quad x\in\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\vec{v}$ is the unit outward normal vector on $\partial\Omega$, $f\colon\Omega\times(0,T)\to\mathbb{R}$ is a given measurable function, $\mathbf{F},\mathbf{b}\colon\Omega\times(0,T)\to\mathbb{R}^n$ are given vector field functions, and u is an unknown solution with a given initial condition u_0 for which we do not require any regularity. Moreover, T is a given fixed positive number, and the principal term

$$\mathbf{A} = \mathbf{A}(x,t,s,\xi)\colon \Omega\times(0,T)\times\mathbb{K}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

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is a given vector field. We assume that $\mathbf{A}(\cdot,\cdot,s,\xi)$ is measurable in $\Omega_T = \Omega \times (0,T)$ for every $(s,\xi) \in \mathbb{K} \times \mathbb{R}^n$; $\mathbf{A}(x,t,\cdot,\xi)$ Hölder continuous in $\mathbb{K}$ for a.e. $(x,t) \in \Omega_T$ and for all $\xi \in \mathbb{R}^n$; and $\mathbf{A}(x,t,s,\cdot)$ differentiable in $\mathbb{R}^n$ for each $s \in \mathbb{K}$ and for a.e. $(x,t) \in \Omega_T$. Here, $\mathbb{K}$ is an open interval in $\mathbb{R}$, which could be the same as $\mathbb{R}$. We assume in addition that there exist constants $\Lambda > 0$ and $\alpha_0 \in (0,1]$ such that $\mathbf{A}$ satisfies the following natural growth conditions:

$$\langle \mathbf{A}(x,t,s,\eta) - \mathbf{A}(x,t,s,\xi), \eta - \xi\rangle \geq \Lambda^{-1}|\eta-\xi|^2 \quad \text{for a.e. } (x,t) \in \Omega_T \text{ for all } s \in \mathbb{K}, \text{ for all } \xi,\eta \in \mathbb{R}^n, \tag{1.2}$$

$$|\mathbf{A}(x,t,s,\xi)| + |\xi||\partial_\xi \mathbf{A}(x,t,s,\xi)| \leq \Lambda|\xi| \quad \text{for a.e. } (x,t) \in \Omega_T \text{ for all } s \in \mathbb{K}, \text{ for all } \xi \in \mathbb{R}^n, \tag{1.3}$$

$$|\mathbf{A}(x,t,s_1,\xi) - \mathbf{A}(x,t,s_2,\xi)| \leq \Lambda|\xi||s_1-s_2|^{\alpha_0} \quad \text{for a.e. } (x,t) \in \Omega_T \text{ for all } s_1,s_2 \in \mathbb{K},\ \xi \in \mathbb{R}^n. \tag{1.4}$$

Under the conditions (1.2)–(1.4) and with $\mathbf{F} = \mathbf{b} = 0$, the class of equations (1.1) contains the well-known quasi-linear parabolic equations with zero-flux boundary condition. If $\mathbf{F} = 0$, but $\mathbf{b} \neq 0$, equation (1.1) is the standard nonlinear advection-diffusion equations. The drift term $\mathbf{b}$ considered in this paper could be singular. Due to its relevance in many applications such as in fluid dynamics and mathematical biology (see [4,11,27,43,44,47] for examples), we are particularly interested in the case where $\mathbf{b}$ is divergence-free, *i.e.*, $\operatorname{div}[\mathbf{b}(\cdot,t)] = 0$, in the sense of distributions in Ω, for a.e. $t \in (0,T)$.

On one hand, when $\mathbf{b} = 0$ and $\mathbf{F}, f$ are sufficiently regular, the $C^{1,\alpha}$-regularity theory for *bounded, weak solutions* of this class of equations (1.1) has been investigated extensively in the classical work; see for example [22, 23, 31, 33, 45], assuming some regularity of $\mathbf{A}$ in $(x,t,s,\xi) \in \Omega_T \times \mathbb{K} \times \mathbb{R}^n$. On the other hand, when $\mathbf{b}, \mathbf{F}, f$ are not so regular or when $\mathbf{A}$ is discontinuous in (x,t), one does not expect those mentioned Schauder's type estimates for weak solutions of (1.1) to hold. It is therefore mathematically interesting and essentially important to search for regularity estimates of Calderón–Zygmund type for gradients of weak solutions in Lebesgue spaces. In particular, in these situations, this kind of Calderón–Zygmund regularity estimates is vital in studying many questions in nonlinear equations and systems of equations; see [27] for example. In this perspective, it is known that to establish the Calderón–Zygmund theory, the class of considered equations must be invariant under the scalings and dilations, see [46] for more geometric intuition of this issue. However, due to the fact that the nonlinearity of the principal term $\mathbf{A}$ depends on u as its variable, the class of this equations (1.1) is not invariant under the scalings and dilations

$$u \mapsto u/\lambda, \quad \text{and} \quad u(x,t) \mapsto \frac{u(rx,r^2t)}{r} \quad \text{for all positive numbers} \quad r,\lambda. \tag{1.5}$$

Due to the lack of this homogeneity, Calderón–Zygmund type regularity theory for weak solutions of (1.1) becomes delicate and is still not completely understood. In a simpler case when $\mathbf{A}$ is independent on the variable $s \in \mathbb{K}$, and $\mathbf{b} = f = 0$, equation (1.1) is reduced to

$$u_t - \operatorname{div}[\mathbf{A}(x,t,\nabla u)] = \operatorname{div}[\mathbf{F}] \quad \text{in} \quad \Omega_T, \tag{1.6}$$

and the $W^{1,q}$-regularity estimate for weak solutions of equations (1.6) has been non-trivially and extensively developed by many authors for both elliptic, parabolic settings and also for p-Laplacian type equations; for example, see [6, 8–10, 12, 14, 17, 18, 30, 34, 37].

In recent work [27,38,39], the $W^{1,q}$-regularity estimates for weak solutions of equations (1.1) with $\mathbf{b} = 0$ is addressed, and the $W^{1,q}$-regularity estimates are established for bounded weak solutions. To overcome the loss of homogeneity that we mentioned, in [27, 38, 39], we introduced some “double-scaling parameter” techniques. Essentially, we studied an enlarged class of “double-scaling parameter" equations of the type (1.1). Then, by some compactness argument, we successfully applied the perturbation method in [10] to tackle the problem. Careful analysis is required to ensure that all intermediate steps in the perturbation process are uniform with respect to the scaling parameters. See also a very recent work [7] for further implementation of this idea for which global regularity theory for bounded weak solutions of some class of degenerate elliptic equations is obtained. In all mentioned papers [7, 27, 38, 39], the boundedness assumption on the solutions is essential to start the investigation of $W^{1,q}$-theory. This is because the approach uses maximum principle for the unperturbed equations to implement the perturbation technique. We would like to refer also to [5], in which the $W^{1,q}$-theory for parabolic p-Laplacian type equations of the form (1.1) is also achieved, but only for continuous weak solutions plus other assumptions on $\mathbf{A}$.

In this paper, we establish regularity estimates in Lorentz spaces for gradients of weak solutions of (1.1) by assuming that the solutions are in the BMO space, *i.e.*, the critical borderline case, and including the singular drifts $\mathbf{b} \neq 0$. We achieve this in Theorems 1.1–1.3. Our paper therefore generalizes the results in [5, 7, 27, 39] for (1.1) by relaxing the boundedness assumption on solutions, and putting into the context of Lorentz space setting. Even in the linear case, and with $f = 0$, our results are also stronger than the classical Calderón–Zygmund results. Precisely, in this case, (1.1) is reduced to

$$u_t - \operatorname{div}[\mathbf{A}_0(x,t)\nabla u] = \operatorname{div}[\mathbf{b}u + \mathbf{F}], \tag{1.7}$$

and the classical Calderón–Zygmund theory gives

$$\|\nabla u\|_{L^p} \le C\big[\, \|\mathbf{F}\|_{L^p} + \|u\|_{L^\infty}\|\mathbf{b}\|_{L^p}\,\big].$$

Our results in Theorems 1.1–1.3 below improve this estimate by replacing $\|u\|_{L^\infty}$ by its borderline case $[[u]]_{\mathrm{BMO}}$. See also [43] for some similar results in this direction for linear equations and with more regularity assumptions on $\mathbf{b}$. At this point, we also would like to note that when $\mathbf{b}, \mathbf{F}, f$ satisfies some certain regularity conditions, weak solutions of (1.7) are proved in [44, 47] to be in C^α, with some $\alpha \in (0,1)$. The results in this paper can therefore be considered as the Sobolev counter part of this result, but for more general nonlinear equations.

Unlike [7, 27, 38, 39], which used “double-scaling parameter", we only use “single scaling parameter" in the class of our equations (see [41, 42]). To be precise, we will investigate the following class of equations

$$
\begin{aligned}
u_t - \operatorname{div}\Big[\mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)\Big] &= f(x,t), \quad &&\text{in } \Omega_T,\\
\langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{v}\rangle &= 0, &&\text{on } \partial\Omega\times(0,T),\\
u(\cdot,0) &= u_0(\cdot), &&\text{in } \Omega,
\end{aligned} \tag{1.8}
$$

with the scaling parameter $\lambda \geq 0$. As we will see in Subsection 2.1, this class of equations is the smallest one that is invariant with respect to scalings and dilations (1.5) and contains the class of equations (1.1). When $\lambda = 0$, $f = 0$, and $\mathbf{b} = 0$, equation (1.8) clearly becomes equation (1.6). This paper, therefore, recovers all known results for (1.6) such as [9, 27, 43].

In this paper, $B_R(y)$ denotes the ball in $\mathbb{R}^n$ with radius $R > 0$ and centered at $y \in \mathbb{R}^n$. If $y = 0$, we write $B_R = B_R(0)$. Also, for each $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, we write

$$
Q_R(z_0) = B_R(x_0) \times \Gamma_R(t_0), \quad \text{with} \quad \Gamma_R(t_0) = (t_0 - R^2, t_0 + R^2).
$$

When $z_0 = 0$, we write

$$
Q_R = Q_R(0,0), \quad \Gamma_R = \Gamma_R(0).
$$

For a measurable set $U \subset \mathbb{R}^{n+1}$, for some $\rho_0 > 0$, and for a locally integrable $f\colon U \to \mathbb{R}^n$, the bounded mean oscillation semi-norm of f is defined by

$$
[[f]]_{\mathrm{BMO}(U,\rho_0)} = \sup_{\substack{z_0=(x_0,t_0)\in U\\ 0<\rho<\rho_0}} \frac{1}{|Q_\rho(z_0)\cap U|} \int_{Q_\rho(z_0)\cap U} |f(x,t) - \overline{f}_{Q_\rho(z_0)\cap U}|\, dxdt,
$$

where

$$
\overline{f}_{Q_\rho(z_0)\cap U} = \frac{1}{|Q_\rho(z_0)\cap U|} \int_{Q_\rho(z_0)\cap U} f(x,t)\, dxdt.
$$

For each $p > 0$ and $q \in (0, \infty]$, the Lorentz quasi-norm of f on U is defined by

$$
\|f\|_{L^{p,q}(U)} = \begin{cases} \Big\{ p \int_0^\infty s^q \Big| \big\{ (x,t) \in U : |f(x,t)| > s \big\} \Big|^{q/p} \frac{ds}{s} \Big\}^{1/q} & \text{if } q < \infty,\\ \sup_{s>0} s \Big| \big\{ (x,t) \in U : |f(x,t)| > s \big\} \Big|^{1/p} & \text{if } q = \infty. \end{cases} \tag{1.9}
$$

The set of all measurable functions f defined on U so that $\|f\|_{L^{p,q}(U)} < \infty$ is denoted by $L^{p,q}(U)$ and called Lorentz space with indices p and q. It is clear that $L^{p,p}(U) = L^p(U)$, the usual Lebesgue space. Moreover, $L^{p,q}(U) \subset L^{p,r}$ for all $p > 0$ and $0 < q < r \leq \infty$. When $q = \infty$, the space $L^{p,\infty}(U)$ is usually called "weak-$L^p(U)$" space or Lorentz–Marcinkiewicz space. See [24, Chapter 1.4], for example, for more details on Lorentz spaces.

Our first main result is the interior regularity estimates for the gradients of solutions of (1.1).

Theorem 1.1 *Let $\Lambda > 0, M > 0, p > 2, q \in (0,\infty]$, and $\alpha_0 \in (0,1]$. Then there exists a sufficiently small constant $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ such that the following*

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statement holds. For every $R > 0$, let $\mathbf{A}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map satisfying (1.2)–(1.4) on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some open interval $\mathbb{K} \subset \mathbb{R}$, and

$$(1.10)\quad [\mathbf{A}]_{\mathrm{BMO}(Q_R,R)} := \sup_{\substack{z_0=(x_0,t_0)\in Q_R,\\ 0<\rho\le R}} \frac{1}{|Q_\rho(z_0)|} \times \int_{Q_\rho(z_0)} \Big[\sup_{\substack{\xi\in\mathbb{R}^n\setminus\{0\},\\ s\in\mathbb{K}}} \frac{|\mathbf{A}(x,t,s,\xi) - \overline{\mathbf{A}}_{B_\rho(x_0)}(t,s,\xi)|}{|\xi|} \Big]\, dxdt \le \delta.$$

Then if $\mathbf{F} \in L^{p,q}(Q_{2R},\mathbb{R}^n)$, $f \in L^{n_* p, n_* q}(Q_{2R})$, and u is a weak solution in Q_{2R} of

$$u_t - \mathrm{div}[\mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}] = f(x,t)$$

with $[[\lambda u]]_{\mathrm{BMO}(Q_R,R)} \le M$ for some $\lambda \ge 0$, and $[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b} \in L^{p,q}(Q_{2R},\mathbb{R}^n)$ for some given divergence-free vector field $\mathbf{b}$ defined on Q_{2R}, there holds

$$\|\nabla u\|_{L^{p,q}(Q_R)} \le C\Big[\|\mathbf{F}\|_{L^{p,q}(Q_{2R})} + R|Q_{2R}|^{\frac{1}{p}-\frac{1}{pn_*}} \|f\|_{L^{n_* p, n_* q}(Q_{2R})} + \big\|[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b}\big\|_{L^{p,q}(Q_{2R})} + |Q_{2R}|^{\frac{1}{p}-\frac{1}{2}} \|\nabla u\|_{L^2(Q_{2R})}\Big],$$

where $n_* = \frac{n+2}{n+4}$, and C is a constant depending only on q, p, n, Λ, α_0, M, $\mathbb{K}$.

Local regularity estimates near the boundary are not only interesting by themselves, but are also important in many problems, because they only require local information on data. Our next result is the local regularity estimate on the boundary $\partial\Omega$ for weak solutions u of (1.1). In this theorem, for $z = (y,t) \in \Omega \times \mathbb{R}$, and $R > 0$, we write

$$\Omega_R(y) = \Omega \cap B_R(y), \quad K_R(z) = \Omega_R(y) \times \Gamma_R(t), \quad T_R(z_0) = (\partial\Omega \cap B_R(y)) \times \Gamma_R(t)$$

When $z = (0,0)$, we write

$$\Omega_R = \Omega_R(0), \quad K_R = K_R(0,0), \quad T_R = T_R(0,0).$$

For each $\widehat{x} \in \partial\Omega$, we assume that $\mathrm{div}[\mathbf{b}] = 0$ in $\Omega_{2R}(\widehat{x})$ and $\langle \mathbf{b}, \vec{\nu}\rangle = 0$ on $B_{2R}(\widehat{x}) \cap \Omega$ in the sense that

$$(1.11)\quad \int_{\Omega_{2R}(\widehat{x})} \langle \mathbf{b}(x,t), \nabla\varphi(x)\rangle dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_{2R}(\widehat{x})), \text{ for a.e. } t \in (0,T).$$

Theorem 1.2 *Let $M > 0, \Lambda > 0, p > 2, q \in (0,\infty]$, and $\alpha_0 \in (0,1]$. Then there exists a sufficiently small constant $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ such that the following statement holds true. Suppose that $0 \in \partial\Omega$ and for some $R > 0$, $\partial\Omega \cap B_{2R}$ is C^1, and suppose that $\mathbf{A}: K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1.2)–(1.4) on $K_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some open interval $\mathbb{K} \subset \mathbb{R}$, and*

$$(1.12)\quad [\mathbf{A}]_{\mathrm{BMO}(K_R,R)} := \sup_{\substack{z_0=(x_0,t_0)\in K_R,\\ 0<\rho\le R}} \frac{1}{|K_\rho(z_0)|} \times \int_{K_\rho(z_0)} \Big[\sup_{\substack{\xi\in\mathbb{R}^n\setminus\{0\},\\ s\in\mathbb{K}}} \frac{|\mathbf{A}(x,t,s,\xi) - \overline{\mathbf{A}}_{\Omega_\rho(x_0)}(t,s,\xi)|}{|\xi|} \Big]\, dxdt \le \delta.$$

Then for every $\mathbf{F} \in L^{p,q}(K_{2R}, \mathbb{R}^n)$, $f \in L^{n_* p, n_* q}(K_{2R})$, *if* u *is a weak solution of*

$$\begin{aligned} &u_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}] = f(x,t), \quad &&\text{in } K_{2R}, \\ &\langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu}\rangle = 0, &&\text{on } T_{2R}, \end{aligned} \tag{1.13}$$

satisfying $[[\lambda u]]_{\mathrm{BMO}(K_R,R)} \le M$, *and* $[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b} \in L^{p,q}(K_{2R}, \mathbb{R}^n)$ *with some* $\lambda \ge 0$ *and some given divergence-free vector field* $\mathbf{b}$ *defined on* K_{2R} *and satisfying* (1.11) *at* $\widehat{x} = 0$, *there holds*

$$\begin{aligned} \|\nabla u\|_{L^{p,q}(K_R)} \le C\Big[\ &\|\mathbf{F}\|_{L^{p,q}(K_{2R})} + R|K_{2R}|^{\frac{1}{p} - \frac{1}{pn_*}} \|f\|_{L^{n_* p, n_* q}(K_{2R})} \\ &+ \big\|[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b}\big\|_{L^{p,q}(K_{2R})} + |K_{2R}|^{\frac{1}{p}-\frac{1}{2}} \|\nabla u\|_{L^2(K_{2R})}\Big], \end{aligned} \tag{1.14}$$

where $n_* = \frac{n+2}{n+4}$, *and* C *is a constant depending only on* q, p, n, Λ, α_0, M, $\mathbb{K}$.

Theorems 1.1 and 1.3 are still valid if we replace $Q_\rho(z_0)$ by $\widehat{Q}_\rho(z_0) = B_\rho(x_0) \times (t_0 - r^2, t_0]$ and $K_\rho(z_0)$ by $\widehat{K}_\rho(z_0) = \Omega_\rho(x_0) \times (t_0 - \rho^2, t_0]$. As a consequence, the following global regularity estimates in Lorentz space for gradients of weak solutions of (1.8) can be obtained.

Theorem 1.3 *Let* $M > 0, \Lambda > 0, p > 2, q \in (0, \infty]$, *and* $\alpha_0 \in (0,1]$. *Then there exists a sufficiently small constant* $\delta = \delta(p,q,n,\Lambda,M,\alpha_0) > 0$ *such that the following statement holds true. Suppose that* $\partial\Omega$ *is* C^1, *and suppose that* $\mathbf{A}\colon \Omega_T \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory map satisfying* (1.2)–(1.4) *on* $\Omega_T \times \mathbb{R} \times \mathbb{R}^n$ *for some* $T > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$, *and*

$$\begin{aligned} \sup_{\substack{z_0 = (x_0,t_0) \in \Omega \times (\bar{t}, T),\\ 0<\rho\le r}} &\frac{1}{|\widehat{Q}_\rho(z_0) \cap (\Omega \times (\bar{t}, T))|} \\ &\times \int_{\widehat{Q}_\rho(z_0)\cap\Omega_T} \Big[\sup_{\substack{\xi \in \mathbb{R}^n \setminus \{0\},\\ s \in \mathbb{K}}} \frac{|\mathbf{A}(x,t,s,\xi) - \overline{\mathbf{A}}_{\Omega_\rho(x_0)}(t,s,\xi)|}{|\xi|}\Big]\, dxdt \le \delta, \end{aligned}$$

for some $r > 0, \bar{t} \in (0,T)$. *Then, for every* $\mathbf{F} \in L^{p,q}(\Omega_T, \mathbb{R}^n)$, $f \in L^{n_* p, n_* q}(\Omega_T)$, *if* u *is a weak solution of* (1.8) *satisfying* $[[\lambda u]]_{\mathrm{BMO}(\Omega_T, r)} \le M$ *and* $[[u]]_{\mathrm{BMO}(\Omega_T, r)}\mathbf{b} \in L^{p,q}(\Omega_T, \mathbb{R}^n)$ *with some* $\lambda \ge 0$ *and some given vector field* $\mathbf{b}$ *satisfying* (1.11) *at every* $\widehat{x} \in \partial\Omega$, *there holds*

$$\|\nabla u\|_{L^{p,q}(\Omega \times (\bar{t}, T))} \le C\Big[\ \|\mathbf{F}\|_{L^{p,q}(\Omega_T)} + \|f\|_{L^{n_* p, n_* q}(\Omega_T)} + \big\|[[u]]_{\mathrm{BMO}(\Omega_T, r)}\mathbf{b}\big\|_{L^{p,q}(\Omega_T)}\Big],$$

where $n_* = \frac{n+2}{n+4}$, *and* C *is a constant depending only on* q, p, n, Λ, α_0, M, $\mathbb{K}$, r, Ω, $\bar{t}$, T.

