FORCED CUBIC SCHRÖDINGER EQUATION WITH ROBIN BOUNDARY DATA: CONTINUOUS DEPENDENCY RESULT

CHARLES BU[1]

Abstract

For the cubic Schrödinger equation $iu_t = u_{xx} + k|u|^2 u$, $0 \le x$, $t < \infty$, initial data $u(x,0) = u_0(x) \in H^2[0, \infty)$, and Robin boundary data $u_x(0, t) + \alpha u(0, t) = R(t) \in C^2[0, \infty)$ (where α is real), we show that the solution u depends continuously on u_0 and R.

1. Preliminaries

In recent years, a vast amount of work has been done on pure initial value problems for important nonlinear evolution equations such as NLS and KdV. No comparable attention has been given to mixed initial boundary-value problems for these equations. Yet in many cases of physical interest, the mathematical models lead precisely to problems where boundary data is non-zero (sometimes called "forced problems"). For example, the launching of solitary waves in a shallow water channel, and the excitation of ionacoustic solutions in a double plasma machine, belong to this class. In ionospheric modification experiments, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron-plasma wave is excited to make the nonlinear behavior important [9]. This may be described by the cubic NLS equation with Dirichlet boundary conditions

$$iu_t = u_{xx} \pm 2|u|^2 u, \qquad 0 \le x, \quad t < \infty, \tag{1.1}$$
$$u(x, 0) = u_0(x), \quad u(0, t) = Q(t).$$

Under the assumption that $u_0 \in H^2[0, \infty)$, $Q \in C^2[0, \infty)$, global existence, well-posedness and blow-up of the solution to (1.1) were established in [2, 4, 6]. The

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[1]Department of Mathematics, Wellesley College, Wellesley, MA 02482, USA

following forced nonlinear Schrödinger equation is also important:

$$iu_t = u_{xx} + k|u|^2 u, \qquad 0 \le x, \quad t < \infty, \tag{1.2}$$
$$u(x, 0) = u_0(x), \qquad u_x(0, t) + \alpha u(0, t) = R(t),$$

where α and k are real. This problem has significant physical implications. It arises from the propagation of optical solitons [5]. Also, it models water waves [1] when there is an additional term u_x on the right-hand side and $\alpha \to \infty$. We notice that the boundary condition is of Robin type.

It was shown in [3] that for this equation, there exists a unique classical solution $u \in C^0(H^2) \cap C^1(L^2)$ provided that $u_0 \in H^2[0, \infty)$, $R \in C^2[0, \infty)$. In this article, we will show that the solution u depends continuously on the initial and boundary data.

The following notation will be used throughout:

$$P(t) = u_x(0, t), \qquad Q(t) = u(0, t), \qquad R(0, t) = P(t) + \alpha Q(t), \tag{1.3}$$

$$\|u'\|_2 = \left[\int_0^\infty |u_x(x, t)|^2 \, dx \right]^{1/2}, \tag{1.4}$$

$$\|u\|_4 = \left[\int_0^\infty |u(x, t)|^4 \, dx \right]^{1/4}, \tag{1.5}$$

$$\|u\|_{m,p} = \left[\int_0^\infty \sum_{|\alpha| \le m} |D_x^\alpha u(x, t)|^p \, dx \right]^{1/p}, \tag{1.6}$$

$$\|R\|_{C^1[0,T]} = \sup_{0 \le t \le T} \left[|R(t)| + |R'(t)| \right]. \tag{1.7}$$

In addition, we need the following Gagliardo-Nirenberg estimates [7] where λ is a constant:

$$\|u\|_4^4 \le \lambda \|u'\|_2 \|u\|_2^3. \tag{1.8}$$

There are three interesting identities for (1.2):

$$\|u\|_2^2 = \|u_0\|_2^2 - 2\Im \left(\int_0^t P(\tau) \overline{Q(\tau)} \, d\tau \right), \tag{1.9}$$

$$\|u'\|_2^2 - \frac{k}{2} \|u\|_4^4 = \|u'\|_2^2 - \frac{k}{2} \|u_0\|_4^4 - 2\Re \left(\int_0^t P(\tau) \overline{Q'(\tau)} \, d\tau \right), \tag{1.10}$$

$$\int_0^\infty u \bar{u}' \, dx = \int_0^\infty \bar{u}_0 \bar{u}_0' \, dx - \int_0^t Q(\tau) \overline{Q'(\tau)} \, d\tau \tag{1.11}$$
$$+ i \int_0^\infty |P(\tau)|^2 \, d\tau + \frac{k}{2} i \int_0^t |Q(\tau)|^4 \, d\tau.$$

Based on these identities and Gagliardo-Nirenberg estimates, the following lemma can be derived directly from [3].

