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REPRESENTATION-DIRECTED DIAMONDS

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Abstract

A module over a finite-dimensional algebra is called a 'diamond' if it has a simple top and a simple socle.Using covering theory, the classification of all diamonds for algebras of finite representation type over algebraically closed fields can be reduced to representation-directed algebras. We prove a criterion referring to the positive roots of the corresponding Tits quadratic form, which makes it easy to check whether a representation-directed algebra has a faithful diamond. Using an implementation of this criterion in the CREP program system on representation theory, we are able to classify all exceptional representation-directed algebras having a faithful diamond. We obtain a list of 157 algebras up to isomorphism and duality. The 52 maximal members of this list are presented at the end of this paper.

1. *Introduction and main result*

Following the conventions used in [**11**], a (right) module D over an associative ring A is said to be a *diamond*, provided that it has an essential simple submodule and a superfluous maximal submodule. Obviously, any diamond is indecomposable. If A happens to be a finite-dimensional algebra over a field k, then a module D is a diamond if and only D is a finite-dimensional module with a simple socle and a simple top. (Recall that the top of a module is the factor module by the Jacobson radical.) Since any indecomposable module of length 2 is a diamond, a finite-dimensional algebra A will usually have infinitely many isomorphism classes of diamonds. On the other hand, an algebra A of finite representation type (that is, A has only finitely many indecomposable modules up to isomorphism) can have only finitely many isomorphism classes of diamonds. At least, if the field k is algebraically closed, the algebras of finite representation type have been well studied. We refer to [**9**] for an introduction to this theory.

Using the covering theory developed in [**3**], we may reduce the study of modules over finite-dimensional algebras A of finite representation type over an algebraically closed field k to the case that A is representation-directed. In particular, any diamond over an algebra of finite representation type is obtained from a diamond over a representation-directed algebra by the application of the push-down functor associated with the universal Galois covering. Recall that, following [**10**], an algebra A is said to be *representation-directed* if there exists no sequence $X_0, \ldots, X_n$ of indecomposable finite-dimensional A-modules with $n > 0$ and $X_0 \cong X_n$ such that for each $i = 1, \ldots, n$ there is a homomorphism $X_{i-1} \to X_i$ which is neither an isomorphism nor zero.

Since factor algebras of representation-directed algebras are representation-directed, in order to find all diamonds over representation-directed algebras it suffices to look at

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all representation-directed algebras which have faithful indecomposable modules, and to check which of these modules are diamonds. Fortunately, all the representation-directed algebras over an algebraically closed field which have an indecomposable faithful module have been classified. They appear in 24 families (see [1]), together with many exceptions in low dimensions (see [4]). These exceptions were found by a computer program, and are accessible via a data base in the CREP system (see [6]). Hence, it remains only to find out which of the algebras appearing in the families and in the data base have a faithful indecomposable module which is a diamond. It is the aim of this paper to present a convenient criterion to determine when this happens.

Theorem. *Let A be a representation-directed algebra over an algebraically closed field k. Then A is obtained from a representation-directed algebra having a faithful diamond by a reorientation of the arms if and only if the vector $\varepsilon = (1, \ldots, 1)$ is the only sincere positive 1-root of the Tits form of A.*

We explain all the notation in the next section. Here, we note that the lists mentioned above give, for each algebra, the maximal (with respect to the natural product order on $\mathbb{Z}^r$) positive roots of the associated Tits form. It is observed in [11] that, of the 24 families, only those which are listed in [10] as (Bo1), (Bo15), (Bo16), (Bo17), (Bo19), (Bo20) and (Bo21) have a faithful diamond. An implementation of the above criterion, which searches the data base in the CREP system, yields a list of 157 exceptional algebras (up to isomorphism and duality) having a faithful diamond. As we shall explain in the next section, any such algebra is the incidence algebra of a finite, partially ordered set. In order to present our list in a compact way, we display the Hasse diagrams of its 52 maximal members in Figures 2 and 3, at the end of this paper.