Several remarks are worth mentioning regarding Theorems 1.1–1.3. Firstly, we reinforce that the most important improvement in Theorems 1.1–1.3 is that they relax and do not require the solutions to be bounded as in the known work [5, 7, 27, 38, 39]. This is completely new even for the case $\mathbf{b} = 0$ and $f = 0$, in comparison to the known work that we already mentioned for both Schauder's regularity theory and Sobolev's theory regarding weak solutions of equations (1.1). To overcome the loss of boundedness from the assumption, instead of applying the maximum principle during the approximation process, we directly derive and carefully use some delicate analysis estimates and Hölder's regularity estimates for solutions of the corresponding homogeneous equations; see the estimates (3.5) and (3.16) for examples. These estimates are first observed in [41, 42] but for elliptic equations. In a related context, interested readers may see [15, 32] for further study of C^α-regularity of weak, BMO solutions. Secondly, we also note that due to the availability of f, which is scaled differently compared to $\mathbf{F}$ and ∇u, the approach based on the Hardy–Littlewood maximal function and harmonic analysis used in [8–10, 41, 42, 46] does not seem to produce our desired estimates here. Instead, we use the maximal-function free approach introduced in [1], and also used in [3, 5, 6]. This paper seems to be the first one that treats the equations (1.1) with inhomogeneous f in the Lorentz space setting. In addition, this paper also treats quasi-linear equations with inhomogeneous singular drifts $\mathbf{b}$, which has not been done before. As one will find in the proof, to deal with $\mathbf{b}$, we introduce the function $\mathbf{G}(x,t) \approx [[u]]_{\mathrm{BMO}}\mathbf{b}(x,t)$, which has the same scaling properties as $\mathbf{F}, \nabla u$. This key fact plays an essential role in the proof. Thirdly, we note that when $\lambda = 0$, $f = 0$, and $\mathbf{b} = 0$, Theorems 1.1–1.3 recover and sharpen results in [6, 8–10, 14, 17, 18, 27, 30, 34, 37, 43] when restricted to the class of equations (1.1) in which $\mathbf{A}$ is independent of $u \in \mathbb{K}$, see Remark 1.4 for more details on this. See also [16, 29, 40] for some other related work with more regular $f, \mathbf{F}$. This paper therefore not only unifies both $W^{1,q}$-theories for (1.1) and (1.6) but also extends the theory to the Lorentz regularity estimate setting. Lastly, observe that papers such as [6–8, 37], among others, regarding the $W^{1,q}$-regularity estimates in nonsmooth domains, only establish global regularity estimates. Our paper provides the regularity estimates locally for both the interior and the boundary. Our Theorem 1.1, Theorem 1.2 can be considered as giving some high regularity estimates of Caccioppoli type, which are important for many practical purposes for which local information is available and required. Certainly, our local regularity estimates imply the global ones, as Theorem 1.3 shows. However, it is generally impossible to derive local estimates directly from the global ones; see [6–9, 37].

Remark 1.4 Two important points are worth pointing out.

(i) This paper does not require any regularity assumption on the initial data u_0 in (1.1), compared to [6, 9, 37] in which it is assumed that $u_0 = 0$. Moreover, M is not required to be small. Note also that the condition $[[u]]_{\mathrm{BMO}}\mathbf{b} \in L^{p,q}$ is trivial if $\mathbf{b} = 0$. Similarly, the condition $[[\lambda u]]_{\mathrm{BMO}} \le M$ is always satisfied when $\lambda = 0$.

(ii) If $\lambda = 0, \mathbf{b} = 0$ and $f = 0$, known results for (1.6) such as [5, 6, 9, 37] provide the estimates of the form

$$\|\nabla u\|_{L^p} \le C\big[\ \|\mathbf{F}\|_{L^p} + 1\big]. \tag{1.15}$$

Our estimates in Theorems 1.1-1.3 are invariant under the scalings and dilations, and they do not contain the inhomogeneous constant, *i.e.,* the number 1 in the right-hand side of (1.15). Our results are natural and sharp.

We conclude this section by outlining the organization of this paper. Section 2 reviews some definitions and proves preliminaries needed in the paper. Perturbation arguments, and approximation estimates are given in Section 3. Section 4 establishes estimates of level sets of gradients of solutions. The proofs of Theorems 1.1–1.3 are given in Section 5. The paper concludes with Appendix A, which gives proofs for some reverse Hölder's inequalities needed in the paper.

2 Definitions and Preliminaries

2.1 Invariant Properties and Definitions of Weak Solutions

Let $\lambda' \geq 0$, and let $Q_{2R} \subset \mathbb{R}^{n+1}$ be the parabolic cylinder of radius $2R$. Let us consider a weak solution u of

$$u_t - \operatorname{div}[\mathbf{A}(x,t,\lambda' u, \nabla u) - \mathbf{b}u(x,t) - \mathbf{F}(x,t)] = f(x,t)$$

in Q_{2R}. Then it is simple to check that for some fixed $\lambda > 0$, the rescaled function

$$v(x,t) = \frac{u(x,t)}{\lambda} \quad \text{for } (x,t) \in Q_{2R},$$

is a weak solution in Q_{2R} of

$$v_t - \operatorname{div}\left[\widehat{\mathbf{A}}(x,\widehat{\lambda} v, \nabla v) - \mathbf{b}(x,t)v(x,t) - \widehat{\mathbf{F}}(x,t)\right] = \widehat{f}$$

for $\widehat{\lambda} = \lambda\lambda' \geq 0$, $\widehat{\mathbf{A}}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\widehat{\mathbf{A}}(x,t,s,\xi) = \frac{\mathbf{A}(x,t,s,\lambda\xi)}{\lambda}, \quad \widehat{f}(x,t) = \frac{f(x,t)}{\lambda}, \widehat{\mathbf{F}}(x,t) = \frac{\mathbf{F}(x,t)}{\lambda}, \quad (x,t) \in Q_{2R}. \tag{2.1}$$

Moreover, let $\widetilde{\lambda} = R\lambda'$,

$$\begin{gathered} \widetilde{v}(x,t) = \frac{u(Rx,R^2t)}{R}, \quad \widetilde{\mathbf{A}}(x,t,s,\xi) = \mathbf{A}(Rx,R^2t,s,\xi) \\ \widetilde{\mathbf{F}}(x,t) = \mathbf{F}(Rx,R^2t), \quad \widetilde{f}(x,t) = Rf(Rx,R^2t), \quad \widetilde{\mathbf{b}}(x,t) = R\mathbf{b}(Rx,R^2t) \end{gathered} \tag{2.2}$$

for $(x,t) \in Q_2$, $s \in \mathbb{K}$, and $\xi \in \mathbb{R}^n$. Then $\widetilde{v}$ is a weak solution in Q_2 of

$$\widetilde{v}_t - \operatorname{div}[\widetilde{\mathbf{A}}(x,t,\widetilde{\lambda}\widetilde{v}, \nabla\widetilde{v}) - \widetilde{\mathbf{b}}\widetilde{v} - \widetilde{\mathbf{F}}] = \widetilde{f}.$$

This is the main reason that we study the class of equation (1.8) with a parameter λ, instead of (1.1).

Remark 2.1 It is not too hard to see that if $\mathbf{A}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies conditions (1.2)–(1.4) on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n$, then the rescaled vector field $\widehat{\mathbf{A}}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.1) also satisfies the conditions (1.2)–(1.4) on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ with the same constants Λ, α_0. The same conclusion also holds for $\widetilde{\mathbf{A}}: Q_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (2.2). Moreover,

$$\begin{gathered} [\widehat{\mathbf{A}}]_{\mathrm{BMO}(Q_R,R)} = [\widetilde{\mathbf{A}}]_{\mathrm{BMO}(Q_1,1)} = [\mathbf{A}]_{\mathrm{BMO}(Q_R,R)}, \\ [[\widehat{\lambda} v]]_{\mathrm{BMO}(Q_R,R)} = [[\widetilde{\lambda}\widetilde{v}]]_{\mathrm{BMO}(Q_1,1)} = [[\lambda' u]]_{\mathrm{BMO}(Q_R,R)}. \end{gathered}$$

With respect to the scalings and dilations, the following remark follows directly from (1.9); see also [24, Remark 1.4.7].

Remark 2.2 For all $0 < p, r < \infty$ and for all $0 < q \le \infty$, if f is a measurable function defined on a measurable set $U \subset \mathbb{R}^{n+1}$, then

$$\| |f|^r \|_{L^{p,q}(U)} = \|f\|^r_{L^{rp,rq}(U)}.$$

Moreover, for a measurable function f defined on Q_R with some $R > 0$, then

$$\|\widetilde{f}\|_{L^{p,q}(Q_1)} = R^{-(n+2)/p} \|f\|_{L^{p,q}(Q_R)},$$

where

$$\widetilde{f}(x,t) = f(Rx, R^2 t), \quad (x,t) \in Q_1.$$

Let us now give the precise definition of weak solution that is used throughout the paper.

Definition 2.3 Let $\mathbb{K} \subset \mathbb{R}$ be an interval, let $\Lambda > 0$, $\alpha_0 \in (0,1]$ and $\alpha > 2$. Also, let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with boundary $\partial\Omega$, and let $\mathbf{A}: \Omega_T \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy conditions (1.2)–(1.4) on Ω_T. For each $\mathbf{F} \in L^2(\Omega_T; \mathbb{R}^n)$, $f \in L^{\frac{2(n+2)}{n+4}}(\Omega_T)$, and $\lambda \ge 0$, a function u is called a *weak solution* of

$$\begin{aligned} u_t - \operatorname{div}\Big[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}\Big] &= f(x,t), && \text{in } \Omega_T,\\ \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}, \vec{\nu}\rangle &= 0, && \text{on } \partial\Omega \times (0,T), \end{aligned}$$

if $\lambda u(x,t) \in \mathbb{K}$ for a.e. $(x,t) \in \Omega_T$, $u \in L^\infty((0,T), L^2(\Omega)) \cap L^2((0,T), W^{1,2}(\Omega))$, $[[u]]_{\mathrm{BMO}(\Omega_T)}\mathbf{b} \in L^\alpha_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$, and for all $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi(\cdot,0) = \varphi(\cdot,T) = 0$

$$-\int_{\Omega_T} u\varphi_t \,dxdt + \int_{\Omega_T} \Big\langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}, \nabla\varphi \Big\rangle \,dxdt = \int_{\Omega_T} f(x,t)\varphi(x,t)\,dxdt.$$

Here, $L^p(U,\mathbb{R}^n)$ for $1 \le p < \infty$ is the Lebesgue space consisting all measurable functions $f: U \to \mathbb{R}^n$ such that $|f|^p$ is integrable on U, and $W^{1,p}(U)$ is the standard Sobolev space on U. Moreover, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$.

Remark 2.4 When $\mathbf{b} \ne 0$, we require that the solution $u \in \mathrm{BMO}(\Omega_T)$ to insure that $\int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi\rangle dz$ is well defined for a singular vector field $\mathbf{b}$. Indeed, for $\mathbf{b} \in L^\alpha_{\mathrm{loc}}(\Omega_T)$ with some $\alpha > 2$, if $\varphi \in C^\infty_0(Q)$ for some cube $Q \subset \Omega_T$, since $\operatorname{div}[\mathbf{b}(\cdot,t)] = 0$, we can write

$$\int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi\rangle dz = \int_Q \langle \mathbf{b}(u - \overline{u}_Q), \nabla\varphi\rangle dz.$$

Then it follows from the Hölder's inequality that

$$\Big| \int_{\Omega_T} \langle \mathbf{b}u, \nabla\varphi\rangle dz \Big| \le \Big(\int_Q |\mathbf{b}|^\alpha dz \Big)^{1/\alpha} \Big(\int_Q |u - \overline{u}_Q|^{\alpha'} dz \Big)^{\alpha'} \Big(\int_Q |\nabla\varphi|^2 dz \Big)^{1/2} < \infty,$$

where α' is defined as

$$\frac{1}{\alpha} + \frac{1}{\alpha'} + \frac{1}{2} = 1.$$

2.2 Some Technical Lemmas

Several technical lemmas from analysis are needed in the paper. Our first is a standard iteration lemma that can be found, for example, in [25, Lemma 4.3] or [23, Lemma 6.1].

Lemma 2.5 *Let $\phi : [r,R]$ be a bounded, nonnegative function. Assume that for all $r < s < t \le R$,*

$$\phi(t) \le \theta\phi(s) + \frac{A}{(t-s)^\kappa} + B,$$

where $A, B \ge 0$, $\kappa > 0$ and $\theta \in (0,1)$. Then

$$\phi(r) \le C(\kappa,\theta)\Big[\frac{A}{(R-r)^\kappa} + B\Big].$$

Our next lemma is the classical Hardy inequality, which can be found, for example, in [26, Theorem 330], [3, Lemma 3.4], and [28].

Lemma 2.6 *Let $h: [0,\infty) \to [0,\infty)$ be a measurable function such that*

$$\int_0^\infty h(\lambda)d\lambda < \infty.$$

Then, for every $\kappa \ge 1$ and for every $r > 0$, there holds

$$\int_0^\infty \lambda^r \Big(\int_\lambda^\infty h(\mu)d\mu\Big)^\kappa \frac{d\lambda}{\lambda} \le \Big(\frac{\kappa}{r}\Big)^\kappa \int_0^\infty \lambda^r\big[\lambda h(\lambda)\big]^\kappa \frac{d\lambda}{\lambda}.$$

The following variant of the reverse-Hölder inequality can be found in [3, Lemma 3.5] and will be useful for this paper.

Lemma 2.7 *Let $h: [0,\infty) \to [0,\infty)$ be a nonincreasing, measurable function, and let $\kappa \in [1,\infty)$, $r > 0$. Then there is $C > 0$ such that*

$$\Big(\int_\lambda^\infty \big[t^r h(t)\big]^\kappa \frac{dt}{t}\Big)^{1/\kappa} \le \lambda^r h(\lambda) + C\int_\lambda^\infty t^r h(t)\frac{dt}{t}, \quad \text{for any } \lambda \ge 0.$$

2.3 Hölder Regularity of Weak Solutions of Homogeneous Equations

We recall some results on Hölder's regularity for weak solutions of homogeneous equations that will be needed in the paper. Those results are indeed consequences of the well-known, classical De Giorgi–Nash–Möser theory. Our first lemma is about the interior Hölder regularity estimate, whose proof, for example, can be found in [31, Theorems 1.1 (p. 419) and 2.1 (p. 425)], and also in [2, Theorems 2 and 4] and [45, Theorem 2.2].

Lemma 2.8 *Let $\Lambda > 0$, and let $\mathbb{A}_0: Q_r \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and satisfy (1.2) and (1.3) on Q_{2r} with some $r > 0$. If v is a weak solution of the equation*

$$v_t - \operatorname{div}[\mathbb{A}_0(x,t,\nabla v)] = 0, \quad \text{in } Q_r.$$

Then there exists $C_0 > 0$ depending only on Λ, n such that

$$\|v\|_{L^\infty(Q_{5r/6})} \le C_0 \Big(\fint_{Q_r} |v|^2 dz \Big)^{1/2}.$$

Moreover, there exists $\beta_0 \in (0,1)$ depending on Λ, n and $\|v\|_{L^\infty(Q_{5r/6})}$ such that

$$|v(z) - v(z')| \le C_0 \, \|v\|_{L^\infty(Q_{5r/6})} \Big[\frac{|x - x'| + |t - t'|^{1/2}}{r} \Big]^{\beta_0}$$

for all $z = (x,t), z' = (x',t') \in \overline{Q}_{2r/3}$.

To state the boundary regularity, we recall that for some domain $\Omega \subset \mathbb{R}^n$, and for each $r > 0$, $z_0 = (x_0, t_0) \in \partial\Omega \times \mathbb{R}$, we define

$$\Omega_r(x_0) = \Omega \cap B_r(x_0), \quad \Omega_r = \Omega_r(0), \quad K_r(z_0) = \Omega_r(x_0) \times \Gamma_r(t_0), \quad K_r = K_r(0,0).$$

Moreover, we also write

$$T_r(z_0) = (\partial\Omega \cap B_r(x_0)) \times \Gamma_r(t_0), \quad T_r = T_r(0,0).$$

The following classical boundary Hölder regularity result can be found in [31, Theorems 1.1 (p. 419) and 2.1 (p. 425)], and [45, Theorem 4.2].

Lemma 2.9 *Let $\Lambda > 0$ be fixed and let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\partial\Omega \in C^1$. Assume that $\mathbb{A}_0: K_r \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and satisfies (1.2) and (1.3) on $K_r \times \mathbb{R}^n$ for some $r > 0$. Assume also that $T_r \neq \emptyset$ and v is a weak solution of the equation*

$$\begin{aligned} v_t - \operatorname{div}[\mathbb{A}_0(x,t,\nabla v)] &= 0 \qquad \text{in } K_r, \\ \langle \mathbb{A}_0(x,t,\nabla v), \vec{v} \rangle &= 0 \qquad \text{on } T_r. \end{aligned}$$

Then there exists $C_0 > 0$ depending only on Λ, n such that

$$\|v\|_{L^\infty(K_{5r/6})} \le C_0 \Big(\fint_{K_r} |v|^2 dz \Big)^{1/2}.$$

Moreover, there exists a constant β_0 depending only on Λ, n and $\|v\|_{L^\infty(K_{5r/6})}$ such that $v \in C^{\beta_0}(\overline{K}_{5r/6})$, and

$$|v(z) - v(z')| \le C_0 \, \|v\|_{L^\infty(K_{5r/6})} \Big[\frac{|x - x'| + |t - t'|^{1/2}}{r} \Big]^{\beta_0}$$

for all $z = (x,t)$, $z' = (x',t') \in \overline{K}_{2r/3}$.

2.4 Self-improving Regularity Estimates of Meyers–Gehring's Type

We need to establish two higher regularity estimates of Meyers–Gehring's type; see [19–21, 35, 36, 44], for weak solutions of (1.1). To begin, let us introduce the following notation, which will be used frequently in the paper. For each function f defined on $U \subset \mathbb{R}^{n+1}$, we write

$$\mathcal{G}_U(f) = \Big(\int_U |f|^{2n_*} dz \Big)^{\frac{1-n_*}{2n_*}}, \quad \text{with } n_* = \frac{n+2}{n+4}.$$

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Our first lemma is the interior one.

Lemma 2.10 *Let $\Lambda > 0$. Then there exists $\epsilon_0 = \epsilon_0(\Lambda, n) > 2$ such that the following statement holds. Suppose that $\mathbf{A}: Q_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1.2) and (1.3) on Q_2. If u is a weak solution of the equation*

$$u_t - \operatorname{div}[\mathbf{A}(x, t, \lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}] = f(x, t), \quad \text{in } Q_2,$$

with some $\lambda \geq 0$, then for every $p \in [2, 2 + \epsilon_0]$ and $\gamma_0 > 0$, there exists a constant $C = C(\Lambda, p, n) > 0$ such that

$$\begin{aligned}
&\Big(\fint_{Q_r(z_0)} |\nabla u|^p dz \Big)^{1/p} \\
&\quad \leq C \Big[\Big(\fint_{Q_{2r}(z_0)} |\nabla u|^2 dx \Big)^{1/2} \Big(\fint_{Q_{2r}(z_0)} |\mathbf{F}|^p dx \Big)^{1/p} \\
&\qquad + \Big(\fint_{Q_{2r}(z_0)} |\mathbf{G}|^{p(1+\gamma_0)} dz \Big)^{\frac{1}{(1+\gamma_0)p}} + \mathcal{G}_{Q_2}(f) \Big(\fint_{Q_{2r}(z_0)} |f|^{n_* p} dx \Big)^{1/p} \Big],
\end{aligned}$$

where $z_0 = (x_0, t_0) \in Q_1$, $r \in (0, 1/2)$, $\mathbf{G}(x, t) = \widehat{C}_0(n, \gamma_0)[[u]]_{\mathrm{BMO}(Q_1, 1)}\mathbf{b}$ and $\widehat{C}_0(n, \gamma_0)$ is some definite constant.

