# ON INTEGER MATRICES AND INCIDENCE MATRICES OF CERTAIN COMBINATORIAL CONFIGURATIONS, I: SQUARE MATRICES

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**Introduction.** A few years ago, in a short paper **(4)** Ryser introduced an interesting topic in number theory, viz. the connection between integer matrices (i.e., matrices having only integers as their elements) satisfying certain conditions and 0–1 matrices (i.e., matrices that have no element different from 0 and 1). In this series of papers we shall pursue this topic further.

To make the statements of our theorems short we introduce some terminology. We need the definitions of certain 0–1 matrices related to a few well-known combinatorial configurations. By an *incidence matrix* of a balanced incomplete block (b.i.b. for conciseness) design we mean a 0–1 matrix with $v$ rows and $b$ columns, such that the sum of the elements in each column of $A$ is $k$, $k < v$, and the scalar product of any two row vectors of $A$ is $\lambda$, $\lambda \neq 0$. It easily follows that the sum of the elements in any row of $A$ is a constant, say $r$, and then

$$bk = vr \quad \text{and} \quad \lambda(v-1) = r(k-1).$$

One can also infer that $b > v$; see **(3)**. Clearly $AA'$ has $r$ down its main diagonal and $\lambda$ elsewhere. If $b = v$, $A$ is called an incidence matrix of a symmetrical b.i.b. design (or a $v$–$k$–$\lambda$ configuration). If $b = nr$, where $n > 2$ is an integer, and $A$ is such that its $nr$ columns can be divided into $r$ disjoint classes, each class containing $n$ column vectors, and the scalar product of any two column vectors in each class is 0, then $A$ is defined to be an incidence matrix of a resolvable b.i.b. design. Such an incidence matrix can be written as $A = (A_1, A_2, \ldots\ldots A_r)$ where each $A_i$ is a $v \times n$ submatrix of $A$ and each row sum of $A_i$ is 1 $(i = 1, 2, \ldots, r)$. If the incidence matrix of a resolvable b.i.b. design satisfies the extra condition $b = v + r - 1$, then it is called an incidence matrix of an affine resolvable b.i.b. design. The genesis of these terms can be found in **(3)**.

If $r$ and $\lambda$ are positive integers, $r > \lambda$, Hasse–Minkowski's classical theorem in the arithmetical theory of quadratic forms directly furnishes necessary and sufficient conditions on $r$, $\lambda$, and $v$ for the existence of a $v \times v$ matrix $A$ with rational numbers as its elements such that $AA'$ has $r$ down its main diagonal and $\lambda$ elsewhere. These conditions are the following:

(i) $(r - \lambda)^{v-1}$ is a perfect square,

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(ii) $(r - \lambda)x^2 + (-1)^{\frac{1}{2}(v-1)}\lambda y^2 = z^2$ is solvable in rational integers $x, y, z$, not all 0; see (**5**; **1**; **2**). Plainly these conditions become necessary for the existence of an incidence matrix of a symmetrical b.i.b. design.

Hasse–Minkowski's theorem deals with square matrices whose elements are in the field of rational numbers. An extension of their theorem for rectangular matrices has not yet been obtained. The discovery of an analogous theorem for square matrices with their elements in the ring of integers, as is well known, is a classical unsolved problem in number theory. In view of this lack of knowledge, we have assumed, in each of the following theorems, the existence of a matrix (without stating conditions on its elements) which can be factored into an integer matrix and its transpose; then we have sought conditions on the elements so as to make one of the factors an incidence matrix of one of the combinatorial configurations given above.

Our first theorem shows a connection between an integer matrix and an incidence matrix of a symmetrical b.i.b. design.