2. *Representation-directed algebras*

We use the terminology of [10]. For the study of diamonds, we may assume, without loss of generality, that our given algebra A is basic and connected. It is well-known (see [8]) that any basic finite-dimensional algebra A over an algebraically closed field k up to isomorphism can be written as $k\vec{\Delta}/I$, where $\vec{\Delta}$ is a finite quiver and I is an admissible ideal of the path algebra $k\vec{\Delta}$. We denote by $a(x, y)$ the number of arrows from x to y in $\vec{\Delta}$, and by $b(x, y)$ the number of minimal generators of I starting in x and ending in y. After labelling the vertices of $\vec{\Delta}$ by $1, \ldots, r$ we obtain a quadratic form $q_A : \mathbb{Z}^r \to \mathbb{Z}$, called the *Tits form*, given by $q_A(x) = \sum_{i=1}^r x_i^2 - \sum_{i,j=1}^r a(i, j)x_i x_j + \sum_{i,j=1}^r b(i, j)x_i x_j$ for $x = (x_1, \ldots, x_r) \in \mathbb{Z}^r$. If A is representation-directed, then by [2] q_A is *weakly positive* (that is, $q_A(x) > 0$ for all $0 \neq x \in \mathbb{Z}^r$ with non-negative entries). Consequently, in this case q_A has only finitely many *positive* 1-*roots*; that is, vectors $x \in \mathbb{Z}^r$ with non-negative entries satisfying $q_A(x) = 1$.

The A-modules can be identified with the contravariant representations X of $\vec{\Delta}$, such that $X(\varrho) = 0$ for all elements ϱ of I (see [10]). Using this identification, the *dimension vector* $\mathbf{dim}\, X \in \mathbb{Z}^r$ is given by $(\mathbf{dim}\, X)_i = \dim_k X(i)$ for all vertices $i = 1, \ldots, r$. By [2], for representation-directed A, the map **dim** yields a bijection from the set of isomorphism classes of indecomposable A-modules to the set of positive 1-roots of q_A. A vector x in $\mathbb{Z}^r$ is called *sincere* if $x_i \neq 0$ for all $i = 1, \ldots, r$. Analogously, an A-module X is called *sincere* provided that $X(i) \neq 0$ for all $i = 1, \ldots, r$. Thus the map **dim** also yields a bijection between the set of isomorphism classes of sincere indecomposable A-modules and the set of sincere positive 1-roots of q_A.

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It is well-known (see, for example, [10]) that an indecomposable module over a representation-directed algebra is faithful if and only if it is sincere. Moreover, it is shown in [10] that a representation-directed algebra which has an indecomposable sincere module is *simply connected* (see [3]). Hence A is *completely separating* in the terminology of [5], and can therefore be written as kS/J, where S is a finite partially ordered set and J is an ideal of the incidence algebra kS generated by elements (y, x) such that there is z in S satisfying $y < z < x$. Recall that the incidence algebra kS is the vector space with the basis given by all pairs (y, x) such that $y \leqslant x$ in S. The product $(z, y)(y', x)$ in kS is (z, x) for $y = y'$, and 0 otherwise. For $A = kS/J$, the quiver $\vec{\Delta}$ of A is the Hasse diagram of S, and we can also write A as $A = k\vec{\Delta}/I$, where I is the ideal of $k\vec{\Delta}$ generated by all differences $u - v$ of paths in $\vec{\Delta}$ with the same origin and terminus, together with all paths w starting in x and ending in y such that there is a generator (y, x) of J.

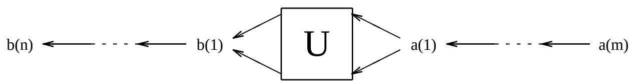
Let A be an algebra of the shape $A = kS/J$ for a finite partially ordered set S. With any subset T of S which is convex and *relation-free* (that is, for each generator (y, x) of J, x and y may not both lie in T) there is an associated *indicator module* δ_T, which is defined by $\delta_T(x) = k$ for all $x \in T$, and $\delta_T(x) = 0$ otherwise. Moreover, the arrows $\alpha : x \to y$ of $\vec{\Delta}$ are sent to the identity of k for x and y in T, and 0 otherwise.