The next lemma is a self-improving regularity estimate on the boundary.

Lemma 2.11 *For every $\Lambda > 0$, there exists $\epsilon_0 = \epsilon_0(\Lambda, n) > 2$ such that the following statement holds. Suppose that $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega \in C^1$. Suppose that $\mathbf{A}: K_2 \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1.2) and (1.3) on $K_2 \times \mathbb{K} \times \mathbb{R}^n$ and (1.11) holds on Ω_2 with $T_2 \neq \varnothing$. Suppose also that u is a weak solution of the equation*

$$\begin{aligned}
u_t - \operatorname{div}[\mathbf{A}(x, t, \lambda u, \nabla u) - \mathbf{b}u - \mathbf{F}] &= f(x, t), && \text{in } K_2, \\
\langle \mathbf{A}(x, t, u, \nabla u) - \mathbf{b}u - \mathbf{F}, \vec{v} \rangle &= 0, && \text{on } T_2,
\end{aligned}$$

with some $\lambda \geq 0$. Then, for every $p \in [2, 2 + \epsilon_0]$, and $\gamma_0 > 0$, there exists a constant $C = C(\Lambda, p, \gamma_0, n) > 0$ such that

$$\begin{aligned}
&\Big(\fint_{K_r(z_0)} |\nabla u|^p dz \Big)^{1/p} \\
&\quad \leq C \Big[\Big(\fint_{K_{2r}(z_0)} |\nabla u|^2 dz \Big)^{1/2} + \Big(\fint_{K_{2r}(z_0)} |\mathbf{G}|^{p(1+\gamma_0)} dz \Big)^{\frac{1}{p(1+\gamma_0)}} \\
&\qquad + \Big(\fint_{K_{2r}(z_0)} |\mathbf{F}(x, t)|^p dz \Big)^{1/p} + \mathcal{G}_{K_2}(f) \Big(\fint_{K_{2r}(z_0)} |f(x, t)|^{n_* p} dz \Big)^{1/p} \Big],
\end{aligned}$$

for every $z_0 = (z_0, t_0) \in T_1$, $r \in (0, 1/2)$, $\mathbf{G}(x, t) = \widehat{C}_0(n, \gamma_0)[[u]]_{\mathrm{BMO}(K_1, 1)}\mathbf{b}$, $n_* = \frac{n+2}{n+4}$, and where $\widehat{C}_0(n, \gamma_0)$ is some definite constant.

Remark 2.12 Two remarks on Lemmas 2.10 and 2.11 are in order.

(i) Observe that when $\mathbf{b} \in L^q(Q)$ and $u \in \mathrm{BMO}$, it does not follow that $u\mathbf{b} \in L^q(Q)$. Therefore, the above self-improving regularity estimates are new and could not be directly deduced from the known self-improving regularity estimates.

(ii) If $\mathbf{b} \in L^{\infty}(\mathrm{BMO}^{-1})$ and $\mathbf{F} = f = 0$, then a similar self-improving regularity estimate as in Lemma 2.10 for linear equations is established in [44].

From Remark 2.12, proofs of Lemmas 2.10 and 2.11 are needed. We follow the standard approach using Caccioppoli's estimates as in [20, 21]. Details will be given in Appendix A.

3 Approximation Estimates

3.1 Interior Approximation Estimates

In this section, let $\mathbf{A}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1.2)–(1.4) on $Q_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some $R > 0$. We also recall that $\partial_p Q_R$ is the parabolic boundary of Q_R. We study a weak solution u of the class of equations

$$u_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{b}(x,t)u - \mathbf{F}] = f(x,t), \quad \text{in } Q_{2R}, \tag{3.1}$$

with the parameter $\lambda \geq 0$. The following number is used frequently in the paper:

$$n_* = \frac{n+2}{n+4}. \tag{3.2}$$

In the sequel, for each $\alpha > 2$, let $\alpha' > 2$ be the number such that

$$\frac{1}{\alpha'} + \frac{1}{\alpha} = \frac{1}{2}, \quad \textit{i.e.,} \quad \alpha' = \frac{2\alpha}{\alpha - 2}.$$

Moreover, if u is a weak solution of (3.1), we define

$$\mathbf{G}(x,t) = \widehat{C}_0(n,\alpha)[[u]]_{\mathrm{BMO}(Q_R,R)}\mathbf{b}(x,t), \quad (x,t) \in Q_{2R},$$

with some definite constant $\widehat{C}_0(n,\alpha)$, In our first step, we freeze u in $\mathbf{A}$, and then approximate the solution u of (3.1) by a solution of the corresponding homogeneous equations with frozen u. See also [5, 7, 27, 38, 41, 42] for some similar approaches.

Lemma 3.1 *Let $\Lambda, \alpha > 2$ be fixed. Assume that $\mathbf{A}: Q_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.4), and assume that $\mathbf{F} \in L^2(Q_{2R}, \mathbb{R}^n)$, $f \in L^{2n_*}(Q_{2R})$, and $\mathbf{G} \in L^{\alpha}(Q_{2R})$. Assume also that $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ is a weak solution of (3.1) with some $\lambda \geq 0$. Then for each $z_0 = (x_0, t_0) \in Q_R$, $r \in (0, R)$,*

$$\begin{aligned} \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dx \leq C(\Lambda, n)\Big[\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2 \Big(\fint_{Q_r(z_0)} |f|^{2n_*} dz\Big)^{1/n_*} \\ + \Big(\fint_{Q_r(z_0)} |u - \overline{u}_{Q_r(z_0)}|^{\alpha'} dz\Big)^{\frac{2}{\alpha'}} \Big(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dz\Big)^{\frac{2}{\alpha}} \Big], \end{aligned} \tag{3.3}$$

where $v \in C\big(\Gamma_r(t_0), L^2(B_r(x_0))\big) \cap L^2\big(\Gamma_r(t_0), W^{1,2}(B_r(x_0))\big)$ is the weak solution of

$$\begin{cases} v_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u, \nabla v)] = 0, & \textit{in } Q_r(z_0), \\ v = u, & \textit{on } \partial_p Q_r(z_0). \end{cases} \tag{3.4}$$

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Moreover, it also holds that

(3.5)
$$\Big(\fint_{Q_r(z_0)} |v - \bar{u}_{Q_r(z_0)}|^2 dz \Big)^{1/2} \leq C(n,p)\Big[r\Big(\fint_{Q_r(z_0)} |\nabla v - \nabla u|^2 dx\Big)^{1/2} + \Big(\fint_{Q_r(z_0)} |u - \bar{u}_{Q_r(z_0)}|^2 dz\Big)^{1/2}\Big].$$

Proof Even though the proof is similar and simpler than that of Lemma 3.5 below, we give the proof for the sake of clarity and completeness. Observe that for a given weak solution $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ of (3.1), by taking $\mathbf{A}_0(x,t,\xi) := \mathbf{A}(x,t,\lambda u(x,t),\xi)$, we see that $\mathbf{A}_0 \colon Q_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ is independent of the variable $s \in \mathbb{K}$, and it satisfies all assumptions in (1.2) and (1.3). Therefore, the existence of a weak solution $v \in C(\Gamma_{2r}, L^2(B_{2r})) \cap L^2(\Gamma_{2r}, W^{1,2}(B_{2r}))$ of (3.4) can be obtained using the Galerkin's method [31, pp. 466–475]. It remains to prove the estimates (3.3) and (3.5). Through the procedure using the Skelov's average (see [5,13,38], for examples), we can formally use $v-u$ as a test function for equations (3.4) and (3.1). We obtain

(3.6)
$$\frac{1}{2}\frac{d}{dt}\int_{B_{r(x_0)}} |v-u|^2 dx + \int_{B_r(x_0)} \langle \mathbf{A}(x,t,\lambda u, \nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla v - \nabla u\rangle dx = \int_{B_r(x_0)} \langle \mathbf{b}u + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{B_r(x_0)} f(x,t)(v-u)dx.$$

Also, because $\operatorname{div}[b(\cdot,t)] = 0$, it follows that

$$\begin{aligned}\Big| \int_{B_r(x_0)} \big\langle \mathbf{b}(x,t)u(x,t), \nabla u - \nabla v\big\rangle\, dx\Big| &= \Big| \int_{B_r(x_0)} \big\langle \mathbf{b}(x,t)[u(x,t) - \bar{u}_{Q_r(z_0)}], \nabla u - \nabla v\big\rangle\, dx\Big| \\ &\leq \Big(\int_{B_r(x_0)} |\nabla u - \nabla v|^2 dx\Big)^{1/2} \Big(\int_{B_r(x_0)} |u - \bar{u}_{Q_r(z_0)}|^{\alpha'} dx\Big)^{1/\alpha'} \\ &\quad \times \Big(\int_{B_r(x_0)} |\mathbf{b}(x,t)|^{\alpha}\Big)^{1/\alpha}.\end{aligned}$$

Then it follows from an integration in time, Remark 2.1, (3.6), and Young's inequality that

$$\begin{aligned}\frac{1}{2}\sup_{\Gamma_r(t_0)} \int_{B_{r(x_0)}} |v-u|^2 dx + \int_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz &\leq C(\Lambda)\Big[\frac{1}{2}\sup_{\Gamma_r(t_0)}\int_{B_{r(x_0)}} |v-u|^2 dx \\ &\quad + \int_{Q_r(z_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u, \nabla v), \nabla v - \nabla u\rangle dz\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq C(\Lambda)\Big[\int_{Q_r(x_0)} |\langle \mathbf{b}u + \mathbf{F}, \nabla u - \nabla v\rangle| dz + \int_{Q_r(z_0)} |f||v-u| dz \Big] \\
&\leq \frac{1}{2}\int_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz + C(\Lambda)\Big\{ \int_{Q_r(z_0)} |\mathbf{F}|^2 dz + \int_{Q_r(z_0)} |f||v-u| dz \\
&\quad + \Big(\int_{Q_r(z_0)} |u - \overline{u}_{Q_r(z_0)}|^{\alpha'} dxdt \Big)^{2/\alpha'} \Big(\int_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt \Big)^{2/\alpha} \Big\}.
\end{aligned}
$$

Hence,

(3.7)

$$
\begin{aligned}
&\sup_{\Gamma_r(t_0)} r^{-2} \int_{B_r(x_0)} |v-u|^2 dx + \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\quad \leq C(\Lambda)\Big\{ \fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + 2\fint_{Q_r(z_0)} |f||v-u| dz \\
&\qquad + \Big(\fint_{Q_r(z_0)} |u - \overline{u}_{Q_r(z_0)}|^{\alpha'} dxdt \Big)^{2/\alpha'} \Big(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt \Big)^{2/\alpha} \Big\}.
\end{aligned}
$$

Now let $p_0 = 2n_* > 1$, where n_* is the number defined in (3.2). Also, let p_0' be the number such that $\frac{1}{p_0} + \frac{1}{p_0'} = 1$, *i.e.*, $p_0' = \frac{2(n+2)}{n}$. It follows from Hölder's inequality, the parabolic Sobolev imbedding (see [31, eqn (3.2), p. 74] or [13, Proposition 3.1, p. 7]), and Young's inequality that

$$
\begin{aligned}
(3.8)\quad &\fint_{Q_r(z_0)} |f||v-u| dz \\
&\leq \Big(\fint_{Q_r(z_0)} |v-u|^{p_0'} dz \Big)^{1/p_0'} \Big(\fint_{Q_r(z_0)} |f|^{p_0} dz \Big)^{1/p_0} \\
&\leq C(n) r \Big(\fint_{Q_r(z_0)} |\nabla v - \nabla u|^2 dz \Big)^{1/p_0'} \Big(\sup_{\Gamma_r(t_0)} r^{-2} \fint_{B_r(x_0)} |v-u|^2 dx \Big)^{\frac{2}{np_0'}} \\
&\quad \times \Big(\fint_{Q_r(z_0)} |f|^{p_0} dz \Big)^{1/p_0} \\
&\leq \frac{1}{4} \fint_{Q_r(z_0)} |\nabla v - \nabla u)|^2 dz + \frac{1}{4} \sup_{\Gamma_r(t_0)} r^{-2} \fint_{B_r(x_0)} |v-u|^2 dx \\
&\quad + C(n) r^2 \Big(\fint_{Q_r(z_0)} |f|^{p_0} dz \Big)^{2/p_0}.
\end{aligned}
$$

The estimates (3.8) and (3.7) imply that

$$
\begin{aligned}
&\sup_{\Gamma_r(t_0)} r^{-2} \int_{B_r(x_0)} |v-u|^2 dx + \fint_{Q_r(z_0)} |\nabla u - \nabla v|^2 dz \\
&\quad \leq C_0(\Lambda, n)\Big[\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2 \Big(\fint_{Q_r(z_0)} |f|^{p_0} dz \Big)^{2/p_0} \\
&\qquad + \Big(\fint_{Q_r(z_0)} |u - \overline{u}_{Q_r(z_0)}|^{\alpha'} dxdt \Big)^{\frac{2}{\alpha'}} \Big(\fint_{Q_r(z_0)} |\mathbf{b}(x,t)|^{\alpha} dxdt \Big)^{\frac{2}{\alpha}} \Big].
\end{aligned}
$$

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This proves estimate (3.3). Also, by the Poincaré's inequality, we see that

$$\Big(\fint_{Q_r(z_0)} |v-\overline{u}_{Q_r(z_0)}|^2 dz\Big)^{1/2}$$
$$\le \Big[\Big(\fint_{Q_r(z_0)} |v-u|^2 dz\Big)^{1/2} + \Big(\fint_{Q_r(z_0)} |u-\overline{u}_{Q_r(z_0)}|^2 dz\Big)^{1/2}\Big]$$
$$\le \Big[C(n,p)r\Big(\fint_{Q_r(z_0)} |\nabla v-\nabla u|^2 dz\Big)^{1/2} + \Big(\fint_{Q_r(x_0)} |u-\overline{u}_{Q_r(z_0)}|^2 dz\Big)^{1/2}\Big].$$

This proves (3.5) and completes the proof of the lemma. ∎

The next step is to approximate the solution u in $Q_{\kappa r}(z_0)$ by the solution w of

$$(3.9)\qquad \begin{aligned} w_t - \operatorname{div}[\mathbf{A}(x,t,\lambda\overline{u}_{Q_{\kappa r}(z_0)},\nabla w)] &= 0, \quad \text{in} \quad Q_{\kappa r}(z_0),\\ w &= v, \quad \text{on} \quad \partial_p Q_{\kappa r}(z_0),\end{aligned}$$

where in (3.9) v is a weak solution of (3.4), and $\kappa \in (0,1/3)$ is some sufficiently small constant that will be determined.

Lemma 3.2 *Let $\Lambda, M > 0, \alpha > 2$, and $\alpha_0 \in (0,1]$ be fixed, and let $\epsilon \in (0,1)$. There exist positive, sufficiently small numbers*

$$\overline{\kappa} = \overline{\kappa}(\Lambda,M,n,\alpha_0,\epsilon) \quad \textit{and} \quad \delta_1 = \delta_1(\epsilon,\Lambda,M,n,\alpha_0) \textit{ in } (0,\epsilon)$$

such that the following holds. Assume that $\mathbf{A}\colon Q_{2R}\times\mathbb{K}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.4), and assume that $\mathbf{F} \in L^2(Q_{2R},\mathbb{R}^n), \mathbf{G} \in L^\alpha(Q_{2R},\mathbb{R}^n)$, $f \in L^{2n_*}(Q_{2R})$, and

$$\fint_{Q_r(z_0)} |\mathbf{F}|^2 dz + r^2\Big(\fint_{Q_r(z_0)} |f|^{2n_*} dz\Big)^{1/n_*} + \Big(\fint_{Q_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\Big)^{\frac{2}{\alpha}} \le \delta_1^2$$

for some $z_0 = (x_0,t_0) \in Q_R$ and some $r \in (0,R)$. Then, for every $\lambda > 0$, if $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^p(\Gamma_{2R}, W^{1,2}(B_{2R}))$ is a weak solution of (3.1) satisfying

$$\fint_{Q_{2\overline{\kappa}r}(z_0)} |\nabla u|^2 dz \le 1 \quad \textit{and} \quad [[\lambda u]]_{\mathrm{BMO}(Q_R,R)} \le M,$$

it holds that

$$\fint_{Q_{\overline{\kappa}r}(z_0)} |\nabla v - \nabla w|^2 dz \le \epsilon^2,$$

where w is the weak solution of (3.9).

Proof The proof is similar that of Lemma 3.6 in the next subsection, but using Lemma 2.8 and (3.5) instead of Lemma 2.9 and (3.16), respectively. We therefore skip the proof. ∎

The next lemma is a general result that compares the solution w of (3.9) with a solution of the corresponding constant coefficient equation.

Lemma 3.3 *Let $\Lambda > 0$ be fixed; then there is some $\gamma = \gamma(\Lambda,n) > 2$ such that the following statement holds. For some $z_0 = (x_0,t_0) \in Q_R$, assume that $\mathbf{A}_0\colon Q_{2R}\times\mathbb{R}^n \to \mathbb{R}^n$ such that (1.2) and (1.3) hold for some $R > 0$. Assume also that for some $\rho \in (0,R/2)$,*

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w is a weak solution of $w_t - \operatorname{div}[\mathbf{A}_0(x,t,\nabla w)] = 0$ in $Q_{2\rho}(z_0)$. Then there is some function

$$h \in L^\infty\big((\Gamma_{7\rho/4}(t_0), L^2(B_{7\rho/4}(x_0)\big) \cap L^2\big(\Gamma_{7\rho/4}(t_0), W^{1,2}(B_{7\rho/4}(x_0))\big)$$

such that

$$\Big(\frac{1}{|Q_{7\rho/4}(z_0)|}\int_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\Big)^{1/2} \le C(\Lambda,n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_{2R},R)}\Big(\frac{1}{|Q_{2\rho}(z_0)|}\int_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\Big)^{1/2}.$$

Moreover,

$$\|\nabla h\|_{L^\infty(Q_{3\rho/2}(z_0))} \le C(\Lambda,n)\Big(\frac{1}{|Q_{2\rho}(z_0)|}\int_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\Big)^{1/2}.$$

Proof The proof is simple, and we give it here for the sake of completeness. Observe that from Lemma 2.11, there is $p_1 = p_1(\Lambda,n) > 2$ such that

$$\Big(\fint_{Q_{7\rho/4}} |\nabla w|^{p_1} dz\Big)^{1/p_1} \le C(\Lambda,p,n)\Big(\fint_{Q_{2\rho}} |\nabla w|^2 dz\Big)^{1/2}. \tag{3.10}$$

Let us denote

$$\mathbf{a}(t,\xi) = \fint_{B_{7\rho/4}(x_0)} \mathbf{A}_0(x,t,\xi)dx,$$
$$\Theta_{\mathbf{A}_0,B_{7\rho/4}(x_0)} = \frac{|\mathbf{A}_0(x,t,\xi) - \mathbf{a}(t,\xi)|}{|\xi|}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then let h be a weak solution of

$$\begin{aligned} h_t - \operatorname{div}[\mathbf{a}(t,\nabla v)] &= 0, && \text{in } Q_{7\rho/4}(z_0), \\ h &= w, && \text{on } \partial_p Q_{7\rho/4}(z_0). \end{aligned} \tag{3.11}$$

Observe that the existence of h can be obtained by a standard method using Galerkin's approximation. Also, from Skelov's average as in [5,13], we can formally use $w - h$ as a test function for both the equations of w and of h to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{B_{7\rho/4}(x_0)} |w-h|^2 dx + \int_{B_{7\rho/4}(x_0)} \langle \mathbf{a}(t,\nabla w) - \mathbf{a}(t,\nabla h), \nabla w - \nabla h\rangle dx = \int_{B_{7\rho/4}(x_0)} \langle \mathbf{a}(t,\nabla w) - \mathbf{A}_0(x,t,\nabla h), \nabla w - \nabla h\rangle dx.$$

This and (1.2) imply that

$$\begin{aligned} &\frac{1}{2}\sup_{t\in\Gamma_{7\rho/4}(t_0)} \int_{B_{7\rho/4}(x_0)} |w-h|^2 dx + \int_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz \\ &\le C(\Lambda)\int_{Q_{7\rho/4}(z_0)} |\mathbf{a}(t,\nabla w) - \mathbf{A}_0(x,t,\nabla w)||\nabla w - \nabla h| dz \\ &\le C(\Lambda)\int_{Q_{7\rho/4}(z_0)} \Theta_{\mathbf{A}_0,B_{7\rho/4}(x_0)}(x,t)|\nabla w||\nabla w - \nabla h| dz. \end{aligned}$$

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Now let $\gamma > 2$ satisfy

$$\frac{1}{\gamma} + \frac{1}{p_1} = \frac{1}{2}.$$

Then, by using Hölder's inequality and (3.10), we see that

$$\begin{aligned}
&\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz \\
&\quad \le C(\Lambda)\Big(\fint_{Q_{7\rho/4}(z_0)} |\Theta_{\mathbf{A}_0,\Omega_{7\rho/4}(x_0)}(x,t)|^\gamma dz\Big)^{1/\gamma} \Big(\fint_{Q_{7\rho/4}(z_0)} |\nabla w|^p dz\Big)^{1/p} \\
&\qquad\quad \times \Big(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\Big)^{1/2} \\
&\quad \le C(\Lambda, n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_R,R)}\Big(\fint_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\Big)^{1/2}\Big(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\Big)^{1/2}.
\end{aligned}$$

Hence,

$$\Big(\fint_{Q_{7\rho/4}(z_0)} |\nabla w - \nabla h|^2 dz\Big)^{1/2} \le C(\Lambda, n)[\mathbf{A}_0]^{2/\gamma}_{\mathrm{BMO}(Q_R,R)}\Big(\fint_{Q_{2\rho}(z_0)} |\nabla w|^2 dz\Big)^{1/2},$$

and this proves the first assertion of the lemma. To prove the last estimate of Lemma 3.3, we can use standard regularity theory for equation (3.11) to obtain

$$\|\nabla h\|_{L^\infty(Q_{3\rho/2}(z_0))} \le C(\Lambda, n)\Big(\fint_{Q_{7\rho/4}(z_0)} |\nabla h|^2 dz\Big)^{1/2}.$$

This, together with the fact that $[\mathbf{A}_0]_{\mathrm{BMO}(Q_R,R)} \le C(\Lambda, n)$, the triangle inequality, and (3.28) imply (3.11). The proof of the lemma is complete. ■

Our next result is the main result of the section.