LEMMA 1. *For any $T > 0$, assume $R(t) \in C^1$, $u_0 \in H^1$ and the initial-boundary data satisfy necessary compatibility conditions at $x = t = 0$. If $\|u_0\|_{1,2} < M$, $\|R\|_{C^1[0,T]} < M$, then there exists $\tilde{\lambda} > 0$ which only depends on M, T such that $\|u\|_{1,2} \leq \tilde{\lambda}$ for $0 \leq t \leq T$ and $\|u\|_\infty \leq c\|u'\|_2^{1/2}\|u\|_2^{1/2} \leq \lambda_0$.*

We will consider (1.2) for $0 \leq t \leq T$ and $\|u_0\|_{1,2} < M$ (M, $T < \infty$, arbitrary).

2. Continuous dependency result

Throughout this section we shall assume that u, v solve (2.1) with data (R, u_0) and (R_1, v_0) lying in $C^2[0, T] \times H^2[0, \infty) = X$. According to the global existence theorem in [3], the map

$$f : X \to Y = C^1(L^2, [0, T]) \cap C^0(H^2, [0, T]) \tag{2.1}$$

via $(R, u_0) \to u$ is well-defined. To prove continuous dependency, we shall fix $z = (R, u_0) \in X$ and $z_1 = (R_1, v_0) \in X$. Let $\|z\|_X = \max\{\|R\|_{C^2[0,T]}, \|u_0\|_{2,2}\} < M$, $\|z_1\|_X < M$ and

$$w = \Delta u = v - u, \quad \Delta z = z_1 - z = (\Delta R, w_0) = (R_1 - R, v_0 - u_0). \tag{2.2}$$

Since $v = w + u$ solves the forced NLS (1.2), we have

$$i(w_t + u_t) = w_{xx} + u_{xx} + k|w + u|^2(w + u) \tag{2.3}$$
$$= w_{xx} + u_{xx} + k(w + u)^2(\bar{w} + \bar{u})$$
$$= w_{xx} + u_{xx} + k(|w|^2w + 2u|w|^2 + u^2\bar{w} + 2|u|^2w + \bar{u}w^2 + |u|^2u).$$

But u solves the forced NLS (1.2) as well, therefore w solves the following initial-boundary value problem:

$$iw_t = w_{xx} + k|w|^2w + 2ku|w|^2 + ku^2\bar{w} + 2k|u|^2w + k\bar{u}w^2, \tag{2.4}$$
$$w_x(0, t) + \alpha w(0, t) = \Delta R, \quad w_0 = v_0 - u_0. \tag{2.5}$$

We have the following lemma.

LEMMA 2. *There exists $m > 0$ such that $\sup_{0 \leq t \leq T} \|v - u\|_2 \leq m\|z_1 - z\|_{X_0}^{1/2}$ where $X_0 = C[0, T] \times L^2[0, \infty)$.*

PROOF. Write $\Delta R = R_1(t) - R(t)$, $\Delta P = P_1(t) - P(t) = v_x(0, t) - u_x(0, t)$, $\Delta Q = Q_1(t) - Q(t)$. We calculate

$$i\partial_t |w|^2 = iw_t \bar{w} + iw\bar{w}_t \tag{2.6}$$
$$= \left[w_{xx} + k|w|^2 w + 2k|w|^2 u + ku^2 \bar{w} + 2kw|u|^2 + kw^2 \bar{u} \right] \bar{w}$$
$$- w \left[\bar{w}_{xx} + k|w|^2 \bar{w} + 2k|w|^2 \bar{u} + k\bar{u}^2 w + 2k\bar{w}|u|^2 + k\bar{w}^2 u \right]$$
$$= w_{xx}\bar{w} - w\bar{w}_{xx} + 2k|w|^2 (\bar{w}u - \bar{u}w) + k(u^2 \bar{w}^2 - \bar{u}^2 w^2)$$
$$+ k|w|^2 (\bar{u}w - u\bar{w})$$
$$= 2i\Im \left(w_{xx}\bar{w} + 2k|w|^2 \bar{w}u + ku^2 \bar{w}^2 + k|w|^2 \bar{u}w \right).$$