THEOREM 1. *Let $A$ be a $v \times v$ integer matrix such that*

$$AA' = \begin{bmatrix} r_1 & \lambda & \lambda & \dots & \lambda \\ \lambda & r_2 & \lambda & \dots & \lambda \\ . & . & . & \dots & . \\ \lambda & \lambda & \lambda & \dots & r_v \end{bmatrix} \tag{1.1}$$

*and* (i) *each $r_i - \lambda$ is odd, $i = 1, 2, \dots, v$ and $\lambda \neq 0$,*

(ii) $$\sum_{i=1}^{v} (r_i - \lambda)^{-1} = \frac{v}{r - \lambda},$$

*where $r - \lambda$ is a positive integer,*

(iii) *$r - \lambda + \lambda v = \tilde{r}^2 h \neq 0$, where $h$ is square-free, and*

(iv) *the greatest square dividing $a$ is relatively prime to $a$ where $a$, $a'$, $b$, $b'$, are integers defined by*

$$l\lambda h/(r - \lambda) = a/b, \qquad l\tilde{r}h/(r - \lambda) = a'/b'$$

*with $(a, b) = (a', b') = 1$ and $l$ denoting the least common multiple of the $(r_i - \lambda)$'s. Then either $A$ is the incidence matrix of a symmetrical b.i.b. design or becomes one when some of its columns are multiplied by $-1$.*

*Proof.* Let $A = (a_{ij})$, $i, j = 1, 2, \dots, v$. Put $d_i = r_i - \lambda$ $(i = 1, 2 \dots v)$ and let $\sum$ stand for $\sum^v{}_{i=1}$.

We notice that $A$ is non-singular; for by (1.1), (i), (ii), and (iii)

$$0 \leqslant |AA'| = \prod_{i=1}^{v} (1 + \lambda \sum d_i^{-1}) d_i = \left(\prod_{i=1}^{v} d_i\right) (r - \lambda + \lambda v)(r - \lambda)^{-1} \neq 0.$$

Next, no two $d_i$'s can be negative, for $d_i + d_j$ gives the square of the length of a vector which is the difference of the $i$th and $j$th row vectors of $A$, $i \neq j$.

We introduce a set of rational variables $x_1, x_2, \ldots, x_v$. Let

$$(1.2) \qquad A_0 = \begin{bmatrix} & & & & (-\lambda)^{\frac{1}{2}} \\ & & A & & (-\lambda)^{\frac{1}{2}} \\ & & & & \vdots \\ & & & & (-\lambda)^{\frac{1}{2}} \\ x_1 & x_2 & \ldots & x_v & -\tilde{r}(-\lambda)^{\frac{1}{2}} \end{bmatrix},$$

where the last row vector is such that its scalar product with the previous rows is 0. In other words, the $x_i$'s are the roots of the consistent system of equations

$$(1.3) \qquad A \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_v \end{bmatrix} = \begin{bmatrix} \tilde{r} \\ \tilde{r} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{r} \end{bmatrix}.$$

Clearly the $x_i$'s are rational numbers. If one or more of them are negative we multiply the corresponding columns by $-1$ and get a modified matrix that satisfies all the conditions of the hypothesis, but in (1.3) all the $x_i$'s are non-negative. Without changing the notation, we shall assume that $A$ itself is the modified matrix and in (1.3), $x_i \geqslant 0$ $(i = 1, 2, \ldots, v)$. With this convention we now show that $A$ is an incidence matrix.

It is easy to see that as $d_i \neq 0$, the row vectors of $A_0$ are linearly independent in the field of complex numbers. So $A_0$ is non-singular. Now $A_0 A_0'$ is a diagonal matrix $D$ with $d_1, d_2, \ldots, d_v, w$ down the main diagonal, where

$$(1.4) \qquad w = \sum x_i^2 - \tilde{r}^2\lambda^{-1} \neq 0.$$

From $A_0' D^{-1} A_0 = I$, where $I$ is the identity matrix, we obtain, by comparing terms on both sides,

$$(1.5) \qquad a_{1i}^2 d_1^{-1} + a_{2i}^2 d_2^{-1} + \ldots + a_{vi}^2 d_v^{-1} + x_i^2 w^{-1} = 1 \qquad (i = 1, 2, \ldots, v),$$

$$(1.6) \qquad a_{1i} a_{1j} d_1^{-1} + a_{2i} a_{2j} d_2^{-1} + \ldots + a_{vi} a_{vj} d_v^{-1} + x_i x_j w^{-1} = 0 \qquad (i \neq j; i, j = 1, 2, \ldots, v),$$

$$(1.7) \qquad a_{1i} d_1^{-1} + a_{2i} d_2^{-1} + \ldots + a_{vi} d_v^{-1} + x_i \tilde{r}(\lambda w)^{-1} = 0 \qquad (i = 1, 2, \ldots, v),$$

$$(1.8) \qquad -\lambda d_1^{-1} - \lambda d_2^{-1} - \ldots - \lambda d_v^{-1} - \tilde{r}^2(\lambda w)^{-1} = 1.$$