Proposition 3.4 *Let $\Lambda > 0, \alpha > 2$ and $\alpha_0 \in (0,1]$ be fixed. Then, for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\overline{\kappa} = \overline{\kappa}(\Lambda, M, n, \alpha_0, \epsilon) \in (0,1/3)$ and $\delta_0 = \delta_0(\epsilon, \Lambda, M, n, \alpha_0) \in (0,\epsilon)$ such that the following holds. Assume that $\mathbf{A}\colon Q_{2R}\times\mathbb{K}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.4) and (1.10) holds with δ replaced by δ_0. Assume also that*

$$\fint_{Q_{2r}(z_0)} |\mathbf{F}|^2 dz + r^2\Big(\fint_{Q_{2r}(z_0)} |f|^{2n_*} dz\Big)^{1/n_*} + \Big(\fint_{Q_{2r}(z_0)} |\mathbf{G}(x,t)|^\alpha dz\Big)^{\frac{2}{\alpha}} \le \delta_0^2$$

for some $z_0 = (x_0, t_0) \in \overline{Q}_R$ and some $r \in (0, R/2)$. Then for every $\lambda \ge 0$, if $u \in C(\Gamma_{2R}, L^2(B_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(B_{2R}))$ is a weak solution of (3.1) satisfying

$$\fint_{Q_{4\overline{\kappa}r}(z_0)} |\nabla u|^2 dz \le 1 \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(Q_R,R)} \le M,$$

then there is $h \in C(\Gamma_{7\overline{\kappa}r/2}(t_0), L^2(B_{7\overline{\kappa}r/2}(x_0))) \cap L^2(\Gamma_{7\overline{\kappa}r/2}(t_0), W^{1,2}(B_{7\overline{\kappa}r/2}(x_0)))$ such that the following estimate holds:

$$\frac{1}{|Q_{7\overline{\kappa}r/2}(z_0)|}\int_{Q_{7\overline{\kappa}r/2}(z_0)} |\nabla u - \nabla h|^2 dz \le \epsilon^2, \quad \|\nabla h\|_{L^\infty(Q_{3\overline{\kappa}r}(z_0))} \le C(\Lambda, n).$$

Proof The proposition follows directly by applying Lemma 3.2 with r replaced by $2r$, Lemma 3.3 with $\mathbf{A}_0(x,t,\xi) = \mathbf{A}(x,t,\lambda\overline{u}_{Q_{4\overline{\kappa}r}(z_0)},\xi)$ and $\rho = 2\overline{\kappa}r$, and the triangle inequality. ∎

3.2 Boundary Approximation Estimates

To be convenient for the readers and self-contained, we recall some frequently used notation. For each $R > 0$, we write $B_R = B_R(0)$ for the ball in $\mathbb{R}^n$ centered at the origin with radius R. Moreover, for an open set $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$, we write

$$\Omega_R = B_R \cap \Omega, \quad K_R = \Omega_R \times \Gamma_R, \quad \text{and} \quad T_R = (\partial\Omega \cap B_R) \times \Gamma_R.$$

We also denote by $\partial_p K_R$ the parabolic boundary of K_R, moreover, for each $z_0 = (x_0, t_0)$, we define

$$\Omega_R(x_0) = x_0 + \Omega_R, K_R(z_0) = z_0 + K_R, \quad T_R(z_0) = z_0 + T_R.$$

We can assume that $T_{2R} \neq \varnothing$, and we will investigate weak solutions u of the equation

$$\begin{aligned} &u_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t)] = f(x,t), && \text{in } K_{2R}, \\ &\langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}(x,t)u - \mathbf{F}(x,t), \vec{\nu}\rangle = 0, && \text{on } T_{2R}. \end{aligned} \tag{3.12}$$

with the parameter $\lambda \geq 0$. By a weak solution of (3.12) we mean any

$$u \in C(\Gamma_{2R}, L^2(\Omega_{2R})) \cap L^2(\Gamma_{2R}, W^{1,2}(\Omega_{2R}))$$

such that $\mathbf{G} \in L^\alpha(K_{2R})$ for some $\alpha > 2$, $\lambda u(x,t) \in \mathbb{K}$ for a.e. $(x,t) \in K_{2R}$, and

$$-\int_{K_{2R}} u(x,t)\partial_t\varphi(x,t)dz + \int_{K_{2R}} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{b}u - \mathbf{F}, \nabla\varphi\rangle dz = \int_{K_{2R}} f(x,t)\varphi(x,t)dz$$

for all $\varphi \in C_0^\infty(\overline{Q}_{2R})$. Here,

$$\mathbf{G}(x,t) = \widehat{C}_0(n,\alpha)[[u]]_{\mathrm{BMO}(K_R,R)}\mathbf{b}(x,t), \quad z = (x,t) \in K_{2R},$$

for some definite constant $\widehat{C}_0(n,\alpha)$. As before, for each $\alpha > 2$, let α' be the number satisfying

$$\frac{1}{\alpha'} + \frac{1}{\alpha} = \frac{1}{2}, \quad \textit{i.e.}, \quad \alpha' = \frac{2\alpha}{\alpha - 2}. \tag{3.13}$$

Recall that the number n_* is defined in (3.2). Our first step is to approximate u by the solution v of the homogeneous equation with frozen u in $\mathbf{A}$.

Lemma 3.5 *Let $\Lambda, \alpha > 2$ be fixed. Assume that $\mathbf{A}\colon K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.4). Assume that $\mathbf{G} \in L^\alpha(K_R)$, and u is a weak solution of (3.12) for some $\lambda \geq 0$. Then for each $z_0 = (x_0, t_0) \in K_R$ and each $r \in (0, R)$, it holds that*

$$\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \leq C_0(\Lambda,n)\Big[\fint_{K_r(z_0)} |\mathbf{F}|^2 dz + \Big(\fint_{K_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\Big)^{\frac{2}{\alpha}} + r^2\Big(\fint_{K_r(z_0)} |f|^{2n_*} dz\Big)^{1/n_*}\Big], \tag{3.14}$$

where $v \in C(\Gamma_r(t_0), L^2(\Omega_r(x_0))) \cap L^2(\Gamma_r(t_0), W^{1,2}(\Omega_r(x_0)))$ *is the weak solution of*

$$
(3.15)\qquad
\begin{aligned}
v_t - \operatorname{div}[\mathbf{A}(x,t,\lambda u,\nabla v)] &= 0, && \text{in } K_r(z_0),\\
v &= u, && \text{on } \partial_p K_r(z_0) \smallsetminus T_r(z_0),\\
\langle \mathbf{A}(x,t,\lambda u,\nabla v), \vec{v}\rangle &= 0, && \text{on } T_r(z_0), \text{ if } T_r(z_0) \neq \varnothing.
\end{aligned}
$$

Moreover,

$$
(3.16)\quad \Big(\fint_{K_r(z_0)} |v - \overline{u}_{K_r(z_0)}|^2 dz \Big)^{1/2} \leq
C(n)\Big[r\Big(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\Big)^{1/2} + \Big(\fint_{K_r(z_0)} |u - \overline{u}_{K_r(z_0)}|^2 dz\Big)^{1/2}\Big].
$$

Proof If $T_r(z_0) = \varnothing$, this lemma follows directly from Lemma 3.1. Therefore, we only consider the case where $T_r(z_0) \neq \varnothing$. The proof is similar to that of Lemma 3.1 with some modification dealing with the boundary. Observe that since $\partial\Omega \in C^1$, $\Omega_r(x_0)$ is a Lipschitz domain. Therefore, $W^{1,2}(\Omega_r(x_0))$ is well defined with all imbedding and compact imbedding properties. Therefore, the existence of the solution v of (3.16) can be obtained by Galerkin's method (see [31, pp. 466–475]). It then remains to prove the estimates (3.14) and (3.16). By proceeding with Steklov's average (see [5, 13, 38]), we can formally use $v - u$ as a test function for the equations (3.15) and (3.12) to infer that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega_r(x_0)} |v-u|^2 dx + \int_{\Omega_r(x_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\
= \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)u + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{\Omega_r(x_0)} f(v-u) dx.
\end{aligned}
$$

Due to the fact that $\operatorname{div}\mathbf{b} = 0$ on Ω_{2R} and $\langle \mathbf{b}, \vec{v}\rangle = 0$ on $B_{2R} \cap \partial\Omega$ in the sense that

$$
\int_{\Omega_r(x_0)} \mathbf{b}(x,t)\cdot\nabla\varphi(x) dx = 0 \quad \text{for all } \varphi \in C_0^\infty(B_r(x_0)), \text{ for a.e. } t \in \Gamma_R,
$$

we see that

$$
\int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)u, \nabla u - \nabla v\rangle dx = \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)[u - \overline{u}_{K_r(z_0)}], \nabla u - \nabla v\rangle dx.
$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega_r(x_0)} |v-u|^2 dx + \int_{\Omega_r(x_0)} \langle \mathbf{A}(x,t,\lambda u,\nabla u) - \mathbf{A}(x,t,\lambda u,\nabla v), \nabla u - \nabla v\rangle dx \\
= \int_{\Omega_r(x_0)} \langle \mathbf{b}(x,t)[u - \overline{u}_{K_r(z_0)}] + \mathbf{F}, \nabla u - \nabla v\rangle dx + \int_{\Omega_r(x_0)} f(v-u) dx.
\end{aligned}
$$

From this and conditions (1.2)–(1.4), we infer that

$$
\begin{aligned}
&\frac{1}{2}\sup_{t\in\Gamma_r(t_0)}\int_{\Omega_r(x_0)}|v-u|^2dx+\int_{K_r(z_0)}|\nabla u-\nabla v|^2dz\\
&\quad\le\frac{1}{2}\sup_{t\in\Gamma_r(t_0)}\int_{\Omega(x_0)}|v-u|^2dx\\
&\qquad+C(\Lambda)\Big[\int_{K_r(z_0)}\langle\mathbf{A}(x,t,\lambda u,\nabla u)-\mathbf{A}(x,t,\lambda u,\nabla v),\nabla u-\nabla v\rangle dz\Big]\\
&\quad\le C(\Lambda)\Big[\int_{K_r(z_0)}\Big(|\mathbf{F}|+|\mathbf{b}||u-\overline{u}_{K_r(z_0)}|\Big)|\nabla u-\nabla v|+\int_{K_r(z_0)}|f||v-u|dz\Big]\\
&\quad\le\frac{1}{2}\int_{K_r(z_0)}|\nabla u-\nabla v|^2dz\\
&\qquad+C(\Lambda)\Big[\int_{K_r(z_0)}\Big(|\mathbf{F}|^2+|\mathbf{b}|^2|u-\overline{u}_{K_r(z_0)}|^2\Big)dz+\int_{K_r(z_0)}|f||v-u|dz\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sup_{t\in\Gamma_r(t_0)}r^{-2}\fint_{\Omega_r(x_0)}|v-u|^2dx+\fint_{K_r(z_0)}|\nabla u-\nabla v|^2dz\le\\
&\qquad C(\Lambda,n)\Big[\fint_{K_r(z_0)}\Big(|\mathbf{F}|^2+|\mathbf{b}|^2|u-\overline{u}_{K_r(z_0)}|^2\Big)dz+\fint_{K_r(z_0)}|f||v-u|dz\Big].
\end{aligned}
$$

Now we control the last two terms in the right-hand side of the previous estimate. Observe that from (3.13), the John–Nirenberg theorem, and Hölder's inequality, it follows that

$$
\begin{aligned}
&\fint_{K_r(z_0)}|\mathbf{b}|^2|u-\overline{u}_{K_r(z_0)}|^2\Big)dz\\
&\quad\le C(n)\Big(\fint_{K_r(z_0)}|\mathbf{b}|^\alpha dz\Big)^{2/\alpha}\Big(\frac{1}{|Q_r(z_0)|}\int_{K_r(z_0)}|u-\overline{u}_{K_r(z_0)}|^{\alpha'}dz\Big)^{2/\alpha'}\\
&\quad\le\widehat{C}_0(n,\alpha)[[u]]^2_{\mathrm{BMO}(K_R,R)}\Big(\fint_{K_r(z_0)}|\mathbf{b}|^\alpha dz\Big)^{2/\alpha}=\Big(\fint_{K_r(z_0)}|\mathbf{G}(x,t)|^\alpha dz\Big)^{2/\alpha}.
\end{aligned}
$$

On the other hand, as in (3.8), we define $p_0=2n_*$ and p_0' such that $1/p_0+1/p_0'=1$, *i.e.*, $p_0'=\frac{2n}{n+2}$. From Hölder's inequality, the parabolic Sobolev imbedding (see [31, eqn. (3.2), p. 74] or [13, Proposition 3.1, p. 7]), and Young's inequality, it follows that

$$
\begin{aligned}
(3.17)\qquad&\fint_{K_r(z_0)}|f|\,|v-u|\,dz\\
&\quad\le\Big(\fint_{K_r(z_0)}|v-u|^{p_0'}dz\Big)^{1/p_0'}\Big(\fint_{K_r(z_0)}|f|^{p_0}dz\Big)^{1/p_0}\\
&\quad\le\frac{1}{2}\fint_{K_r(z_0)}|\nabla v-\nabla u|^2dz+\frac{1}{4}\sup_{\Gamma_r(t_0)}r^{-2}\fint_{B_r(x_0)}|v-u|^2dx\\
&\qquad+C(n)r^2\Big(\fint_{K_r(z_0)}|f|^{p_0}dz\Big)^{2/p_0}.
\end{aligned}
$$

Therefore,

$$\sup_{t\in\Gamma_r(t_0)} r^{-2} \fint_{\Omega_r(x_0)} |v-u|^2 dx + \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \leq$$
$$C(\Lambda,n)\Big[\fint_{K_r(z_0)} |\mathbf{F}|^2 + \Big(\fint_{K_r(z_0)} |\mathbf{G}(x,t)|^\alpha dz\Big)^{2/\alpha}$$
$$+ r^2 \Big(\fint_{K_r(z_0)} |f(x,t)|^{p_0} dz\Big)^{2/p_0}\Big],$$

and (3.14) follows. Lastly, we prove (3.16). Observe that the triangle inequality gives

$$\fint_{K_r(x_0)} |v - \overline{u}_{K_r(z_0)}|^2 dz \leq C\Big[\fint_{K_r(z_0)} |v-u|^2 dz + \fint_{K_r(z_0)} |u - \overline{u}_{K_r(z_0)}|^2 dx\Big].$$

Then, using the Poincaré's inequality for the first term in the right-hand side of the above inequality, we see that

$$\Big(\fint_{K_r(z_0)} |v - \overline{u}_{K_r(z_0)}|^2 dz\Big)^{1/2} \leq$$
$$C(n)\Big[r\Big(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\Big)^{1/2} + \Big(\fint_{K_r(z_0)} |u - \overline{u}_{K_r(z_0)}|^2 dz\Big)^{1/2}\Big].$$

This proves (3.16) and also completes the proof. ∎

Lemma 3.6 *Let $\Lambda, M > 0, \alpha > 2$ and $\alpha_0 \in (0,1)$ be fixed. Then, for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon) \in (0,1/3)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha_0) \in (0,\epsilon)$ such that the following holds. Assume that $\mathbf{A}\colon K_{2R}\times\mathbb{K}\times\mathbb{R}^n \to \mathbb{R}^n$ such that (1.2)–(1.4) hold with some $R > 0$ and some open set $\mathbb{K} \subset \mathbb{R}$, and assume that*

$$\fint_{K_r(z_0)} |\mathbf{F}|^2 dz + \Big(\fint_{K_r(z_0)} |\mathbf{G}|^\alpha dz\Big)^{\frac{2}{\alpha}} + r^2\Big(\fint_{K_r(z_0)} |f|^{2n_*} dz\Big)^{1/n_*} \leq \delta_2^2, \tag{3.18}$$

for some $z_0 = (x_0, t_0) \in K_R$ and some $r \in (0,R)$. Then, for every $\lambda \geq 0$, if u is a weak solution of (3.12) satisfying

$$\fint_{K_{2\kappa r}(z_0)} |\nabla u|^2 dx \leq 1 \quad \textit{and} \quad [[\lambda u]]_{\mathrm{BMO}(K_R,R)} \leq M,$$

then there is a weak solution w of

$$\begin{aligned} w_t - \operatorname{div}[\mathbf{A}(x,t,\lambda\overline{u}_{K_{\kappa r}(z_0)}, \nabla w)] &= 0, \quad \text{in } K_{\kappa r}(z_0),\\ w &= v, \quad \text{on } \partial_p K_{\kappa r}(z_0) \smallsetminus T_{\kappa r}(z_0),\\ \langle \mathbf{A}(x,t,\lambda\overline{u}_{K_{\kappa r}(z_0)}, \nabla w), \vec{v}\rangle &= 0, \quad \text{on } T_{\kappa r}(z_0),\ \text{if } T_{\kappa r}(z_0) \neq \varnothing \end{aligned} \tag{3.19}$$

such that the following estimate holds

$$\Big(\fint_{K_{\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz\Big)^{1/2} \leq \epsilon \quad \textit{and} \quad \Big(\fint_{K_{\kappa r}(z_0)} |\nabla w|^2 dz\Big)^{1/2} \leq 1 + 2^{\frac{n+2}{2}}, \tag{3.20}$$

where in (3.19) the function v is defined as in Lemma 3.5.