Thus, by applying Lemma 1, we get

$$\int_0^\infty |w|^2 \, dx \tag{2.7}$$
$$= \|w_0\|_2^2 + 2\Im \int_0^t \left[\int_0^\infty \left(w_{xx}\bar{w} + 2k|w|^2 \bar{w}u + ku^2 \bar{w}^2 + k|w|^2 \bar{u}w \right) dx \right] d\tau$$
$$= \|w_0\|_2^2 - 2\Im \int_0^t \Delta P \Delta \bar{Q} \, d\tau$$
$$+ 2\Im \int_0^t \int_0^\infty (2k|w|^2 \bar{w}u + ku^2 \bar{w}^2 + k|w|^2 \bar{u}^2 w) \, dx \, d\tau$$
$$\leq \|w_0\|_2^2 - 2\Im \int_0^t (\Delta R - \alpha \Delta Q) \Delta \bar{Q} \, d\tau$$
$$+ 2|k| \int_0^t \int_0^\infty \left(3|w|^3 |u| + |u|^2 |w|^2 \right) dx \, d\tau$$
$$\leq \|w_0\|_2^2 + 2 \left(\int_0^t |\Delta R(\tau)|^2 \, d\tau \right)^{1/2} \left(\int_0^t |\Delta Q(\tau)|^2 \, d\tau \right)^{1/2}$$
$$+ 2|k| \int_0^t \int_0^\infty \left(3|w|^3 |u| + |u|^2 |w|^2 \right) dx \, d\tau.$$

Direct calculation shows

$$|Q(t)|^2 = |Q(0)|^2 + \Re \int_0^\infty u_0 \bar{u}_0' \, dx - \Re \int_0^\infty u\bar{u}' \, dx \tag{2.8}$$
$$= |u_0(0)|^2 + \|u_0'\|_2 \|u_0\|_2 - \Re \int_0^\infty u\bar{u}' \, dx$$
$$\leq c_0 + \|u\|_2 \|u'\|_2$$
$$\leq c_0 + \tilde{\lambda}^2,$$

and, from Lemma 1, we also get the following

$$|Q_1(t)|^2 = |Q_1(0)|^2 + \Re \int_0^\infty v_0 \bar{v}_0' \, dx - \Re \int_0^\infty v\bar{v}' \, dx \tag{2.9}$$

$$= |v_0(0)|^2 + \|v_0'\|_2 \|v_0\|_2 - \Re \int_0^\infty v\bar{v}' \, dx$$

$$\leq c_0' + \|v\|_2 \|v'\|_2$$

$$\leq c_0' + \tilde{\lambda}^2.$$

Thus

$$\left(\int_0^t |\Delta Q(\tau)|^2 \, d\tau\right)^{1/2} \leq \left[\int_0^t \left(2|Q_1(\tau)|^2 + 2|Q(\tau)|^2\right) d\tau\right]^{1/2} \tag{2.10}$$

$$\leq \left(2T(c_0 + \tilde{\lambda}^2) + 2T(c_0' + \tilde{\lambda}^2)\right)^{1/2}$$

$$= m_0.$$

Substitute (2.10) in (2.7):

$$\int_0^\infty |w|^2 \, dx \leq \|w_0\|_2^2 + \hat{c}\|\Delta R\|_{C[0,T]} + \tilde{c} \int_0^t \int_0^\infty |w|^2 \, dx \, d\tau. \tag{2.11}$$