If S is a finite partially ordered set with a unique maximal and a unique minimal element, then the algebra kS has a sincere diamond, namely δ_S. The following proposition establishes the converse of this observation for representation-directed algebras.

Proposition 1. *If $A = kS/J$ is a representation-directed algebra with a sincere diamond X, then $J = 0$, S has a unique minimal and a unique maximal element, X is isomorphic to δ_S, and X is up to isomorphism the only sincere indecomposable A-module.*

Proof. Since X is a diamond, there is an epimorphism $\phi : P(x) \to X$ for some element x of S where $P(x)$ comprises the indecomposable projective modules associated with x. It is easy to see that $P(x) \cong \delta_T$, where T is the subset of all y in S such that $y \geqslant x$ and $(x, y) \notin J$. The sincerity of X shows that $S = \operatorname{supp} X \subseteq \operatorname{supp} P(x) = T$. (For an A-module X, we denote by ‘$\operatorname{supp} X$’ the set of all elements y of S with $X(y) \neq 0$.) Hence x is the unique minimal element of S, and moreover $J = 0$ because $T = S$ is relation-free. Dually, S has a unique maximal element. If ϕ were not an isomorphism, then its kernel would be non-zero, and X would not be sincere. Finally, let N be another sincere indecomposable A-module. Since X is projective and dually also injective, there exist non-zero homomorphisms $X \to N$ and $N \to X$. Consequently, X and N have to be isomorphic because A is representation-directed. □

The above proposition shows that, if $A = kS$ is a representation-directed algebra with a sincere diamond, then there are two possible cases. Either S is a finite chain, or the Hasse diagram $\vec{\Delta}$ of S has the shape shown below, where $a(1)$ has at least two lower neighbors, $b(1)$ has at least two upper neighbors, and all elements u of U satisfy $a(1) \geqslant u \geqslant b(1)$.



If $A = k\vec{\Delta}/I$, where $\vec{\Delta}$ is a quiver formed by attaching one end of a quiver C of type $\mathbb{A}_r$ to a quiver $\vec{\Delta}'$, and the admissible ideal I is generated by elements in $k\vec{\Delta}'$, then the Tits form for A is independent of the orientation of C. Algebras obtained from a given algebra A by changing the orientation for various subquivers C in this fashion are said to be obtained

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from A by a *reorientation of arms*. Thus we have shown the following corollary to the theorem.

Corollary. *If $A = kS$ is a representation-directed algebra with a sincere diamond, and if $A' = kS'$ is obtained from A by a reorientation of arms, then A' is a representation-directed algebra such that $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of the Tits form of A'.*

3. *The combinatorial part of the proof*

Lemma 1. *If $A = kS/J$ is a representation-directed algebra, and if $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of q_A, then $J = 0$.*

Proof. If X is the indecomposable A-module with $\mathbf{dim}\, X = \varepsilon$, then by [5] we know that $X \cong \delta_{\operatorname{supp} X} = \delta_S$. Hence S is relation-free, and therefore $J = 0$. □

Lemma 2. *Suppose that $A = kS$ is a representation-directed algebra, that $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of q_A, and that x is an element of S which is neither minimal nor maximal. If S' is the full subposet of S associated with $S \setminus \{x\}$, then $A' = kS'$ is representation-directed, and $\varepsilon' = (1, \dots, 1)$ is the only positive sincere 1-root of $q_{A'}$.*

Proof. It is clear that S' is connected, kS' is representation-directed, and $X' = \delta_{S'}$ is a sincere indecomposable kS'-module. We assume that there is another sincere indecomposable kS'-module Y', different from X'. From [5], there has to be an element y of S', satisfying $\dim_k Y'(y) \geqslant 2$. Let L be the left adjoint of the restriction functor from the category of kS-modules to the category of kS'-modules. Hence LY' is an indecomposable kS-module such that $LY'(z) = Y'(z)$ for all elements z of S different from x. From [2], the support of LY' is convex, and therefore LY' is a sincere module, not isomorphic to δ_S, a contradiction. □