Proof For a given sufficiently small $\epsilon > 0$, let $\epsilon' \in (0, \epsilon/2)$ and $\kappa \in (0, 1/3)$, both sufficiently small depending on $\epsilon, \Lambda, M, n, \alpha_0$, which will be determined. Now, by Lemma 3.5 with ϵ', we can find $\delta_2 = \delta_2(\epsilon', \Lambda, \kappa) > 0$ sufficiently small such that if (3.18) holds, then

$$(3.21)\qquad \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz \le (\epsilon')^2 \kappa^{n+2},$$
$$\lambda\Big(\fint_{K_r(z_0)} |v - \overline{u}_{K_r(z_0)}|^2 dz\Big)^{1/2} \le C(n,p)\big[r\epsilon' \kappa^{\frac{n+2}{2}}\lambda + M\big],$$

where v is the solution of (3.15). Observe that the first inequality in (3.21) and the fact that $\epsilon', \kappa \in (0,1)$ imply

$$(3.22)\qquad \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\Big)^{1/2} \le \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla u - \nabla v|^2 dz\Big)^{1/2} + \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla u|^2 dz\Big)^{1/2}$$
$$\le \Big(\frac{1}{(2\kappa)^{n+2}} \fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\Big)^{1/2} + 1 \le 2.$$

Note that when $\lambda = 0$, $w = v$. The lemma is then trivial with every $\kappa \in (0, 1/3)$. Therefore, we only need to consider the case where $\lambda > 0$. From the standard Caccioppli's type estimate for the solution v of (3.15) and (3.21), we also see that

$$(3.23)\qquad \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\Big)^{1/2} \le \frac{C(\Lambda, n)}{(1-2\kappa)\kappa^{\frac{n+2}{2}} r}\Big(\fint_{K_r(z_0)} |v - \overline{u}_{K_r(z_0)}|^2 dz\Big)^{1/2}$$
$$\le C(\Lambda, n)\big[\epsilon' + M(\lambda \kappa^{\frac{n+2}{2}} r)^{-1}\big],$$

where in the last estimate we have used the fact that $\kappa \in (0, 1/3)$ to control the factor $1 - 2\kappa$. Now, let w be the weak solution of (3.19), whose existence can be obtained by a standard procedure using Galerkin's method; see [31, pp. 466–475]. It remains to prove the estimate (3.20). By using the Skelov average as in [5, 13], we can formally take $v - w$ as a test function for equations (3.19) and (3.15) to obtain

$$(3.24)\qquad \frac{1}{2}\frac{d}{dt} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx + \int_{\Omega_{\kappa r}(x_0)} \big\langle \mathbf{A}(x,t,\lambda u, \nabla v), \nabla w - \nabla v\big\rangle dx$$
$$= \int_{\Omega_{\kappa r}(x_0)} \big\langle \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla w - \nabla v\big\rangle dx$$

From this, it follows that

$$\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx + \int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz$$
$$\le C(\Lambda)\Big[\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v - w|^2 dx$$
$$+ \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla v - \nabla w\rangle dz\Big]$$

$$\leq C(\Lambda)\Big[\int_{K_{\kappa r}(z_0)}\Big|\langle \mathbf{A}(x,t,\lambda\bar{u}_{K_{\kappa r}(z_0)},\nabla v),\nabla v-\nabla w\rangle\Big|dz$$

$$+\int_{\mathbb{K}_{\kappa r}(z_0)}\Big|\langle \mathbf{A}(x,t,\lambda u,\nabla v),\nabla v-\nabla w\rangle\Big|dz\Big]$$

$$\leq C(\Lambda)\int_{K_{\kappa r}(z_0)}|\nabla v||\nabla v-\nabla w|dz$$

$$\leq C(\Lambda)\int_{K_{\kappa r}(z_0)}|\nabla v|^2dz+\frac{1}{2}\int_{K_{\kappa r}(z_0)}|\nabla v-\nabla w|^2dz.$$

This last estimate together with (3.23) imply that

$$\Big(\fint_{K_{\kappa r}(z_0)}|\nabla v-\nabla w|^2dz\Big)^{1/2}$$

$$\leq C(\Lambda,n)\Big(\fint_{K_{\kappa r}(z_0)}|\nabla v|^2dz\Big)^{1/2}$$

$$\leq C(\Lambda,n)\Big(\fint_{K_{2\kappa r}(z_0)}|\nabla v|^2dz\Big)^{1/2}\leq C_1(\Lambda,n)\big[\epsilon'+M(r\kappa^{\frac{n+2}{2}}\lambda)^{-1}\big].$$

Hence, if $MC_1(\Lambda,n)(\lambda\kappa^{\frac{n+2}{2}}r)^{-1}\leq\frac{\epsilon}{4}$, we can choose ϵ' sufficiently small such that

$$4C_1(\Lambda,n)\epsilon'<\epsilon.$$

From these choices, it follows that

$$\Big(\fint_{K_{\kappa r}(z_0)}|\nabla v-\nabla w|^2dz\Big)^{1/2}\leq\epsilon/2.$$

This estimate, the triangle inequality, and the first estimate in (3.21) give

$$\Big(\fint_{K_{\kappa r}(z_0)}|\nabla u-\nabla w|^2dz\Big)^{1/2}$$

$$\leq\Big(\fint_{K_{\kappa r}(z_0)}|\nabla u-\nabla v|^2dz\Big)^{1/2}+\Big(\fint_{K_{\kappa r}(z_0)}|\nabla v-\nabla w|^2dz\Big)^{1/2}$$

$$\leq\Big(\frac{1}{\kappa^{n+2}}\fint_{K_r(z_0)}|\nabla u-\nabla v|^2dz\Big)^{1/2}+\epsilon/2\leq\epsilon'+\epsilon/2\leq\epsilon,$$

which is the first estimate in (3.20). It therefore remains to consider the case

$$\lambda r\kappa^{\frac{n+2}{2}}\epsilon\leq 2C_1(\Lambda,n)M. \tag{3.25}$$

In this case, we note that from our choice that $\epsilon'\leq\epsilon$, we particularly have

$$\lambda\kappa^{\frac{n+2}{2}}\epsilon' r\leq C(\Lambda,M,n).$$

Then it follows from (3.21) that

$$\lambda\Big(\fint_{K_r(x_0)}|v-\bar{u}_{K_r(z_0)}|^2dz\Big)^{1/2}\leq C(\Lambda,M,n).$$

From this and equation (3.15), we can apply Hölder's regularity theory in Lemmas 2.8 and 2.9 for the function

$$\widetilde{v}(x,t):=\lambda[v(x-x_0,t-t_0)-\bar{u}_{K_r(z_0)}],\quad (x,t)\in K_r,$$

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to find that there is $\beta_0 \in (0,1)$ depending only on Λ, n such that $\widetilde{v} \in C^{\beta_0}(\overline{K}_{2r/3})$. Then, by scaling back, we obtain the following estimates

$$\lambda \left\| v - \overline{u}_{K_r(z_0)} \right\|_{L^\infty(K_{5r/6}(z_0))} \leq C(\Lambda, M, n), \tag{3.26}$$

$$\lambda|v(z) - v(z')| \leq C(\Lambda, M, n)\Big[\frac{|x-x'| + |t-t'|^{1/2}}{r}\Big]^{\beta_0} \leq \kappa^{\beta_0}$$
$$\text{for all} z = (x,t), z' = (x',t') \in \overline{K}_{\kappa r}(z_0).$$

From now on, for simplicity, we write $\widehat{u} = u - \overline{u}_{K_{\kappa r}(z_0)}$. We can use (3.24) again to obtain

$$\begin{aligned}
&\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v-w|^2 dx + \int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \\
&\quad \leq C(\Lambda)\Big[\frac{1}{2}\sup_{\Gamma_{\kappa r}(t_0)} \int_{\Omega_{\kappa r}(x_0)} |v-w|^2 dx \\
&\qquad + \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla w), \nabla v - \nabla w\rangle dz\Big] \\
&\quad \leq C(\Lambda) \int_{K_{\kappa r}(z_0)} \langle \mathbf{A}(x,t,\lambda \overline{u}_{K_{\kappa r}(z_0)}, \nabla v) - \mathbf{A}(x,t,\lambda u, \nabla v), \nabla v - \nabla w\rangle dz \\
&\quad \leq C(\Lambda) \int_{K_{\kappa r}(z_0)} [\lambda \widehat{u}]^{\alpha_0} |\nabla v||\nabla v - \nabla w| dz \\
&\quad \leq \frac{1}{2}\int_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz + C(\Lambda) \int_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^{2\alpha_0} |\nabla v|^2 dz.
\end{aligned}$$

Hence,

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \leq C(\Lambda) \fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^{2\alpha_0} |\nabla v|^2 dz.$$

For $p_1 > 2$ and sufficiently close to 2 depending only on Λ, n, we write $q = \frac{2\alpha_0 p_1}{p_1 - 2} > 2$. Using Hölder's inequality and the self-improving regularity estimate, Lemma 2.11, we then obtain

$$\begin{aligned}
\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz &\leq C(\Lambda)\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^q\Big)^{\frac{p_1-2}{p_1}} \Big(\fint_{K_{\kappa r}(z_0)} |\nabla v|^{p_1} dz\Big)^{\frac{2}{p_1}} \\
&\leq C(\Lambda, n)\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^q dz\Big)^{\frac{p_1-2}{p_1}} \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\Big).
\end{aligned}$$

We further write

$$\begin{aligned}
\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^q dz &= \fint_{K_{\kappa r}(z_0)} |\widehat{u}||\lambda \widehat{u}|^{q-1} dz \\
&\leq \Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{1/2} \Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^{2q-2} dz\Big)^{1/2} \\
&\leq C(n, \alpha_0)[[\lambda u]]^{q-1}_{\mathrm{BMO}(K_R, R)} \Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{1/2} \\
&\leq C(n, M, \alpha_0)\Big(\fint_{K_{\kappa r}(z_0)} \big|\lambda \widehat{u}|^2 dz\Big)^{1/2}.
\end{aligned}$$

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Therefore,

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \le C(\Lambda, M, n, \alpha_0)\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{\frac{p_1-2}{2p_1}} \Big(\fint_{K_{2\kappa r}(z_0)} |\nabla v|^2 dz\Big).$$

This and (3.22) imply that

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \le C(\Lambda, M, n, \alpha_0)\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{\frac{p_1-2}{2p_1}}. \tag{3.27}$$

On the other hand, we also write

$$\begin{aligned}
&\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz \\
&\quad\le C\Big[\fint_{K_{\kappa r}(z_0)} |\lambda(u-v)|^2 dz + \fint_{K_{\kappa r}(z_0)} |\lambda(v - \overline{v}_{K_{\kappa r}(z_0)})|^2 dz \\
&\qquad + \fint_{K_{\kappa r}(z_0)} |\lambda(\overline{u}_{K_{\kappa r}(z_0)} - \overline{v}_{K_{\kappa r}(z_0)})|^2 dz\Big] \\
&\quad\le C(n)\Big[\kappa^{-(n+2)} \fint_{K_r(z_0)} |\lambda(u-v)|^2 dz + \fint_{K_{\kappa r}(z_0)} |\lambda(v - \overline{v}_{K_{\kappa r}(z_0)})|^2 dz\Big].
\end{aligned}$$

Then, by using Poincaré's inequality for the first term on the right-hand side of the last estimate, we obtain

$$\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{1/2} \le C(\Lambda, n)\Big[\lambda r \kappa^{-\frac{n+2}{2}} \Big(\fint_{K_r(z_0)} |\nabla u - \nabla v|^2 dz\Big)^{1/2} + \lambda \sup_{x,y \in \overline{K}_{\kappa r}(z_0)} |v(z) - v(z')|\Big].$$

This, (3.21), and (3.26) imply that

$$\Big(\fint_{K_{\kappa r}(z_0)} |\lambda \widehat{u}|^2 dz\Big)^{1/2} \le C(\Lambda, n)\big[(\lambda r)\epsilon' + \kappa^{\beta_0}\big].$$

From this last estimate, estimate (3.27) can be written as

$$\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \le C(\Lambda, M, n, \alpha_0)(\lambda r \epsilon' + \kappa^{\beta_0})^{\frac{p_1-2}{p_1}}.$$

From this and (3.25), we can further conclude that

$$\Big(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz\Big)^{1/2} \le C_2(\Lambda, M, \alpha_0, n)\Big[\frac{\epsilon'}{\epsilon \kappa^{\frac{n+2}{2}}} + \kappa^{\beta_0}\Big]^{\frac{p_1-2}{2p_1}}.$$

Now we choose κ sufficiently small depending only on Λ, M, n, α_0, and ϵ such that

$$\kappa^{\beta_0} \le \frac{1}{2}\Big[\frac{\epsilon}{2C_2(\Lambda, M, \alpha_0, n)}\Big]^{\frac{2p_1}{p_1-2}}.$$

Then we choose $\epsilon' > 0$ sufficiently small depending only on Λ, n, α_0, and ϵ such that

$$\epsilon' \le \frac{\epsilon \kappa^{\frac{n+2}{2}}}{2}\Big[\frac{\epsilon}{2C_2(\Lambda, M, \alpha_0, n)}\Big]^{\frac{2p_1}{p_1-2}}.$$

From these choices, we obtain

$$\Big(\fint_{K_{\kappa r}(z_0)} |\nabla v - \nabla w|^2 dz \Big)^{1/2} \le \epsilon/2.$$

Then we use the first estimate in (3.21) and the triangle inequality again to obtain the first estimate in (3.20). It now remains to prove the second estimate in (3.20). By the triangle inequality, the assumption in this lemma, and that $\epsilon \in (0,1)$, we see that

$$\begin{aligned}\Big(\fint_{K_{\kappa r}(z_0)} |\nabla w|^2 dz \Big)^{1/2} &\le \Big(\fint_{K_{\kappa r}(z_0)} |\nabla - \nabla u|^2 dz \Big)^{1/2} + \Big(\fint_{K_{\kappa r}(z_0)} |\nabla u|^2 dz \Big)^{1/2} \\ &\le \epsilon + \Big(2^{n+2} \fint_{K_{2\kappa r}(z_0)} |\nabla w|^2 dz \Big)^{1/2} \le 1 + 2^{\frac{n+2}{2}}\end{aligned}$$

as desired. The proof is therefore complete. ∎

Our next result is a standard approximation that, in particular, gives a comparison of the solution w of (3.19) with a constant coefficient solution.

Lemma 3.7 *Let $\Lambda > 0$ be fixed. Then, for every $\epsilon \in (0,1)$, there exists a constant $\delta' = \delta'(\Lambda, n, \epsilon) > 0$ and sufficiently small such that the following statement holds. Assume that Ω is be an open bounded domain with boundary $\partial\Omega \in C^1$. Assume also that $\mathbf{A}_0 : K_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying (1.2) and (1.3) and $[\mathbf{A}_0]_{\mathrm{BMO}(K_R,R)} \le \delta'$ for some $R > 0$. Suppose that w is a weak solution of*

$$\begin{aligned} w_t - \operatorname{div}[\mathbf{A}_0(x,t,\nabla w)] &= 0, \quad \text{in } K_{4\rho}(z_0), \\ \langle \mathbf{A}_0(x,t,\nabla w), \vec{v} \rangle &= 0, \quad \text{on } T_{4\rho}(z_0), \text{ if } T_{4\rho}(z_0) \ne \varnothing, \end{aligned}$$

and it satisfies

$$\fint_{K_{4\rho}} |\nabla w|^2 dz \le 1,$$

with some $0 < \rho < R/4$ and some $z_0 = (x_0, t_0) \in K_R$. Then there is some function h such that

$$\Big(\fint_{K_{2\rho}(z_0)} |\nabla w - \nabla h|^2 dz \Big)^{1/2} \le \epsilon^2 \quad \text{and} \quad \|\nabla h\|_{L^\infty(K_\rho(z_0))} \le C(\Lambda, n). \tag{3.28}$$

Proof The proof can be done exactly the same way as that of Lemma 3.3. Since $\partial\Omega$ is C^1, the Lipschitz regularity estimates for weak solutions of the corresponding homogeneous equation with frozen coefficient hold true if $T_{2\rho}(z_0) \ne \varnothing$. Alternatively, since $\partial\Omega$ is C^1, it is sufficiently flat in the sense of Reifenberg. Therefore, this lemma follows from [9, Lemma 6 and Corollary 1]; see also [6]. One can flatten the boundary as in [27] and prove a similar approximation in the upper-half cube Q_r^+ as in Lemma 3.3. ∎

Finally, we state and prove the main result of the section.

Proposition 3.8 *Let $\Lambda, M > 0, \alpha > 2$ and let $\alpha_0 \in (0,1)$ be fixed. Then for every $\epsilon \in (0,1)$, there exist sufficiently small numbers $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon) \in (0, 1/3)$ and*

$\delta = \delta(\epsilon, \Lambda, M, n, \alpha_0) \in (0, \epsilon)$ *such that the following holds. Assume* $\partial\Omega \in C^1$, *and let* $\mathbf{A}\colon K_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* (1.2)–(1.4) *and* (1.12) *hold, and assume that*

$$\fint_{K_{8r}(z_0)} |\mathbf{F}|^2 dz + \Big(\fint_{K_{8r}(z_0)} |\mathbf{G}|^\alpha dz \Big)^{\frac{2}{\alpha}} + (8r)^2 \Big(\fint_{K_{8r}(z_0)} |f|^{2n_*} dz \Big)^{1/n_*} \le \delta^2$$

for some $z_0 = (x_0, t_0) \in K_R$, *and* $r \in (0, R/8)$. *Then, for every* $\lambda \ge 0$, *if* u *is a weak solution of* (3.12) *satisfying*

$$\fint_{K_{16\kappa r}(z_0)} |\nabla u|^2 dz \le 1 \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(K_R, R)} \le M,$$

then there is $h \in L^\infty\big(\Gamma_{4\kappa r}(t_0), L^2(\Omega_{4\kappa r}(x_0))\big) \cap L^2\big(\Gamma_{4\kappa r}(t_0), W^{1,2}(\Omega_{4\kappa r}(x_0))\big)$ *such that the following estimate holds:*

$$\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla h|^2 dz \le \epsilon^2, \quad \|\nabla h\|_{L^\infty(K_{2\kappa r}(z_0))} \le C(\Lambda, n).$$

Proof Let

$$\delta = \min\Big\{ \delta_2\Big(\tfrac{1}{2}\Big[\tfrac{1}{2} \Big]^{\frac{n+2}{2}} \epsilon, \Lambda, M, n, \alpha \Big), \delta'\Big(\Lambda, n, \epsilon/[2(1 + 2^{(n+2)/2})] \Big) \Big\},$$

where δ_2 is defined as in Lemma 3.6, and δ' is defined as in Lemma 3.7. Moreover, let $\kappa = \kappa(\Lambda, M, n, \alpha_0, \frac{1}{2}\big[\frac{1}{2}\big]^{\frac{n+2}{2}} \epsilon)$ be the number defined in Lemma 3.6. Then, by applying Lemma 3.6 with r replaced by $8r$, we can find $w \in L^\infty\big(\Gamma_{8\kappa r}(t_0), L^2(\Omega_{8\kappa r}(x_0))\big) \cap L^2\big(\Gamma_{8\kappa r}(t_0), W^{1,2}(\Omega_{8\kappa r}(x_0))\big)$ satisfying

$$\Big(\fint_{K_{8\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz \Big)^{1/2} \le \tfrac{1}{2}\Big[\tfrac{1}{2} \Big]^{\frac{n+2}{2}} \epsilon, \qquad \Big(\fint_{K_{8\kappa r}(z_0)} |\nabla w|^2 dz \Big)^{1/2} \le 1 + 2^{\frac{n+2}{2}}. \tag{3.29}$$

Then we can apply Lemma 3.7 with $\mathbf{A}_0(x, t, \xi) = \mathbf{A}(x, t, \lambda \bar{u}_{K_{8\kappa r}(z_0)}, \xi)$, $\rho = 2\kappa r$, and with some suitable scaling, we can find a function $h \in L^\infty\big(\Gamma_{4\kappa r}(t_0), L^2(\Omega_{4\kappa r}(x_0))\big) \cap L^2\big(\Gamma_{4\kappa r}(t_0), W^{1,2}(\Omega_{4\kappa r}(x_0))\big)$ such that the following estimate holds:

$$\Big(\fint_{K_{4\kappa r}(z_0)} |\nabla w - \nabla h|^2 dz \Big)^{1/2} \le \epsilon/2, \quad \|\nabla h\|_{L^\infty(K_{2\kappa r}(z_0))} \le C(\Lambda, n). \tag{3.30}$$

It then follows from (3.29), (3.30), and the triangle inequality that

$$\begin{aligned} &\Big(\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla h|^2 dz \Big)^{1/2} \\ &\quad\le \Big(\fint_{K_{4\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz \Big)^{1/2} + \Big(\fint_{K_{4\kappa r}(z_0)} |\nabla w - \nabla h|^2 dz \Big)^{1/2} \\ &\quad\le \Big[2 \Big]^{\frac{n+2}{2}} \Big(\fint_{K_{8\kappa r}(z_0)} |\nabla u - \nabla w|^2 dz \Big)^{1/2} + \epsilon/2 \le \epsilon. \end{aligned}$$

The proof is therefore complete. ∎

4 Level Set Estimates

This section gives the key level set estimates needed in the proofs of the main theorems Theorems 1.1 and 1.2. We can assume $R = 1$, since the general case $R > 0$ can be achieved by using the dilation (2.2), Remark 2.1, and the dilation property of Lorentz quasi-norms, Remark 2.2.