Apply the Gronwall lemma to (2.11):

$$\int_0^\infty |w|^2 \, dx \leq \left(\|w_0\|_2^2 + \hat{c}\|\Delta R\|_{C[0,T]}\right) e^{\tilde{c}t} \tag{2.12}$$

$$\leq \left(\|w_0\|_2^2 + \hat{c}\|\Delta R\|_{C[0,T]}\right) e^{\tilde{c}T}.$$

Therefore,

$$\sup_{0 \leq t \leq T} \left(\int_0^\infty |w|^2 \, dx\right)^{1/2} \leq \sqrt{\left(\|w_0\|_2^2 + \hat{c}\|\Delta R\|_{C[0,T]}\right) e^{\tilde{c}T}} \tag{2.13}$$

$$\leq m_0 \left(\|w_0\|_2 + c\|\Delta R\|_{C[0,T]}^{1/2}\right)$$

$$= m_0 \left(\|w_0\|_2^{1/2} \|w_0\|_2^{1/2} + c\|\Delta R\|_{C[0,T]}^{1/2}\right)$$

$$\leq m_0 \left(\sqrt{\|u_0\|_2 + \|v_0\|_2}\|w_0\|_2^{1/2} + c\|\Delta R\|_{C[0,T]}^{1/2}\right)$$

$$\leq m_0 \left(\sqrt{2R}\|w_0\|_2^{1/2} + c\|\Delta R\|_{C[0,T]}^{1/2}\right)$$

$$\leq \max\left\{m\|w_0\|_2^{1/2}, m\|\Delta R\|_{C[0,T]}^{1/2}\right\}$$

$$= m\|\Delta z\|_{X_0}^{1/2}$$

$$= m\|z_1 - z\|_{X_0}^{1/2}.$$

This proves the lemma.

LEMMA 3. *There exists $\tilde{c} > 0$ such that $\|v - u\|_{Y_2} \leq \tilde{c}\|z_1 - z\|_X^{1/2}$ where $Y_2 = C^1(L^2, [0, T])$.*

PROOF. The norm of u on Y_2 is $\sup_{0 \leq t \leq T}(\|u_t\|_2 + \|u\|_2)$. Since $\sup_{0 \leq t \leq T} \|u\|_{2,2} < \infty$, it is clear that RHS of (2.4) satisfies the local-Lipschitz condition on w in H^2. Similar to the technique in [3], we show that $w = v - u$ is the unique global solution satisfying (2.4).

For α real, $\Delta R \in C^2[0, \infty)$, $u \in H^2[0, \infty)$, write

$$w = W - S(t)e^{-bx}, \quad b = |\alpha| + 1 > 0, \quad S(t) = \frac{\Delta R(t)}{\alpha - b}. \tag{2.14}$$

One has $W \in H^2[0, \infty)$ and $W_x = w_x + bS(t)e^{-bx}$. Thus

$$\begin{aligned} W_x(0, t) + \alpha W(0, t) &= w_x(0, t) + bS(t) + \alpha(w(0, t) - S(t)) \\ &= \Delta P(t) + bS(t) + \alpha \Delta Q(t) - \alpha S(t) \\ &= \Delta R(t) + (b - \alpha)S(t) \\ &= 0 \end{aligned} \tag{2.15}$$

and (2.4) becomes

$$W_t = -i W_{xx} - ik|W|^2 W + G_0 + G_1 + G_2. \tag{2.16}$$

It might be cumbersome to write exactly what those G_0, G_1, G_2 are, but for now we only need to indicate that

$$\|G_0\|_2 \leq c_1\|R\|_{C^1[0,T]}, \tag{2.17}$$

$$\|G_j(0)\|_2 \leq c_2\|W_0\|_2, \quad j = 1, 2, \tag{2.18}$$

and

$$\|G_0'(t)\|_2 \leq c_3\|\Delta R\|_{C^2[0,T]}, \tag{2.19}$$

$$\|G_j'(t)\|_2 \leq c_4(\|W\|_2 + \|W_t\|_2), \quad j = 1, 2. \tag{2.20}$$

From $W_0 = W(x, 0) = w_0(x) - S(0)e^{-bx}$ we see that $W_0(0) = w_0(0) - S(0) = 0$. Generally G_1, $G_2 \in D(A)$ but $G_0 \notin D(A)$. But since $S \in C^2$, $G_0 = G_0(e^{-bx}, S, S')$, $G_0'(t)$ is continuous. By [3, 8], one has