Before continuing, we have another prerequisite. Let $(-, -)_A$ be the symmetric bilinear form associated with the quadratic form q_A, and let σ_i be the reflection with respect to $(-, -)_A$ along the canonical base vector $e(i)$ for $i = 1, \dots, r$. This means that $\sigma_i(x) = x - 2(e(i), x)_A e(i)$ for all x in $\mathbb{Z}^r$. For x a 1-root, the vector $\sigma_i(x)$ is also a 1-root of q_A. In particular, if ε is the only sincere positive 1-root of q_A, then $2(e(i), \varepsilon)_A \geqslant 0$ for all $i = 1, \dots, r$, because otherwise $\sigma_i(\varepsilon)$ would be another sincere positive 1-root.

Proposition 2. *If $A = kS$ is a representation-directed algebra such that $\varepsilon = (1, \dots, 1)$ is the only positive 1-root of the Tits form q_A, then A is obtained from a representation-directed algebra with a sincere diamond by a reorientation of arms.*

Proof. We proceed by induction on the cardinality r of S, and observe that for $r = 1$ there is nothing to prove. For $r > 1$ we first consider the case where every element of S is either maximal or minimal. Thus $kS = k\vec{\Delta}$ is a hereditary algebra of finite representation type. By Gabriel's theorem (see [7]) the graph Δ underlying $\vec{\Delta}$ has to be one of the Dynkin diagrams $\mathbb{A}_r$, $\mathbb{D}_r$ or $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. But for all of these diagrams except $\mathbb{A}_r$, the corresponding Tits forms have more than one sincere positive 1-root.

Now we have to deal with the case where there exists an element x of S which is neither minimal nor maximal. By Lemma 2, we can apply induction to the full subposet S' of S associated with $S \setminus \{x\}$. Let us consider the Hasse diagram $\vec{\Delta}'$ of S'. The case where Δ' is a graph of type $\mathbb{A}_{r-1}$ is clear. Otherwise, $\vec{\Delta}'$ has the shape given in Figure 1.

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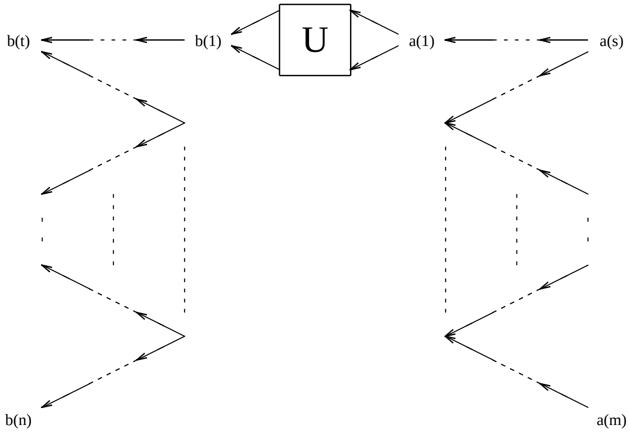

Figure 1: Hasse diagram of S'

We denote by S^u the set of upper neighbours, and by S^l the set of lower neighbours of x in S. The sets S^u and S^l are disjoint non-empty antichains in S', such that $z \leqslant y$ for each z in S^l and y in S^u. That kS is representation-directed implies immediately that $|S^l| + |S^u| \leqslant 3$. If $S^l \cup S^u$ is contained in $U' := U \cup \{a(1), \ldots, a(s), b(1), \ldots, b(t)\}$, then the claim is clear. So we assume otherwise, and distinguish several cases.