Let $\epsilon > 0$ be a sufficiently small number to be determined depending only on the given numbers n, Λ, p, q, and α_0 . Let $\delta = \delta(\epsilon, \Lambda, M, n, \alpha_0)$, $\kappa = \kappa(\Lambda, M, n, \alpha_0, \epsilon)$ be the numbers defined in Proposition 3.8. Note that since ϵ depends on n, Λ, p, q and α_0, the numbers κ and δ also only depend on these numbers. Assume that all assumptions in Theorem 1.2 are valid with this δ and $R = 1$. For each $\lambda \geq 0$, and if u is a weak solution of (1.13), recall that

$$\mathbf{G}(x,t) \approx [[u]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}(x,t), \quad (x,t) \in K_2.$$

We fix $\eta > 2$ such that $\eta < \min\{2+\epsilon_0, p\}$, where $\epsilon_0 = \epsilon_0(\Lambda, n)$ validates Lemmas 2.10 and 2.11. Let us also denote

$$F(x,t) = |\mathbf{F}(x,t)| + |\mathbf{G}(x,t)| + \mathcal{G}(f)|f(x,t)|^{n_*}, \quad (x,t) \in K_2,$$

where n_* is defined in (3.2), and

$$\mathcal{G}(f) = \Big(\fint_{K_2} |f(x,t)|^{2n_*}\,dz \Big)^{\frac{1-n_*}{2n_*}}. \tag{4.1}$$

As we will see in (4.9) and (4.10), the function $\mathcal{G}$ plays an essential role in our proof. Observe also that since $p \geq 2$,

$$\mathcal{G}(f)\, \| |f|^{n_*} \|_{L^{p,q}(K_2)} = \|f\|^{1-n_*}_{L^{2n_*}(K_2)} \, \|f\|^{n_*}_{L^{n_* p, n_* q}(K_2)} \leq C(n)\, \|f\|_{L^{n_* p, n_* q}(K_2)}. \tag{4.2}$$

From now on, let $\tau_0 > 0$ be the number defined by

$$\tau_0 = \Big(\fint_{K_2} |\nabla u|^2 dz \Big)^{1/2} + \frac{1}{\delta} \Big(\fint_{K_2} |F|^{\eta} dz \Big)^{1/\eta} < \infty. \tag{4.3}$$

For fixed numbers $1 \leq \mu \leq 2$, and $\tau > 0$, we denote the upper-level set of ∇u in K_μ by

$$E_\mu(\tau) = \Big\{ \text{Lebesgue point } (x,t) \in K_\mu \text{ of } \nabla u : |\nabla u(x,t)| > \tau \Big\}.$$

The following proposition estimating the upper-level sets of ∇u is the main result of this section.

Proposition 4.1 *There exist $N_0 = N_0(\Lambda, n) > 1$ and $B_0 = B_0(n)$ such that*

$$|E_{s_1}(N_0\tau)| \leq \epsilon^2 \Big[|E_{s_2}(\tau/4)| + \frac{1}{(\delta\tau)^\eta} \int_{\delta\tau/4}^{\infty} s^\eta \big| \big\{ (x,t) \in K_2 : |F(x,t)| > s \big\} \big| \frac{ds}{s} \Big]$$

for all $1 \leq s_1 < s_2 \leq 2$, for every $\tau > \widehat{B}_0 \tau_0$, where $\widehat{B}_0 := B_0[(s_2 - s_1)\kappa]^{-\frac{n+2}{2}}$.

The rest of the section is devoted to proving Proposition 4.1. We follow the approach developed in [1] and used in [3,5,6]. However, some nontrivial modifications are also required to treat the terms $f, \mathbf{b}$ and to obtain the sharp homogeneous estimates; see Remark 1.4. For each $\bar{z} \in \overline{K}_2$ and each $r > 0$, we define

$$CZ_r(\bar{z}) = \Big(\fint_{K_r(\bar{z})} |\nabla u|^2 dz \Big)^{1/2} + \frac{1}{\delta} \Big(\fint_{K_r(\bar{z})} |F|^{\eta} dz \Big)^{1/\eta}.$$

Several lemmas are needed to prove Proposition 4.3. The first is a stopping-time argument lemma.

Lemma 4.2 *There exists a constant $B_0 = B_0(n)$ such that for each $1 \le s_1 < s_2 \le 2$, $\tau > \widehat{B}_0 \tau_0$, and for $\bar{z} \in E_{s_1}(\tau)$, there is $r_{\bar{z}} < \frac{(s_2 - s_1)\kappa}{40}$ such that*

$$CZ_{r_{\bar{z}}}(\bar{z}) = \tau \quad \textit{and} \quad CZ_r(\bar{z}) < \tau \quad \textit{for all } r \in (r_{\bar{z}}, 1).$$

Proof The argument is quite standard; see [1, 3, 5, 6]. Observe that because $r < 1$ and $\eta > 2$, we have

$$\begin{aligned} CZ_r(\bar{z}) &\le C(n) \Big[\Big(\frac{1}{r} \Big)^{(n+2)/2} \Big(\fint_{K_2} |\nabla u|^2 dz \Big)^{1/2} + \Big(\frac{1}{r} \Big)^{(n+2)/\eta} \frac{1}{\delta} \Big(\fint_{K_2} |F|^{\eta} dz \Big)^{1/\eta} \Big] \\ &\le \frac{C(n)\tau_0}{r^{(n+2)/2}}. \end{aligned}$$

Therefore, if $r > \frac{(s_2 - s_1)\kappa}{40}$, then for $B_0 = C(n)[40]^{(n+2)/2}$, we see that

$$\frac{C(n)\tau_0}{r^{(n+2)/2}} \le C(n) \Big(\frac{40}{(s_2 - s_1)\kappa} \Big)^{(n+2)/2} \tau_0 = B_0 [(s_2 - s_1)\kappa]^{-\frac{n+2}{2}} \tau_0 < \tau.$$

Then

$$CZ_r(\bar{z}) < \tau, \quad \text{when} \quad \frac{(s_2 - s_1)\kappa}{40} \le r \le 1, \quad \text{and} \quad \tau > \widehat{B}_0 \tau_0.$$

On the other hand, when $\bar{z} \in E_{s_1}(\tau)$, by Lebesgue's theorem, we see that if r is sufficiently small, then

$$CZ_r(\bar{z}) > \tau.$$

Due to the fact the $CZ_r(\bar{z})$ is absolutely continuous, we can find $r_{\bar{z}}$, which is the largest number in $(0, \frac{(s_2 - s_1)\kappa}{40})$ such that $CZ_{r_{\bar{z}}}(\bar{z}) = \tau$. From this, the conclusion of the lemma follows. ∎

Lemma 4.3 *For each $\tau > \widehat{B}_0 \tau_0$, and each $1 \le s_1 < s_2 \le 2$, there exists a countable, disjoint family $\{K_{r_i}(z_i)\}_{i \in \mathcal{I}}$ with $r_i < \frac{(s_2 - s_1)\kappa}{40}$ and $z_i \in K_{s_1}$ such that the following hold:*

(i) $E_{s_1}(\tau) \subset \bigcup_{i=1}^{\infty} K_{5r_i}(z_i)$;

(ii) $CZ_{r_i}(z_i) = \tau$, *and* $CZ_{r_i}(r) < \tau$ *for all* $r \in (r_i, 1)$.

Moreover, for each $i \in \mathcal{I}$, the following estimate holds:

$$\begin{aligned} (4.4) \quad |K_{r_i}(z_i)| \le C(\Lambda, p, n) \Big[& |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)| \\ & + \frac{1}{(\tau\delta)^{\eta}} \int_{\tau\delta/4}^{\infty} s^{\eta} |\{(x,t) \in K_{r_i}(z_i) : |F(x,t)| > s\}| \frac{ds}{s} \Big]. \end{aligned}$$

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Proof Conclusions (i) and (ii) follow directly from Lemma 4.2 and Vitali's covering lemma. It remains now to prove (4.4). Observe that if

$$\frac{1}{\delta^\eta} \fint_{K_{r_i}(z_i)} |F(x,t)|^\eta dz \geq \frac{\tau^\eta}{2^\eta}, \tag{4.5}$$

then

$$\begin{aligned}
|K_{r_i}(z_i)| &\leq \frac{2^\eta}{\tau^\eta \delta^\eta} \int_{K_{r_i}(z_i)} |F(x,t)|^\eta dz \\
&= \frac{2^\eta}{\tau^\eta \delta^\eta} \int_0^\infty s^\eta \Big| \Big\{ (x,t) \in K_{r_i}(z_i) : |F(x,t)| > s \Big\} \Big| \frac{ds}{s} \\
&= \frac{2^\eta}{\tau^\eta \delta^\eta} \Big[\int_0^{\delta\tau/4} \cdots + \int_{\delta\tau/4}^\infty \cdots \Big] \\
&\leq \frac{|K_{r_i}(z_i)|}{2^\eta} + \frac{2^\eta}{\tau^\eta \delta^\eta} \int_{\delta\tau/4}^\infty s^\eta \Big| \Big\{ (x,t) \in K_{r_i}(z_i) : |F(x,t)| > s \Big\} \Big| \frac{ds}{s}.
\end{aligned}$$

Hence, (4.4) follows.

Otherwise, *i.e.*, if (4.5) is false, it follows from the fact that $CZ_{r_i}(z_i) = \tau$ that

$$\fint_{K_{r_i}(z_i)} |\nabla u|^2 dz \geq \frac{\tau^2}{2^2},$$

and therefore

$$K_{r_i}(z_i) \leq \frac{2^2}{\tau^2} \int_{K_{r_i}(z_i)} |\nabla u|^2 dx dt.$$

Then observe that since $r_i < \frac{(s_2 - s_1)\kappa}{40}$, we have $K_{r_i}(z_i) \subset K_{s_2}$, and hence

$$\begin{aligned}
|K_{r_i}(z_i)| &\leq \frac{2^2}{\tau^2} \int_{K_{r_i}(z_i) \setminus E_{s_2}(\tau/4)} |\nabla u|^2 dz + \frac{2^2}{\tau^2} \int_{K_{r_i}(z_i) \cap E_{s_2}(\tau/4)} |\nabla u|^2 dz \\
&\leq \frac{|K_{r_i}(z_i)|}{4} + \frac{2^2}{\tau^2} \int_{K_{r_i}(z_i) \cap E_{s_2}(\tau/4)} |\nabla u|^2 dz.
\end{aligned}$$

Therefore,

$$|K_{r_i}(z_i)| \leq \frac{16}{3\tau^2} \int_{K_{r_i}(z_i) \cap E_{s_2}(\tau/4)} |\nabla u|^2 dz.$$

This and Hölder's inequality with some $\gamma_0 > 0$ yield that

$$|K_{r_i}(z_i)| \leq \frac{6 |K_{r_i}(z_i)|^{\frac{1}{1+\gamma_0}}}{\tau^2} \Big(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz \Big)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1 - \frac{1}{1+\gamma_0}}.$$

Hence,

$$|K_{r_i}(z_i)|^{1 - \frac{1}{1+\gamma_0}} \leq \frac{6}{\tau^2} \Big(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz \Big)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i) \cap E_{s_2}(\tau/4)|^{1 - \frac{1}{1+\gamma_0}}. \tag{4.6}$$

On the other hand, with γ_0 sufficiently small so that $2(1+\gamma_0) < \eta$, we can apply Lemmas 2.10 and 2.11 to obtain

(4.7)
$$\begin{aligned}&\Big(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\Big)^{\frac{1}{2(1+\gamma_0)}}\\ &\quad\le C\Big[\Big(\fint_{K_{2r_i}(z_i)} |\nabla u|^2 dz\Big)^{1/2} + \Big(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{2(1+\gamma_0)} dz\Big)^{\frac{1}{2(1+\gamma_0)}}\\ &\qquad + \Big(\fint_{K_{2r_i}(z_i)} |\mathbf{G}|^{2(1+\gamma_0)^2} dz\Big)^{\frac{1}{2(1+\gamma_0)^2}} + \mathcal{G}(f)\Big(\fint_{K_{2r_i}(z_i)} |f|^{2n_*(1+\gamma_0)}\Big)^{\frac{1}{2(1+\gamma_0)}}\Big].\end{aligned}$$

Then since $2(1+\gamma_0)^2 < \eta$, we can use Hölder's side inequality to control the last three terms on the right-hand of (4.7) as the following:

$$\begin{aligned}(4.8)\quad &\Big(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{2(1+\gamma_0)} dz\Big)^{\frac{1}{2(1+\gamma_0)}} + \Big(\fint_{K_{2r_i}(z_i)} |\mathbf{G}|^{2(1+\gamma_0)^2} dz\Big)^{\frac{1}{2(1+\gamma_0)^2}}\\ &\qquad + (2r_i)\Big(\fint_{K_{2r_i}(z_i)} |f|^{2n_*(1+\gamma_0)}\Big)^{\frac{1}{2(1+\gamma_0)}}\\ &\le \Big(\fint_{K_{2r_i}(z_i)} |\mathbf{F}|^{\eta} dz\Big)^{\frac{1}{\eta}} + \Big(\fint_{K_{r_i}(z_i)} |\mathbf{G}|^{\eta} dz\Big)^{\frac{1}{\eta}} + \mathcal{G}(f)\Big(\fint_{K_{2r_i}(z_i)} |f|^{n_*\eta}\Big)^{\frac{1}{\eta}}\\ &\le C\Big(\fint_{K_{2r_i}(z_i)} |F|^{\eta} dz\Big)^{\frac{1}{\eta}}.\end{aligned}$$

Collecting the estimates (4.6), (4.7), and (4.8), we conclude that

$$\begin{aligned}|K_{r_i}(z_i)|^{1-\frac{1}{1+\gamma_0}} &\le \frac{6}{\tau^2}\Big(\fint_{K_{r_i}(z_i)} |\nabla u|^{2(1+\gamma_0)} dz\Big)^{\frac{1}{1+\gamma_0}} |K_{r_i}(z_i)\cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}\\ &\le \frac{C(\Lambda,n)}{\tau^2} CZ_{2r_i}(z_i)^2 |K_{r_i}(z_i)\cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}\\ &\le C(\Lambda,n)|K_{r_i}(z_i)\cap E_{s_2}(\tau/4)|^{1-\frac{1}{1+\gamma_0}}.\end{aligned}$$

This implies

$$|K_{r_i}(z_i)| \le C(\Lambda,n)|K_{r_i}(z_i)\cap E_{s_2}(\tau/4)|,$$

and (4.4) follows. The proof is therefore complete. ∎

Proof of Proposition 4.1 Fix s_1, s_2, and τ as in the statement of Proposition 4.1. For each $i \in \mathcal{I}$, observe that from Lemma 4.3(ii), $CZ_{40r_i}(z_i) < \tau$ and $CZ_{20\widehat{r}_i}(z_i) < \tau$, where $\widehat{r}_i = \kappa^{-1} r_i \in (0, 1/40)$. Therefore, we have

$$\Big(\fint_{K_{40\kappa\widehat{r}_i}(z_i)} |\nabla u|^2 dz\Big)^{1/2} \le \tau, \quad \Big(\fint_{K_{20\widehat{r}_i}(z_i)} |F|^{\eta} dz\Big)^{1/\eta} \le \delta\tau.$$

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Moreover, since $K_{20\widehat{r}_i}(z_i) \subset K_2$, it follows that there is some constant $C_0 = C_0(n) > 1$ such that

$$
(4.9)\qquad 20\widehat{r}_i\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{2n_*}dz\Big)^{1/(2n_*)} = 20\widehat{r}_i|K_{20\widehat{r}_i}|^{-(\frac{1}{2n_*}-\frac{1}{2})}\Big(\int_{K_2}|f|^{2n_*}dz\Big)^{\frac{1}{2n_*}-\frac{1}{2}}\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{2n_*}dz\Big)^{1/2} \le C_0(n)\mathcal{G}(f)\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{2n_*}dz\Big)^{1/2}.
$$

Now, with the C_0 defined in (4.9), we define $\tau' = 3C_0\tau$, $\widehat{u} = u/\tau'$, and $\widehat{\lambda} = \tau'\lambda$. We see that $\widehat{u}$ is a weak solution of

$$
\begin{aligned}
\widehat{u}_t - \operatorname{div}\big[\widehat{\mathbf{A}}(x,t,\widehat{\lambda}\widehat{u},\nabla\widehat{u}) - \widehat{u}\mathbf{b} - \widehat{\mathbf{F}}\big] &= \widehat{f}, && \text{in } K_2\\
\big\langle \widehat{\mathbf{A}}(x,t,\widehat{\lambda}\widehat{u},\nabla\widehat{u}) - \widehat{u}\mathbf{b} - \widehat{\mathbf{F}}, \vec{v}\big\rangle &= 0, && \text{on } T_2,
\end{aligned}
$$

where

$$
\widehat{\mathbf{F}} = \frac{\mathbf{F}}{\tau'}, \quad \widehat{f} = \frac{f}{\tau'}, \quad \text{and} \quad \widehat{\mathbf{A}}(x,t,s,\xi) = \frac{\mathbf{A}(x,t,s,\tau'\xi)}{\tau'}.
$$

From Remark 2.1, $\widehat{\mathbf{A}}$ satisfies all conditions (1.2)–(1.4) and

$$
\begin{aligned}
&[\widehat{\mathbf{A}}]_{\mathrm{BMO}(K_1,1)} = [\mathbf{A}]_{\mathrm{BMO}(K_1,1)} \le \delta,\\
&[[\widehat{\lambda}\widehat{u}]]_{\mathrm{BMO}(K_1,1)} = [[\lambda u]]_{\mathrm{BMO}(K_1,1)} \le M, \quad \fint_{K_{40\kappa\widehat{r}_i}(z_i)}|\nabla\widehat{u}|^2dz \le 1.
\end{aligned}
$$

Also, with $\widehat{\mathbf{G}} \approx [[\widehat{u}]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}$, and some $\alpha = 2(1+\gamma_0) \in (2,\eta)$, it follows from (4.9) and Hölder's inequality that

(4.10)

$$
\begin{aligned}
&\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\widehat{\mathbf{F}}|^2dz\Big)^{1/2} + \Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\widehat{\mathbf{G}}|^{\alpha}\big]dz\Big)^{1/\alpha} + 20\widehat{r}_i\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\widehat{f}|^{2n_*}dz\Big)^{1/(2n_*)}\\
&= \frac{1}{\tau'}\Big[\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\mathbf{F}|^2dz\Big)^{1/2} + \Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\mathbf{G}|^{\alpha}dz\Big)^{1/\alpha} + 20\widehat{r}_i\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{2n_*}dz\Big)^{1/(2n_*)}\Big]\\
&= \frac{C_0}{\tau'}\Big[\Big(\fint_{K_{40\widehat{r}_i}(z_i)}|\mathbf{F}|^2dz\Big)^{1/2} + \Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\mathbf{G}|^{\alpha}dz\Big)^{1/\alpha} + \mathcal{G}(f)\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{2n_*}dz\Big)^{1/2}\Big]\\
&\le \frac{1}{3\tau}\Big[\Big(\fint_{K_{40\widehat{r}_i}(z_i)}|\mathbf{F}|^{\eta}dz\Big)^{1/\eta} + \Big(\fint_{K_{20\widehat{r}_i}(z_i)}|\mathbf{G}|^{\eta}dz\Big)^{1/\eta} + \mathcal{G}(f)\Big(\fint_{K_{20\widehat{r}_i}(z_i)}|f|^{n_*\eta}dz\Big)^{1/\eta}\Big]\\
&\le \frac{1}{\tau}\Big(\fint_{K_{20\widehat{r}_i}}|F|^{\eta}dz\Big)^{1/\eta} \le \delta.
\end{aligned}
$$

Therefore, all assumptions in Proposition 3.8 are satisfied with $r = 5\widehat{r}_i/2$. Hence, we can find a function $\widehat{v}_i$ such that

$$
\fint_{K_{10r_i}(z_i)}|\nabla\widehat{u} - \nabla\widehat{v}_i|^2dz \le \epsilon^2, \quad \|\nabla\widehat{v}_i\|_{L^\infty(K_{5r_i}(z_i))} \le C_0(\Lambda,n).
$$

Then, by scaling back with $v_i = \tau'\widehat{v}_i$, we obtain

$$
\fint_{K_{10r_i}(z_i)}|\nabla u - \nabla v_i|^2dz \le 9C_0^2\tau^2\epsilon^2, \quad \|\nabla v_i\|_{L^\infty(K_{5r_i}(z_i))} \le 3C_0(\Lambda,n)\tau.
$$

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Now, let $N_0 = 6C_0(\Lambda, n)\sqrt{C_*(n)}$, where $C_*(n)$ is defined to be

$$C_*(n) \geq \frac{|K_{10r}(z_0)|}{|K_r(z_0)|} \quad \text{for all} r \in (0,1) \quad \text{for all} z_0 \in \Omega \cap B_2.$$

Observe that from Lemma 4.3,

$$|E_{s_1}(N_0\tau)| \leq \sum_{i\in\mathcal{J}} \Big| \Big\{ (x,t) \in K_{5r_i}(z_i) : |\nabla u(x,t)| > N_0\tau \Big\} \Big|.$$

Therefore,

$$\begin{aligned}
|E_{s_1}(N_0\tau)| &\leq \sum_{i\in\mathcal{J}} \Big| \Big\{ (x,t) \in K_{5r_i}(z_i) : |\nabla u(x,t) - \nabla v_i(x,t)| > \frac{N_0\tau}{2} \Big\} \Big| \\
&\quad + \sum_{i\in\mathcal{J}} \Big| \Big\{ (x,t) \in K_{5r_i}(z_i) : |\nabla v_i(x,t)| > \frac{N_0\tau}{2} \Big\} \Big| \\
&\leq \sum_{i\in\mathcal{J}} \Big| \Big\{ (x,t) \in K_{10r_i}(z_i) : |\nabla u(x,t) - \nabla v_i(x,t)| > \frac{N_0\tau}{2} \Big\} \Big| \\
&\leq \Big(\frac{2}{N_0\tau}\Big)^2 \sum_{i\in\mathcal{J}} \int_{K_{10r_i}(z_i)} |\nabla u - \nabla u_i|^2 dz \\
&\leq 9C_0^2\epsilon^2 \Big(\frac{2}{N_0}\Big)^2 \sum_{i\in\mathcal{J}} |K_{10r_i}(z_i)| \leq 9C_0^2\epsilon^2 \Big(\frac{2}{N_0}\Big)^2 C_*(n) \sum_{i\in\mathcal{J}} |K_{r_i}(z_i)|.
\end{aligned}$$

From this and our choice of N_0, it follows that

$$|E_{s_1}(N_0\tau)| \leq \epsilon^2 \sum_{i\in\mathcal{J}} |K_{r_i}(z_i)|,$$

and the conclusion of our proposition follows directly from (4.4) and the fact that $\{K_{r_i}(z_i)\}_{i\in\mathcal{J}}$ is a disjoint family. ∎

5 Proof of Main Theorems

As already discussed, Theorem 1.3 follows immediately from Theorems 1.1 and 1.2 and a standard energy estimate. The proof of Theorem 1.1 is, however, similar to that of Theorem 1.2 by using Proposition 3.4 instead of Proposition 3.8. We therefore skip its proof and focus on proving Theorem 1.2. Again, through the dilation (2.2) and Remarks 2.1 and 2.2, we can assume without loss of generality that $R = 1$.