$$\int_0^t N(t - s)G_0(s)\,ds \in D(A), \tag{2.21}$$

where

$$A = -iD_x^2 + ia, \quad a > \frac{\alpha^2 c^4}{2}, \tag{2.22}$$

with $D(A) = \{W, W_{xx} \in L^2[0, \infty); W(0) = 0\}$ and $N(t) = \exp\{At\}$ being a strongly continuous contraction semigroup in L^2. Here c denotes the coefficient in the Gagliardo-Nirenberg estimate $\|u\|_\infty \leq c\|u'\|_2\|u\|_2$. One then converts (2.16) into an integral equation

$$W(t) = N(t)W_0 + \int_0^t N(t-s)G_0\, ds + \int_0^t \left(G_1 + G_2 - ik|W|^2 W\right) ds \quad (2.23)$$

$$= N(t)W_0 + \int_0^t N(t-s)G(s)\, ds$$

$$= N(t)W_0 + \int_0^t N(s)G(t-s)\, ds.$$

Now $\|w\|_{1,2}$ along with $\|w\|_\infty$ are bounded, because by Lemma 1 one has

$$\|w\|_{1,2} \leq \|u\|_{1,2} + \|v\|_{1,2} \leq 2\tilde{\lambda}, \qquad (2.24)$$

$$\|w\|_\infty \leq \|u\|_\infty + \|v\|_\infty \leq 2\lambda_0, \qquad (2.25)$$

and hence $\|w\|_{1,2}$ and $\|W\|_\infty$ are bounded as well. By using Gagliardo-Nirenberg estimates, one obtains

$$\||k|W|^2 W\|_{2,2} \leq c_0\|W\|_\infty^2 \|W\|_{2,2} \leq \bar{c}\|W\|_{2,2}. \qquad (2.26)$$

Note

$$\|\Delta R\|_{C^2[0,T]} \leq \|R_1\|_{C^2[0,T]} + \|R\|_{C^2[0,T]} \leq 2M. \qquad (2.27)$$

Since $W_0 \in D(A)$, one has $(N(t)W_0)_t = N(t)A W_0$. By (2.23)

$$W_t(t) = (N(t)W_0)_t + N(t)G(0) + \int_0^t N(t)G'(t-s)\, ds \qquad (2.28)$$

$$= N(t)(A W_0) + N(t)G(0) + \int_0^t N(t-s)G'(s)\, ds.$$

Here

$$G(0) = G_0(0) + G_1(0) + G_2(0) - ik|W_0|^2 W_0. \qquad (2.29)$$

Since $N(t)$ is a contraction semigroup on L^2 one has

$$\|N(t)(A W_0)\|_2 \leq c_5\|W_0\|_{2,2}. \qquad (2.30)$$

Put (2.17), (2.18), and (2.26) in (2.29):

$$\|G(0)\|_2 \leq c_0 \left(\|\Delta R\|_{C^1[0,T]} + \|W_0\|_{2,2}\right). \qquad (2.31)$$

From (2.16)

$$G'(t) = G_0'(t) + G_1'(t) + G_2'(t) - ik \left(2|W|^2 W_t + W^2 \bar{W}_t \right). \tag{2.32}$$

Put (2.19) and (2.20) in (2.32) (note $\|W\|_\infty$ is bounded):

$$\|G'(t)\|_2 \leq c_6 \left(\|\Delta R\|_{C^2[0,T]} + \|W\|_2 + \|W_t\|_2 \right). \tag{2.33}$$

Since $N(t)$ is a contraction semigroup, one can use (2.30), (2.33) in (2.32)

$$\|W_t\|_2 \leq \|N(t)(A\,W_0)\|_2 + \|G(0)\|_2 + \int_0^t \|N(t-s)G'(s)\|_2 \, ds \tag{2.34}$$

$$\leq c_5 \|W_0\|_{2,2} + c_0 \left(\|\Delta R\|_{C^1[0,T]} + \|W_0\|_{2,2} \right)$$

$$+ \int_0^t c_6 \left(\|\Delta R\|_{C^2[0,T]} + \|W\|_2 + \|W_t\|_2 \right) \, ds.$$

By the Gronwall lemma,

$$\|W_t\|_2 \leq c_5 \|W_0\|_{2,2} + c_0 \left(\|\Delta R\|_{C^1[0,T]} + \|W_0\|_{2,2} \right) \tag{2.35}$$

$$+ c_6 \left(T \|\Delta R\|_{C^2[0,T]} + \int_0^t \|W\|_2 \, ds \right) \exp\{c_6 T\}$$

$$\leq c' \left(\|\Delta R\|_{C^2[0,T]} + \|W_0\|_{2,2} \right) + \bar{c} \int_0^t \|W\|_2 \, ds.$$