- *Case* 1: $|S^l| + |S^u| = 2$. Since both S^l and S^u are non-empty, we have $S^l = \{z\}$ and $S^u = \{y\}$ with $z < y$ in S'. Up to duality, we may assume that $y \notin U'$.
- *Case* 1.1: $y \in \{b(t+1), \ldots, b(n)\}$; hence $z \in \{b(t), \ldots, b(n)\}$. If there is an arrow $y \to z$ in $\vec{\Delta}'$, then in $\vec{\Delta}$ it is replaced by two arrows $y \to x$ and $x \to z$. Thus $\vec{\Delta}$ has the correct shape. Otherwise, y has two lower neighbours, and z has two upper neighbours in $\vec{\Delta}$. Consequently, Δ contains a subgraph of type $\widetilde{\mathbb{D}}_p$ which is not bound by relations. We arrive at a contradiction to A being of finite representation type.
- *Case* 1.2: $y \in \{a(s+1), \ldots, a(n)\}$; hence $z \in \{a(s+1), \ldots, a(m)\}$. The same arguments as those used in Case 1.1 can be applied.
- *Case* 2: $|S^l| + |S^u| = 3$. Up to duality, we may now assume that $S^l = \{z_1, z_2\}$, and that $S^u = \{y\}$. Thus, either $y \notin U'$, or (without loss of generality) $z_2 \notin U'$. In both situations, we observe that there does not exist any element w of S' satisfying $z_1 \geqslant w \leqslant z_2$. Therefore, $\vec{\Delta}$ contains a full subquiver of the following shape, where no other arrows, and no relations, start or stop at x.

$$\begin{array}{ccccc} & & y & & \\ & & \downarrow & & \\ z_1 & \leftarrow & x & \rightarrow & z_2 \end{array}$$

We obtain a contradiction by the calculation $2(e(x), \varepsilon)_A = 2 - 3 = -1$. □

Remark. If kS is a representation-directed incidence algebra with a sincere diamond, then for any full subposet U of S with a unique minimal and maximal element, the algebra kU is representation-directed with a sincere diamond as well. Thus, in the list given in Figures 2 and 3, we present only the Hasse diagrams of the maximal posets S such that kS is an exceptional representation-directed algebra having a sincere diamond.

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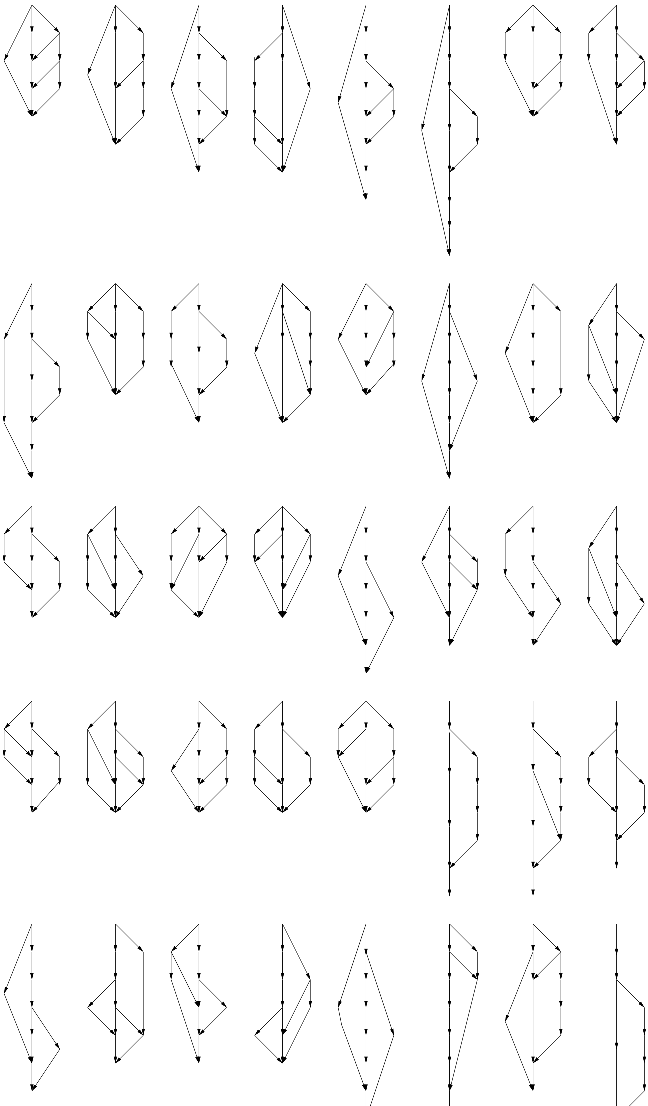