Proof of Theorem 1.2 With Proposition 4.1 in hand, the proof is now standard (see [1, 3, 6]). We give it here for the sake of completeness. For each $k \in \mathbb{N}$, we define $(\nabla u)_k(x,t) = \max\{|\nabla u(x,t)|, k\}$. It should be noted that we do not know yet if ∇u is in $L^{p,q}(K_1)$. However, since $(\nabla u)_k$ is bounded, $(\nabla u)_k \in L^{p,q}(K_2)$ for all $p > 2$ and $0 < q \leq \infty$. For $\mu \in [1,2]$, we let

$$E^k_\mu(\tau) = \Big\{ (x,t) \in K_\mu : (\nabla u)_k(x,t) > \tau \Big\}.$$

By considering the cases $k < N_0\tau$ and $k \geq N_0\tau$, we can conclude from Proposition 4.1 that

$$\text{(5.1)} \quad |E^k_{s_1}(\tau N_0)| \leq \epsilon^2 \Big[|E^k_{s_2}(\tau/4) + \frac{1}{(\delta\tau)^\eta} \int_{\delta\tau/4}^{\infty} s^\eta |\{(x,t) \in K_2 : |F(x,t)| > s\}| \frac{ds}{s} \Big]$$

for all $\tau > \widehat{B}_0\tau_0 = B_0[(s_2 - s_2)\kappa]^{-(n+2)/2}\tau_0$. We now divide the proof into two cases depending on whether or not $q = \infty$.

Case I: We start with the easy case when $q = \infty$. In this case, it is trivial that

$$
\begin{aligned}
(5.2)\qquad &\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_1})}\\
&= \sup_{\tau>0} \tau \Big| \big\{ (x,t) \in K_{s_1} : (\nabla u)_k > \tau \big\} \Big|^{1/p}\\
&\le \Big[\sup_{0<\tau<N_0\widehat{B}_0\tau_0} \tau \Big| \big\{ (x,t) \in K_{s_1} : (\nabla u)_k > \tau \big\} \Big|^{1/p}\\
&\qquad + \sup_{N_0\widehat{B}_0\tau_0<\tau} \tau \Big| \big\{ (x,t) \in K_{s_1} : (\nabla u)_k > \tau \big\} \Big|^{1/p} \Big].
\end{aligned}
$$

From (4.3), the first term on the right-hand side of (5.2) is obviously controlled by

$$
\begin{aligned}
|K_2|^{1/p} N_0\widehat{B}_0\tau_0 &\le C[(s_2 - s_1)\kappa]^{-(n+2)/2}\big[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^\eta(K_2)} \big]\\
&\le C[(s_2 - s_1)\kappa]^{-(n+2)/2}\big[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,\infty}(K_2)} \big].
\end{aligned}
$$

On the other hand, with (5.1), the second term on the right-hand side of (5.2) can be rewritten and then controlled as

$$
\begin{aligned}
&\sup_{N_0\widehat{B}_0\tau_0<\tau} \tau \Big| \big\{ (x,t) \in K_{s_1} : (\nabla u)_k > \tau \big\} \Big|^{1/p}\\
&\quad = N_0 \sup_{\widehat{B}_0\tau_0<\tau} \tau \Big| \big\{ (x,t) \in K_{s_1} : (\nabla u)_k > N_0\tau \big\} \Big|^{1/p}\\
&\quad \le C\epsilon^{2/p} \sup_{\tau>\widehat{B}_0\tau_0} \tau \bigg[\Big| \big\{ (x,t) \in K_{s_2} : (\nabla u)_k > \tau/4 \big\} \Big|\\
&\qquad\qquad + \frac{1}{(\delta\tau)^\eta} \int_{\delta\tau/4}^\infty s^\eta \big| \{(x,t) \in K_2 : |F(x,t)| > s\} \big| \frac{ds}{s} \bigg]^{1/p}\\
&\quad \le C\epsilon^{2/p} \bigg[\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-\eta/p}\\
&\qquad\qquad \times \sup_{\tau>\widehat{B}_0\tau_0} \Big(\tau^{p-\eta} \int_{\delta\tau/4}^\infty s^{\eta-p} s^p \big| \{(x,t) \in K_2 : |F(x,t)| > s\} \big| \frac{ds}{s} \Big)^{1/p} \bigg]\\
&\quad \le C\epsilon^{2/p} \bigg[\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-\eta/p} \|F\|_{L^{p,\infty}(K_2)}\\
&\qquad\qquad \times \sup_{\tau>\widehat{B}_0\tau_0} \Big(\tau^{p-\eta} \int_{\delta\tau/4}^\infty s^{\eta-p-1} ds \Big)^{1/p} \bigg]\\
&\quad \le C\big[\epsilon^{2/p} \|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + \delta^{-1}\|F\|_{L^{p,\infty}(K_2)} \big].
\end{aligned}
$$

Hence, combining the previous two estimates, we see that for every $1 \le s_1 < s_2 \le 2$, there is a constant $C_1 = C_1(\Lambda, n) > 0$ such that

$$\|(\nabla u)_k\|_{L^{p,\infty}(K_{s_1})} \le C_1 \epsilon^{2/p} \|(\nabla u)_k\|_{L^{p,\infty}(K_{s_2})} + C_1[(s_2 - s_1)\kappa]^{-(n+2)/2}\Big[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,\infty}(K_2)}\Big].$$

This and since ϵ is sufficiently small so that $C_1\epsilon^{2/p} \le 1/2$, we can use the iteration Lemma 2.5 to imply that

$$\begin{aligned}\|(\nabla u)_k\|_{L^{p,\infty}(K_1)} &\le C(\Lambda, n, p, \kappa)\Big[\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,\infty}(K_2)}\Big] \\ &\le C\Big[\|\nabla u\|_{L^2(K_2)} + \|\mathbf{F}\|_{L^{p,\infty}(K_2)} + \|\mathbf{G}\|_{L^{p,\infty}(K_2)} + \mathcal{G}(f)\big\||f|^{n_*}\big\|_{L^{p,\infty}(K_2)}\Big].\end{aligned}$$

We note that the Lorentz quasi-norm is lower semi-continuous with respect to the a.e. convergence. Because of this, we can take $k \to \infty$ and use (4.2) to obtain the desired estimate (1.14).

Case II: We consider the case $0 < q < \infty$. In this case,

(5.3)
$$\begin{aligned}\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} &\le C(N_0, p, q)\Big(\int_0^\infty \Big[s^p\big|\{(x,t) \in K_{s_1} : (\nabla u)_k(x,t) > N_0 s\}\big|\Big]^{q/p}\frac{ds}{s}\Big)^{1/q} \\ &\le C\Big[\Big(\int_0^{\widehat{B}_0\tau_0}\cdots\Big)^{1/q} + \Big(\int_{\widehat{B}_0\tau_0}^\infty\cdots\Big)^{1/q}\Big] = I_1 + I_2.\end{aligned}$$

Using (4.3), the first term I_1 is easily controlled as follows:

(5.4)
$$\begin{aligned}I_1 &\le C|K_2|^{1/p}\widehat{B}_0\tau_0 \le C[(s_2 - s_1)\kappa]^{-\frac{n+2}{2}}\Big[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^\eta(K_2)}\Big] \\ &\le C[(s_2 - s_1)\kappa]^{-\frac{n+2}{2}}\Big[\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,q}(K_2)}\Big].\end{aligned}$$

For the term I_2, we use (5.1) to control it as

(5.5)
$$\begin{aligned}I_2 &\le C\epsilon^{2/p}\Big(\int_{\widehat{B}_0\tau_0}^\infty s^q\big|\{(x,t) \in K_{s_2} : (\nabla u)_k(x,t) > s/4\}\big|^{q/p}\frac{ds}{s}\Big)^{1/q} \\ &\quad + C\epsilon^{2/p}\delta^{-\eta/p}\Big(\int_{\widehat{B}_0\tau_0}^\infty s^{(p-\eta)q/p}\Big\{\int_{\delta s/4}^\infty \tau^\eta\big|\{(x,t) \in K_2 : F(x,t) > \tau\}\big|\frac{d\tau}{\tau}\Big\}^{q/p}\frac{ds}{s}\Big)^{1/q} \\ &\le C\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,q}(K_{s_2})} \\ &\quad + C\delta^{-1}\Big(\int_{\widehat{B}_0\tau_0}^\infty (\delta s)^{(p-\eta)q/p}\Big\{\int_{\delta s/4}^\infty \tau^\eta\big|\{(x,t) \in K_2 : F(x,t) > \tau\}\big|\frac{d\tau}{\tau}\Big\}^{q/p}\frac{ds}{s}\Big)^{1/q} \\ &= C\Big[\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,q}(K_{s_2 R})} + J\Big],\end{aligned}$$

where

(5.6)
$$J = \delta^{-1}\Big(\int_{\widehat{B}_0\tau_0}^\infty (\delta s)^{(p-\eta)q/p}\Big\{\int_{\delta s/4}^\infty \tau^\eta\big|\{(x,t) \in K_2 : F(x,t) > \tau\}\big|\frac{d\tau}{\tau}\Big\}^{q/p}\frac{ds}{s}\Big)^{1/q}.$$

To control J, we consider the cases $q > p$ and $q < p$. When $q > p$, we use Hardy's inequality, Lemma 2.6 with

$$\kappa = \frac{q}{p} \geq 1, \quad r = \frac{(p-\eta)q}{p} > 0, \quad \text{and} \quad h(\tau) = \tau^{\eta-1}|\{(x,t) \in K_2 : F(x,t) > \tau\}|.$$

Observe that because $\eta < p$, we have $F \in L^{\eta}(K_2)$, and hence $h \in L^1((0,\infty))$. Therefore, Lemma 2.6 implies

$$\begin{aligned} J &\leq C\delta^{-1}\Big[\int_0^{\infty} s^{(p-\eta)q/p} s^{\eta q/p} \Big|\Big\{(x,t) \in K_2 : F(x,t) > s\Big\}\Big|^{q/p} \frac{ds}{s}\Big]^{1/q} \\ &= C\delta^{-1}\Big[\int_0^{\infty} s^{q} \Big|\Big\{(x,t) \in K_2 : F(x,t) > s\Big\}\Big|^{q/p} \frac{ds}{s}\Big]^{1/q} \\ &= C\delta^{-1} \|F\|_{L^{p,q}(K_2)}. \end{aligned}$$

This estimate, (5.3), (5.4), and (5.5) imply that

$$\begin{aligned} &\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} \leq \\ &\quad C_2\Big[\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,q}(K_{s_2})} + [(s_2-s_1)\kappa]^{-\frac{n+2}{2}}\Big(\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,q}(K_2)}\Big)\Big] \end{aligned}$$

for some constant C_2 depending only on Λ, n, p, q. Using this and taking ϵ sufficiently small such that $C_2\epsilon^{2/p} \leq 1/2$, we can apply the iteration lemma, Lemma 2.5, to obtain

$$\|(\nabla u)_k\|_{L^{p,q}(K_1)} \leq C\Big(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\Big).$$

Then, as before, we can send $k \to \infty$ to infer that

$$\|\nabla u\|_{L^{p,q}(K_1)} \leq C\Big(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\Big).$$

This estimate and (4.2) imply the desired estimate (1.14).

It now only remains to consider the case where $q \leq p$. In this case, by using Lemma 2.7 with

$$\kappa = \frac{p}{q} \geq 1, \quad r = \frac{\eta q}{p}, \quad \text{and} \quad h(\tau) = \Big|\Big\{(x,t) \in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p},$$

we see that

$$\begin{aligned} &\Big(\int_{\delta s/4}^{\infty} \tau^{\eta}\Big|\Big\{(x,t) \in K_2 : F(x,t) > \tau\Big\}\Big|\frac{d\tau}{\tau}\Big)^{q/p} \\ &\quad = \Big(\int_{\delta s/4}^{\infty}\Big[\tau^{\eta q/p}\Big|\Big\{(x,t) \in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\Big]^{p/q}\frac{d\tau}{\tau}\Big)^{q/p} \\ &\quad \leq C\Big[(s\delta)^{\eta q/p}\Big|\Big\{(x,t) \in K_2 : F(x,t) > \delta s/4\Big\}\Big|^{q/p} \\ &\qquad\qquad + \int_{\delta s/4}^{\infty}\tau^{\eta q/p}\Big|\Big\{(x,t) \in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\frac{d\tau}{\tau}\Big] \end{aligned}$$

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Plugging this estimate into the definition of J in (5.6), we infer that

$$J \le C\delta^{-1}\Big[\Big(\int_0^\infty (s\delta)^{(p-\eta)q/p}(s\delta)^{\eta q/p}\Big|\Big\{(x,t)\in K_2 : F(x,t) > \delta s/4\Big\}\Big|^{q/p}\frac{ds}{s}\Big)^{1/q}$$
$$+\Big\{\int_0^\infty (s\delta)^{(p-\eta)q/p}\Big(\int_{\delta s/4}^\infty \tau^{\eta q/p}\Big|\Big\{(x,t)\in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big\}^{1/q}\Big]$$
$$\le C\delta^{-1}\Big[\|F\|_{L^{p,q}(K_2)} + \Big\{\int_0^\infty (s\delta)^{(p-\eta)q/p}$$
$$\times\Big(\int_{\delta s/4}^\infty \tau^{\eta q/p}\Big|\Big\{(x,t)\in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\frac{d\tau}{\tau}\Big)\frac{ds}{s}\Big\}^{1/q}\Big]$$
$$= C\delta^{-1}[\|F\|_{L^{p,q}(K_2)} + J'].$$

We control J' by using Fubini's theorem as follows:

$$J' = \Big(\int_0^\infty \tau^{\eta q/p}\Big|\Big\{(x,t)\in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\Big(\int_0^{\tau/(4\delta)}(s\delta)^{(p-\eta)q/p}\frac{ds}{s}\Big)\frac{d\tau}{\tau}\Big)^{1/q}$$
$$\le C\Big(\int_0^\infty \tau^q\Big|\Big\{(x,t)\in K_2 : F(x,t) > \tau\Big\}\Big|^{q/p}\frac{d\tau}{\tau}\Big)^{1/q} = C\|F\|_{L^{p,q}(K_2)}.$$

Hence, we conclude in this case that $J \le C\|F\|_{L^{p,q}(K_2)}$. From this estimate, (5.3), (5.4), and (5.5), we again conclude that

$$\|(\nabla u)_k\|_{L^{p,q}(K_{s_1})} \le$$
$$C_3\Big[\epsilon^{2/p}\|(\nabla u)_k\|_{L^{p,q}(K_{s_2})} + [(s_2-s_1)\kappa]^{-\frac{n+2}{2}}\Big(|\,\|\nabla u\|_{L^2(K_2)} + \delta^{-1}\|F\|_{L^{p,q}(K_2)}\Big)\Big].$$

Arguing as before, by choosing ϵ such that $C_3\epsilon^{2/p} \le 1/2$ and then sending $k\to\infty$, we also obtain

$$\|\nabla u\|_{L^{p,q}(K_1)} \le C\Big(\|\nabla u\|_{L^2(K_2)} + \|F\|_{L^{p,q}(K_2)}\Big).$$

This and (4.2) give (1.14). The proof is therefore complete once we chose $\epsilon < \min\{\frac{1}{2C_1},\frac{1}{2C_2},\frac{1}{2C_3}\}^{p/2}$, where C_1, C_2, C_3 are constants defined above and dependent only on $\Lambda, M, q, n, \alpha_0$. ■

A Proofs of Lemmas 2.10 and 2.11

We only prove Lemma 2.11, since Lemma 2.10 is similar. We follow the approach used in [21,44]. To this end, some notation is needed. We fix a cut-off function $\varphi \in C_0^\infty(B_2)$ with the properties $\varphi(x) = 1$, for $x \in B_1$. For each $r > 0$, and each $x_0 \in \mathbb{R}^n$, we also define

$$\varphi_{x_0,2r}(x) = \varphi\big((x-x_0)/r\big).$$

As in [21], the following mean value of u will be used:

$$u_{x_0,2r}(t) = \Big(\int_{\Omega_{2r}(x_0)}\varphi_{x_0,2r}^2(x)dx\Big)^{-1}\int_{\Omega_{2r}(x_0)}u(x,t)\varphi_{x_0,2r}^2(x)dx. \tag{A.1}$$

Without loss of generality, we can assume that $R = 1$. Hence, we consider the equation

$$\begin{aligned} u_t - \operatorname{div}[\mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}] &= f, \quad \text{in } K_2,\\ \langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}, \vec{v}\rangle &= 0, \quad \text{on } T_2. \end{aligned} \tag{A.2}$$

We recall that if u is a solution of (A.2), we define

$$\mathbf{G}(x,t) = C_0^*(\gamma_0,n)[[u]]_{\mathrm{BMO}(K_1,1)}\mathbf{b}(x,t), \quad (x,t) \in K_2,$$

for some constant $C_0^*(\gamma_0,n) \geq 1$ defined as in (A.4) below.