Since $w = W + S(t)e^{-bx}$, $w_t = W_t + S'(t)e^{-bx}$, one has

$$\|w_t\|_2 \leq \|W_t\|_2 + \|\Delta R\|_{C^1[0,T]} \tag{2.36}$$

$$\leq c' \left(\|\Delta R\|_{C^2[0,T]} + \|w_0\|_{2,2} + c_0' \|\Delta R\|_{C^1[0,T]} \right)$$

$$+ \bar{c} \left(\int_0^t \left(\|w\|_2 + \|\Delta R\|_{C^1[0,T]} \right) \, ds \right) + \|\Delta R\|_{C^1[0,T]}$$

$$\leq c \left(\|\Delta R\|_{C^2[0,T]} + \|w_0\|_{2,2} \right) + \bar{c} \int_0^t \|w\|_2 \, ds.$$

Now we can use Lemma 2

$$\|v - u\|_{Y_2} = \sup_{0 \leq t \leq T} \left(\|w_t\|_2 + \|w\|_2 \right) \tag{2.37}$$

$$\leq \sup_{0 \leq t \leq T} \left(c \left(\|\Delta R\|_{C^2[0,T]} + \|w_0\|_{2,2} \right) + \bar{c} \int_0^t m \|\Delta z\|_{X_1}^{1/2} \, ds + m \|\Delta z\|_{X_1}^{1/2} \right)$$

$$\leq c \left(\|\Delta R\|_{C^2[0,T]} + \|w_0\|_{2,2} \right) + \bar{c} Tm \|\Delta z\|_{X_1}^{1/2} + m \|\Delta z\|_{X_1}^{1/2}$$

$$\leq \hat{c} \|\Delta z\|_X^{1/2}$$

and Lemma 3 is proved.

THEOREM. *The map* $f : X \to Y$ *is continuous and thus* (1.2) *is well-posed.*

PROOF. By Lemma 3, it suffices to show that there exists $M > 0$ such that $\|v - u\|_{Y_3} \le M \|z_1 - z\|_X^{1/2}$ where $Y_3 = C^0 (H^2, [0, T])$. From (2.4)

$$\|w_{xx}\|_2 \le \|w_t\|_2 + |k| \left(\|w^3\|_2 + 2\|uw^2\|_2 + \|u^2 w\|_2 + 2\|u^2 w\|_2 + \|uw^2\| \right) \quad (2.38)$$
$$\le \|w\|_{Y_2} + |k| \left(\|w^3\|_2 + 3\|u^2 w\|_2 + 3\|w^2 u\|_2 \right).$$

Put (2.13) and (2.37) in (2.38):

$$\|w_{xx}\| \le \hat{c} \|\Delta z\|_X^{1/2} + |k| \left(\|w\|_\infty^2 \|w\|_2 + 3\|u\|_\infty^2 \|w\|_2 + 3\|w\|_\infty \|u\|_\infty \|w\|_2 \right) \quad (2.39)$$
$$\le \hat{c} \|\Delta z\|_X^{1/2} + |k| \left((2c)^2 \|w\|_2 + 3c^2 \|w\|_2 + 3(2c)c\|w\|_2 \right)$$
$$= \hat{c} \|\Delta z\|_X^2 + m' \|w\|_2$$
$$\le \hat{c} \|\Delta z\|_X^2 + m'm \|\Delta z\|_{X_0}^{1/2}$$
$$\le \tilde{c} \|\Delta z\|_X^{1/2}.$$

By (2.13) and (2.39)

$$\|v - u\|_{Y_3} = \sup_{0 \le t \le T} \left(\|w_{xx}\|_2 + \|w\|_2 \right) \quad (2.40)$$
$$\le \tilde{c} \|\Delta z\|_X^{1/2} + m \|\Delta z\|_{X_0}^{1/2}$$
$$\le M \|\Delta z\|_X^{1/2}.$$

Hence (2.40) combined with Lemma 3 shows that $f : X \to Y = Y_2 \cap Y_3$ is continuous at z. The proof of well-posedness of (1.2) is complete.