From (1.8), (ii) and (iii) of the hypothesis,

$$(1.9) \qquad w^{-1} = -(1 + \lambda \sum d_i^{-1}) = -(1 + \lambda v/(r - \lambda))\lambda \tilde{r}^{-2} = -h\lambda/(r - \lambda).$$

Summing (1.5) over $i$, we obtain

$$(1.10) \qquad \sum r_i d_i^{-1} + \sum x_i^2 w^{-1} = v,$$

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and so by (ii) of the hypothesis and (1.9)

$$\sum x_i^2 = (\lambda v/(r-\lambda))((r-\lambda)/(h\lambda)) = vh^{-1}. \tag{1.11}$$

From (1.11) and (iii) of the hypothesis, $r - \lambda + \lambda v > 0$ and then the second paragraph and (ii) show that all the $d_i$'s are positive. It follows from (1.5) and (1.7) that $x_i^2 lw^{-1}$ and $x_i\tilde{r}l(\lambda w)^{-1}$ are integers, i.e., $x_i^2 l\lambda h/(r-\lambda)$ and $x_i \tilde{r}lh/(r-\lambda)$ are integers, i.e., $x_i^2 a/b$ and $x_i a'/b'$ are integers.

Let $\alpha$ be the greatest square dividing $a$. Thus $a = \alpha^2\beta$, where $\beta$ is square-free. Clearly, $x_i = y_i/\alpha$, where $y_i$ is an integer, and then $y_i a'/(\alpha b')$ is also an integer. But by (iv) of the hypothesis, $(\alpha, a) = 1$; hence $\alpha | y_i$ and thus $x_i$ is an integer. Next, no $x_i$ can be zero, for then we would have from (1.7) and (1.5), on multiplying throughout by the least common multiple $l$ and writing $l_i$ for the integer $l/d_i$,

$$\begin{aligned} 0 &= a_{1i} l_1 + a_{2i} l_2 + \ldots + a_{vi} l_v \\ &\equiv a_{1i}^2 l_1 + a_{2i}^2 l_2 + \ldots + a_{vi}^2 l_v \pmod 2 \\ &\equiv l \equiv 1 \pmod 2 \end{aligned} \tag{1.12}$$

and this is a contradiction. It follows from (1.11) that $h = 1$ and $x_i^2 = 1$ and as $x_i \geqslant 0$ in (1.3), we have $x_i = 1$ for all $i$. Subtracting (1.7) from (1.5) and remembering that $x_i = 1$ and $h = 1$, we obtain (using 1.4)

$$\begin{aligned} a_{1i}(a_{1i}-1)d_1^{-1} + a_{2i}(a_{2i}-1)d_2^{-1} + \ldots + a_{vi}(a_{vi}-1)d_v^{-1} \\ = 1 - w^{-1} + \tilde{r}(\lambda w)^{-1} = (r-\tilde{r})/(r-\lambda). \end{aligned} \tag{1.13}$$

As all $d$'s have been shown to be positive, the terms on the left of (1.13) are non-negative; the quantity on the right is also non-negative, and by (iii) of the hypothesis, $r = \tilde{r}$, and each $a_{ij}$ is either 1 or 0. Now from (1.3) and the fact that $x_i = 1$, we infer that the row sums of $A$ are equal to $r$. Moreover, from (1.1) $r_i = r$ for all $i$. So all $d_i$'s are equal, and from (1.7) it follows that all column sums of $A$ are equal. Consequently, $A$ (or its modification) is an incidence matrix for a symmetrical b.i.b. design.

The above theorem supplements the theorem of Ryser given in **(4)**. The condition that each $r_i - \lambda$ is odd is essential. For example, the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies all the conditions of the theorem except (i) of the hypothesis. Clearly $A$ cannot be an incidence matrix for a b.i.b. design. This example is due to Ryser **(4)**.

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