Lemma A.1 *If u is a weak solution of (A.2), then for every $t_1, t_2 \in (-4,4)$ with $t_1 < t_2$ and every $x_0 \in \Omega_2$, and $\rho \in (0,1)$,*

$$\begin{aligned}
&|u_{x_0,2\rho}(t_2) - u_{x_0,2\rho}(t_1)| \\
&\quad \leq C(\Lambda,n)\Big[\frac{1}{\rho^{n+1}}\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}\Big(|\nabla u| + |\mathbf{F}|\Big)dz + \frac{1}{\rho^n}\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|f(x,t)|dz \\
&\qquad\qquad + \frac{1}{\rho^n}\Big(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|\mathbf{G}|^2dz\Big)^{1/2}\Big].
\end{aligned}$$

Proof With Steklov's average as in [5, 13, 38], we can formally use $\varphi_{x_0,2\rho}$ as a test function for (A.2) to obtain

$$\begin{aligned}
&\int_{\Omega_{2\rho}(x_0)} u(x,t_2)\varphi_{x_0,2\rho}(x)dx - \int_{\Omega_{2\rho}(x_0)} u(x,t_1)\varphi_{x_0,2\rho}(x)dx = \\
&\qquad - \int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}\langle \mathbf{A}(x,t,u,\nabla u) - (u - \overline{u}_{K_2})\mathbf{b} - \mathbf{F}, \nabla\varphi_{x_0,2\rho}\rangle dx \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)} f(x,t)\varphi_{x_0,2\rho}(x)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|u_{x_0,2\rho}(t_2) - u_{x_0,2\rho}(t_2)| \\
&\quad \leq \frac{C(\Lambda,n)}{\rho^{n+1}}\Big[\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}\Big(|\nabla u| + |\mathbf{F}|\Big)dz + \rho\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|f(x,t)|dz \\
&\qquad + \Big(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|\mathbf{b}|^2dz\Big)^{1/2}\Big(\int_{K_2}|u - \overline{u}_{K_2}|^2dz\Big)^{1/2}\Big] \\
&\quad \leq \frac{C(\Lambda,n)}{\rho^{n+1}}\Big[\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}\Big(|\nabla u| + |\mathbf{F}|\Big)dz + \rho\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|f(x,t)|dz \\
&\qquad + \Big(\int_{t_1}^{t_2}\int_{\Omega_{2\rho}(x_0)}|\mathbf{G}|^2dz\Big)^{1/2}\Big].
\end{aligned}$$

The proof is complete. ∎

Lemma A.2 (Caccioppoli type estimate) *For each $z_0 = (x_0,t_0) \in K_1$, for each $\rho \in (0,1)$, if u is a weak solution of (A.2), there holds*

$$\begin{aligned}
&\rho^{-2}\sup_{\tau\in\Gamma_{2\rho}(t_0)}\int_{\Omega_\rho(x_0)}|u(x,t) - u_{x_0,2\rho}|^2dx + \fint_{K_\rho(z_0)}|\nabla u|^2dz \\
&\quad \leq C(\Lambda,n)\Big[\rho^{-2}\fint_{K_{2\rho}(z_0)}|u - u_{x_0,2\rho}|^2dz + \Big(\fint_{K_{2\rho}}|\mathbf{G}|^\alpha dz\Big)^{2/\alpha} \\
&\qquad\qquad + \fint_{K_{2\rho}}|\mathbf{F}(x,t)|^2dz + \rho^2\Big(\fint_{K_{2\rho}}|f(x,t)|^{2n_*}dz\Big)^{1/n_*}\Big],
\end{aligned}$$

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where $\alpha = 2(1+\gamma_0)$ *with* $\gamma_0 > 0$ *is any fixed number and* $n_* = \frac{n+2}{n+4}$.

Proof Let $\sigma \in C_0^\infty(\Gamma_{2\rho}(t_0))$ be a cut-off function satisfying $0 \le \sigma \le 1$ and

$$\sigma(t) = 1 \text{ for all } t \in \Gamma_\rho(t_0), \quad \text{and} \quad |\sigma'(t)| \le \frac{100}{\rho^2} \text{ for all } t \in \Gamma_{2\rho}(t).$$

By using Steklov's average as in [5, 13, 38], we can formally use

$$\big(u - u_{x_0,2\rho}(t)\big)\sigma^2(t)\varphi^2_{x_0,2\rho}$$

as a test function for the equation (A.2) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\Omega_{x_0,2\rho}(x_0)} &\varphi^2_{2\rho}|u - u_{x_0,2\rho}(t)|^2\sigma^2(t)dx \\
&+ \partial_t u_{x_0,2\rho}(t)\sigma^2(t)\int_{\Omega_{2\rho}(x_0)}\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho})dx \\
= -&\int_{\Omega_{2\rho}(x_0)}\sigma^2(t)\big\langle \mathbf{A}(x,t,u,\nabla u) - \mathbf{b}u - \mathbf{F}, \nabla[(u - u_{x_0,2\rho})\varphi^2_{x_0,2\rho}]\big\rangle dx \\
&+ \int_{\Omega_{2\rho}(x_0)} f(x,t)\sigma^2(t)\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho})dx \\
&+ \int_{\Omega_{2\rho}(x_0)}\varphi^2_{2\rho}|u - u_{x_0,2\rho}(t)|^2\sigma'(t)\sigma(t)dx.
\end{aligned}$$

We observe that from Lemma A.1, $\partial_t u_{x_0,2\rho}(t)$ is integrable and is defined a.e. $t \in \Gamma_{2\rho}(t_0)$. Moreover, it follows immediately from (A.1) that

$$\int_{\Omega_{2\rho}(x_0)}\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho})dx = 0.$$

From this and since (1.11) holds on Ω_2, it follows that for each $\tau \in \Gamma_{2\rho}(t_0)$,

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_{x_0,2\rho}(x_0)} &\varphi^2_{x_0,2\rho}|u(x,\tau) - u_{x_0,2\rho}(\tau)|^2\sigma^2(\tau)dx \\
&+ \int_{t_0-4\rho^2}^{\tau}\int_{\Omega_{2\rho}(x_0)}\langle \mathbf{A}(x,t,u,\nabla u), \nabla u\rangle\sigma^2(t)\varphi^2_{x_0,2\rho}dz \\
= -2&\int_{t_0-4\rho^2}^{\tau}\int_{\Omega_{2\rho}(x_0)}\langle \mathbf{A}(x,t,u,\nabla u), \nabla\varphi_{x_0,2\rho}\rangle(u - u_{x_0,2\rho})\varphi_{x_0,2\rho}\sigma^2(t)dz \\
&+ \int_{t_0-4\rho^2}^{\tau}\int_{\Omega_{2\rho}(x_0)}\langle (u - \overline{u}_{K_{2\rho}(z_0)})\mathbf{b} + \mathbf{F}, \nabla[(u - u_{x_0,2\rho})\varphi^2_{x_0,2\rho}]\rangle dz \\
&+ \int_{t_0-4\rho^2}^{\tau}\int_{\Omega_{2\rho}(x_0)} f(x,t)\sigma^2(t)\varphi^2_{x_0,2\rho}(u - u_{x_0,2\rho})dz \\
&+ \int_{t_0-4\rho^2}^{\tau}\int_{\Omega_{2\rho}(x_0)}\varphi^2_{x_0,2\rho}|u - u_{x_0,2\rho}(t)|^2\sigma'(t)\sigma(t)dz.
\end{aligned}$$

This and the conditions (1.2) and (1.3) imply that

$$
\begin{aligned}
(\text{A.3}) \qquad & \sup_{\tau\in\Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} \varphi^2_{x_0,2\rho}|u(x,\tau)-u_{x_0,2\rho}(\tau)|^2\sigma^2(\tau)dx \\
&\quad + \int_{K_{2\rho}(z_0)} |\nabla(u-u_{x_0,2\rho})|^2\varphi^2_{x_0,2\rho}\sigma^2(t)dz \\
&\le C(\Lambda)\Big[\int_{K_{2\rho}(z_0)} |\nabla u|\varphi_{x_0,2\rho}|\nabla\varphi_{x_0,2\rho}|\,|u-u_{x_0,2\rho}|\sigma^2(t)dz \\
&\quad + \int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}|\,|u-\overline{u}_{K_{2\rho}}|+|\mathbf{F}|\Big) \\
&\qquad \times\Big(|\nabla u|\varphi^2_{x_0,2\rho}+2|u-u_{x_0,2\rho}|\,|\nabla\varphi_{x_0,2\rho}|\varphi_{x_0,2\rho}\Big)\sigma^2(t)dz \\
&\quad + \int_{K_{2\rho}(z_0)} |f(x,t)|\sigma^2(t)\varphi^2_{x_0,2\rho}|u-u_{x_0,2\rho}|dz \\
&\quad + \int_{K_{2\rho}(z_0)} \varphi^2_{x_0,2\rho}|u-u_{x_0,2\rho}(t)|^2|\sigma'(t)|\sigma(t)dz\Big].
\end{aligned}
$$

We now control the first two terms on the right-hand side of (A.3). Let $\epsilon > 0$ be sufficiently small, which will be determined. Use Hölder's inequality and Young's inequality, we obtain

$$
\begin{aligned}
&\int_{K_{2\rho}(z_0)} |\nabla u|\varphi_{x_0,2\rho}|\nabla\varphi_{x_0,2\rho}|\,|u-u_{x_0,2\rho}|\sigma^2(t)dz \le \\
&\quad \epsilon\int_{K_{2\rho}(z_0)} |\nabla u|^2\varphi^2_{x_0,2\rho}\sigma^2(t)dz + C(\epsilon)\int_{K_{2\rho}(z_0)} |\nabla\varphi_{x_0,2\rho}|^2|u-u_{x_0,2\rho}|^2\sigma^2(t)dz.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}|\,|u-\overline{u}_{K_{2\rho}(z_0)}|+|\mathbf{F}|\Big)\Big(|\nabla u|\varphi^2_{x_0,2\rho}+2|u-u_{x_0,2\rho}|\,|\nabla\varphi_{x_0,2\rho}|\varphi_{x_0,2\rho}\Big)\sigma^2(t)dz \\
&\quad \le \epsilon\int_{K_{2\rho}(z_0)} |\nabla u|^2\varphi^2_{x_0,2\rho}\sigma^2(t)dz \\
&\qquad + C(\epsilon)\Big[\int_{K_{2\rho}(z_0)} \Big(|\mathbf{b}|^2|u-\overline{u}_{K_{2\rho}(z_0)}|^2+|\mathbf{F}|^2\Big)\varphi^2_{x_0,2\rho}\sigma^2(t)dz \\
&\qquad\qquad + \int_{K_{2\rho}(z_0)} |u-\overline{u}_{x_0,2\rho}|^2|\nabla\varphi_{x_0,2\rho}|^2\sigma^2(t)dz\Big].
\end{aligned}
$$

Applying Hölder's inequality again for the term involving $\mathbf{b}$, we see that

$$
\begin{aligned}
(\text{A.4}) \qquad & ⨍_{K_{2\rho}(z_0)} |\mathbf{b}|^2|u-\overline{u}_{K_{2\rho}(z_0)}|^2\varphi^2_{x_0,2\rho}\sigma^2(t)dz \\
&\le \Big(⨍_{K_{2\rho}(z_0)} |\mathbf{b}|^\alpha dz\Big)^{2/\alpha}\Big(⨍_{K_{2\rho}(z_0)} |u-\overline{u}_{K_{2\rho}(z_0)}|^{\frac{2\alpha}{\alpha-2}}\Big)^{(\alpha-2)/\alpha} \\
&\le C_0^*(n,\gamma_0)[[u]]^2_{\mathrm{BMO}(K_1,1)}\Big(⨍_{K_{2\rho}(z_0)} |\mathbf{b}|^\alpha dz\Big)^{2/\alpha} \\
&= \Big(⨍_{K_{2\rho}(z_0)} |\mathbf{G}|^\alpha dz\Big)^{2/\alpha}.
\end{aligned}
$$

Then, by writing $w = |u(x,t) - u_{x_0,2\rho}(t)||\varphi_{x_0,2\rho}(x)\sigma(t)$ and collecting all last estimates together with (A.3), and the choice of ϵ sufficiently small, we infer that

(A.5)

$$\rho^{-2} \sup_{\tau\in\Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2 dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2 dz \\ \le C(\Lambda,n)\Big[\fint_{K_{2\rho}(z_0)} |u - u_{x_0,2\rho}|^2 \Big(\varphi^2_{x_0,2\rho}|\sigma'(t)|\sigma(t) + |\nabla\varphi_{x_0,2\rho}|^2\sigma^2(t)\Big) \\ + \Big(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^\alpha dz\Big)^{2/\alpha} + \fint_{K_{2\rho}(z_0)} |\mathbf{F}|^2 \varphi^2_{x_0,2\rho}\sigma^2(t) dz \\ + \fint_{K_{2\rho}(z_0)} |f(x,t)||u - u_{x_0,2\rho}|\varphi^2_{x_0,2\rho}\sigma^2(t) dz\Big].$$

Finally, it remains to control the last term on the right hand side of (A.5). This, however, can be done exactly the same way as in (3.17) using Hölder's inequality, Sobolev imbedding [31, eqn (3.2), p. 74]), and Young's inequality with ϵ sufficiently small. We then obtain

$$\fint_{K_{2\rho}(z_0)} |f(x,t)||u - u_{x_0,2\rho}|\varphi^2_{x_0,2\rho}\sigma^2(t) dz \\ \le \epsilon\Big[\rho^{-2} \sup_{t\in\Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2 dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2 dxdt\Big] \\ + C(n,\epsilon)\rho^2 \Big(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*} dz\Big)^{1/n_*}.$$

Then with ϵ sufficiently small, it follows from (A.5) and the last estimate that

$$\rho^{-2} \sup_{\tau\in\Gamma_{2\rho}(t_0)} \int_{\Omega_{2\rho}(x_0)} |w(x,t)|^2 dx + \fint_{K_{2\rho}(z_0)} |\nabla w|^2 dz \\ \le C(\Lambda,n)\Big[\fint_{K_{2\rho}(z_0)} |u - u_{x_0,2\rho}|^2 \Big(\varphi^2_{x_0,2\rho}|\sigma'(t)|\sigma(t) + |\nabla\varphi_{x_0,2\rho}|^2\sigma^2(t)\Big) \\ + \Big(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^\alpha dz\Big)^{2/\alpha} + \fint_{K_{2\rho}(z_0)} |\mathbf{F}|^2 \varphi^2_{x_0,2\rho}\sigma^2(t) dz \\ + \rho^2 \Big(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*} dz\Big)^{1/n_*}\Big].$$

The proof of the lemma is now complete. ∎

Lemma A.3 *There is $\mu \in (1,2)$ depending only on n such that for every $\epsilon > 0$ and $\alpha = 2(1+\gamma_0)$ with some $\gamma_0 > 0$, there exists $C_0 = C_0(\Lambda,n,\epsilon)$ such that the following holds. For every $z_0 = (x_0,t_0) \in K_1$, for each $\rho \in (0,1/4)$, if u is a weak solution of (A.2), then*

$$\fint_{K_\rho(z_0)} |\nabla u|^2 dz \le \epsilon \fint_{K_{4\rho}} |\nabla u|^2 dz + C_0\Big[\Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^\mu dz\Big)^{2/\mu} + \Big(\fint_{K_{4\rho}} |\mathbf{G}|^\alpha dz\Big)^{2/\alpha} \\ + \fint_{K_{4\rho}} |\mathbf{F}(x,t)|^2 dz + \mathcal{G}(f)^2 \Big(\fint_{K_{4\rho}} |f(x,t)|^{2n_*} dz\Big)\Big],$$

where $n_* = \frac{n+2}{n+4}$, *and* $\mathcal{G}(f)$ *is defined in* (4.1).

Proof For simplicity in writing, let us denote

$$\mathcal{F}(2\rho) = \Big(\fint_{K_{2\rho}(z_0)} |\mathbf{G}|^{\alpha} dz \Big)^{2/\alpha} + \fint_{K_{2\rho}(z_0)} |\mathbf{F}(x,t)|^2 dz + \mathcal{G}(f)^2 \Big(\fint_{K_{2\rho}(z_0)} |f(x,t)|^{2n_*} dz \Big).$$

By Poincaré's inequality, we see that

$$\rho^{-2} \fint_{K_{2\rho}(z_0)} |u - u_{x_0,2\rho}|^2 dz \le C(n) \fint_{K_{2\rho}(z_0)} |\nabla u|^2 dz.$$

This estimate, Lemma A.2, and (4.9) imply that

$$\text{(A.6)} \qquad \rho^{-2} \sup_{t \in \Gamma_\rho(t_0)} \fint_{\Omega_\rho(x_0)} |u - u_{x_0,2\rho}|^2 dx \le C(\Lambda, n) \Big[\fint_{K_{2\rho}(z_0)} |\nabla u|^2 dz + \mathcal{F}(2\rho) \Big].$$

We now let $\widehat{u} = u - u_{x_0,2\rho}$. Then observe that

$$\text{(A.7)} \qquad \begin{aligned} \rho^{-2} \fint_{K_{2\rho}(z_0)} |\widehat{u}|^2 dz &\le \rho^{-2} \Big[\sup_{t \in \Gamma_{2\rho}(t_0)} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^2 dx \Big)^{1/2} \Big] \\ &\quad \times \Big[\fint_{\Gamma_{2\rho}(t_0)} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^2 dx \Big)^{1/2} dt \Big] \\ &\le C\rho^{-1} \Big[\Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz \Big)^{1/2} + \mathcal{F}(4\rho)^{1/2} \Big] \\ &\quad \times \Big[\fint_{\Gamma_{2\rho}(t_0)} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^2 dx \Big)^{1/2} dt \Big], \end{aligned}$$

where we have used (A.6) in the last estimate with ρ replaced by 2ρ. We now control the last multiplier on the right-hand side of (A.7). To this end, if $n > 2$, and we take $2^* = \frac{2n}{n-2}$, and when $n = 2$, we can take 2^* to be any number that is greater than 2. Then let $\mu \in (1,2)$ such that $\frac{1}{2^*} + \frac{1}{\mu} = 1$ (observe that $\mu = \frac{2n}{n+2}$ if $n > 2$). From this, Hölder's inequality, Poincaré's inequality, and the Sobolev–Poincaré inequality, it follows that

$$\begin{aligned} &\rho^{-1} \fint_{\Gamma_{2\rho}(t_0)} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^2 dx \Big)^{1/2} dt \\ &\quad \le \rho^{-1} \fint_{\Gamma_{2\rho}(t_0)} \Big[\Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^{\mu} dx \Big)^{\frac{1}{2\mu}} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^{2^*} dx \Big)^{\frac{1}{2 \cdot 2^*}} \Big] dt \\ &\quad \le C(n) \rho^{-1/2} \fint_{\Gamma_{2\rho}(t_0)} \Big[\Big(\fint_{\Omega_{2\rho}(x_0)} |\nabla u|^{\mu} dx \Big)^{\frac{1}{2\mu}} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^{2^*} dx \Big)^{\frac{1}{2 \cdot 2^*}} \Big] dt \\ &\quad \le C(n) \fint_{\Gamma_{2\rho}(t_0)} \Big[\Big(\fint_{\Omega_{2\rho}(x_0)} |\nabla u|^{\mu} dx \Big)^{\frac{1}{2\mu}} \Big(\fint_{\Omega_{2\rho}(x_0)} |\nabla u|^2 dx \Big)^{\frac{1}{4}} \Big] dt. \end{aligned}$$

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We then use Hölder's inequality twice for the time integration in the last estimate to infer that

$$\begin{aligned}
&\rho^{-1} \fint_{\Gamma_{2\rho}(t_0)} \Big(\fint_{\Omega_{2\rho}(x_0)} |\widehat{u}|^2 dx \Big)^{1/2} dt \\
&\quad \le C(n) \Big[\Big(\fint_{K_{2\rho}(z_0)} |\nabla u|^\mu dz \Big)^{\frac{1}{2\mu}} \Big] \Big[\fint_{\Gamma_{2\rho}(t_0)} dt \Big(\fint_{\Omega_{2\rho}(x_0)} |\nabla u|^2 dx \Big)^{\frac{1}{2}\frac{\mu}{2\mu-1}} \Big]^{\frac{2\mu-1}{2\mu}} \\
&\quad \le C(n) \Big(\fint_{K_{2\rho}(z_0)} |\nabla u|^\mu dz \Big)^{\frac{1}{2\mu}} \Big(\fint_{K_{2\rho}(z_0)} |\nabla u|^2 dz \Big)^{\frac{1}{4}}.
\end{aligned}$$

The last estimate together with (A.7) imply that

$$\begin{aligned}
&\rho^{-2} \fint_{K_{2\rho}(z_0)} |\widehat{u}|^2 dz \\
&\quad \le C(n) \Big[\Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz \Big)^{1/2} + \mathcal{F}(4\rho)^{1/2} \Big] \\
&\qquad \times \Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^\mu dz \Big)^{\frac{1}{2\mu}} \Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz \Big)^{\frac{1}{4}} \\
&\quad \le \epsilon \fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz + C(n,\epsilon) \Big[\Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^\mu dz \Big)^{\frac{2}{\mu}} + \mathcal{F}(4\rho) \Big],
\end{aligned}$$

From this estimate and Lemma A.2, we see that

$$\fint_{K_\rho(z_0)} |\nabla u|^2 dz \le \epsilon \fint_{K_{4\rho}(z_0)} |\nabla u|^2 dz + C(\Lambda, n, \epsilon) \Big[\Big(\fint_{K_{4\rho}(z_0)} |\nabla u|^\mu dz \Big)^{\frac{2}{\mu}} + \mathcal{F}(4\rho) \Big].$$

Hence, Lemma A.3 is proved. ■

Proof of Lemma 2.11 The proof follows from Lemma A.3 and the standard Gehring type lemma (for example, see [21, Proposition 1.3], [20, Proposition 5.1], or [23, Corollary 6.1, p. 204]). ■

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