3. Some remarks

We wish to make a few remarks about possible blow-up situations for the forced NLS with Robin boundary data. First we give the following definition.

DEFINITION. A solution to the forced NLS (1.2) blows up at T if $\sup_{[0,T]} \|u\|_{H^1} \to \infty$ as $t \to T^-$.

For $T > 0$ consider

$$iu_t = u_{xx} + k|u|^2 u, \qquad 0 \le x < \infty, \quad 0 \le t < T, \quad (3.1)$$
$$u(x, 0) = u_0(x), \quad u_x(0, t) + \alpha u(0, t) = R(t),$$

where $R(t) \in C^2[0, T)$. The following proposition is a direct consequence of the local existence theorem in [3].

PROPOSITION 1. *Assume $u_0 \in H^2[0, \infty)$, $R \in C^2[0, T)$. Then there exists a unique classical solution for (3.1), $u \in C^1([0, T_M), L^2) \cap C^0([0, T_M), H^2)$, such that either $T_M = T$ or $T_M < T$ and $\|u\|_{H^2} \to \infty$ as $t \to T_M^-$.*

Similarly, by the global existence theorem in [3], we conclude that $T_M = T$. We are interested in the conditions on R that would trigger a blow-up at T. For the same problem with Dirichlet boundary data, it was suspected in [2] that a necessary and sufficient for such blow-up is $\int_0^T |Q(\tau)|^2 \, d\tau = \infty$. This conjecture was proved by Guo and Wu [6].

For (3.1) we do have the following proposition.

PROPOSITION 2. *Assume that $u_0 \in H^2[0, \infty)$, $R \in C^2[0, T)$. Then the solution to (3.1) blows up at T if $\sup_{[0, t]} |R(\tau)| \to \infty$ as $t \to T^-$.*

PROOF. If $\sup_{[0, t]} |R(t)| \to \infty$ as $t \to T^-$, then either $\sup |Q(t)|$ or $\sup |P(t)|$ must approach to ∞ as $t \to T^-$, regardless of the value of α. Suppose $\lim_{t \to T^-} \sup_{[0, T]} |Q(\tau)| = \infty$. By the Gagliardo-Nirenberg estimate

$$|Q(t)| = |u(0, t)| \le \|u\|_\infty \le c\|u'\|_2^{1/2} \|u\|_2^{1/2} = c\|u\|_{H^1}; \tag{3.2}$$

thus $\sup_{[0, t]} \|u\|_{H^1} \to \infty$ as $t \to T^-$ and u must blow up at T.

On the other hand, assume that $\lim_{t \to T^-} \sup_{[0, t]} |P(\tau)| = \infty$. By the Gagliardo-Nirenberg estimate again

$$|P(t)| = |u_x(0, t)| \le \|u_x\|_\infty \le c'\|u''\|_2^{3/4} \|u\|_2^{1/4} = c\|u\|_{H^2}; \tag{3.3}$$

thus $\sup_{[0, t]} \|u\|_{H^2} \to \infty$ as $t \to T^-$. In fact, by the local existence theorem in [3], this implies that $\|u\|_{H^2} \to \infty$ as $t \to T^-$. If $\|u\|_{H^1}$ remains bounded on $[0, T)$, a re-examination of the proof of the global existence theorem in [3] reveals that $\|u\|_{H^2}$ must be bounded as well. Thus $\|u\|_{H^2} \to \infty$ as $t \to T^-$ implies that $\|u\|_{H^1} \to \infty$ as $t \to T^-$. Again, this means that the solution u blows up at T. Either way our proposition is proved.

Evidently, $\int_0^t |R(\tau)|^2 \, d\tau = \infty$ implies that $\sup_{[0, t]} |R(t)| \to \infty$ as $t \to T^-$, since $R \in C^2[0, T)$. Therefore, if $\int_0^T |R(\tau)|^2 \, d\tau = \infty$ then u blows up at T. The reverse might be true as well.

CONJECTURE. *The necessary and sufficient condition for the solution to (1.3) to blow up at T is $\int_0^t |R(\tau)|^2 \, d\tau = \infty$.*

Acknowledgments

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