Figure 2: Hasse diagrams for the maximal exceptional representation-directed algebras having a faithful diamond (*Continued in Figure* 3)

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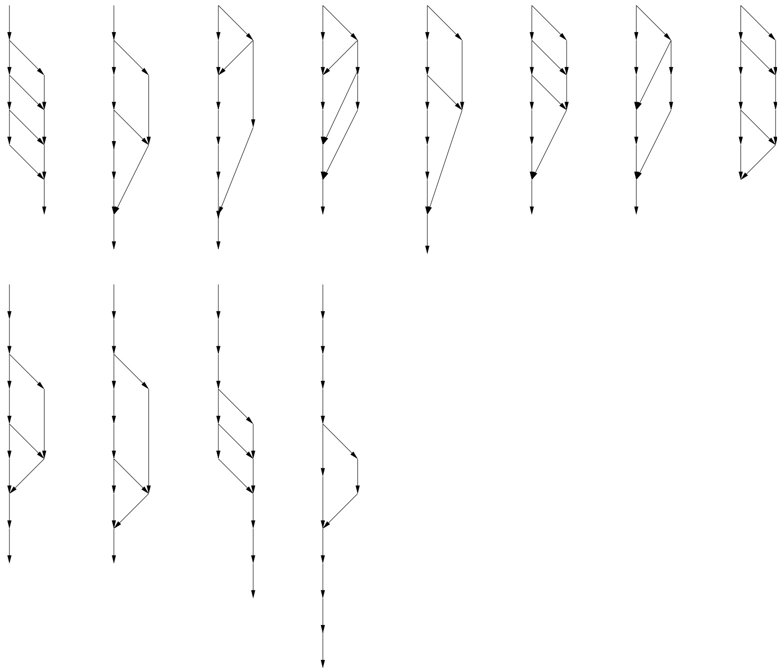

Figure 3: Hasse diagrams for the maximal exceptional representation-directed algebras having a faithful diamond (*Continued from Figure 2*)

References

1. K. Bongartz, 'Treue einfach zusammenhängende Algebren I', *Comment Math. Helv.* 57 (1982) 282–330. 15
2. K. Bongartz, 'Algebras and quadratic forms', *J. London Math. Soc.* 28 (1983) 461–469. 15, 15, 17
3. K. Bongartz and P. Gabriel, 'Covering spaces in representation theory', *Invent. Math.* 65 (1982) 331–378. 14, 16
4. P. Dräxler, 'Aufrichtige gerichtete Ausnahmealgebren', *Bayreuth. Math. Schr.* 29 (1989). 15
5. P. Dräxler, Sur les algèbres exceptionalles de Bongartz', *C. R. Acad. Sci. Paris Sér. I Math.* 311 (1990) 495–498. 16, 17, 17
6. P. Dräxler and R. Nörenberg, 'Classification problems in the combinatorial representation theory of finite-dimensional algebras', *Computational methods for representations of groups and algebras*, Progr. Math. 73 (Birkhaüser, Boston, MA, 1999) 3–28. 15
7. P. Gabriel, 'Unzerlegbare Darstellungen I', *Manuscr. Math.* 6 (1972) 71–103. 17
8. P. Gabriel, *Auslander–Reiten sequences and representation-finite algebras*, Lecture Notes in Math. 831 (Springer, New York, 1980) 1–71. 15

https://doi.org/10.1112/S1461157000000784 Published online by Cambridge University Press

9. P. Gabriel and A. V. Roiter, ‘Representations of finite-dimensional algebras’, *Encyclopedia Math. Sci.* Vol. 73, Algebra VIII (Ed. A. I. Kostrikin and I. V. Shafarevich, Springer, Berlin/Heidelberg/New York, 1992). 14

10. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Math. 1099 (Springer, New York, 1984). 14, 15, 15, 15, 16, 16

11. C. M. Ringel, ‘Infinite length modules. Some examples as introduction’, *Infinite length modules*, Trends in Mathematics (Ed. H. Krause and C. M. Ringelin, Birkhäuser, Basel/Boston/Berlin, 2000) 1–73. 14, 15

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