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QUADRATIC IRRATIONALS IN THE LOWER LAGRANGE SPECTRUM

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1. We let $\xi = \{x_i\}$, where the x_i are positive integers and $i \in N$, the set of all integers. We define $\bar{\xi} = \{x_i'\}$, where $x_i' = x_{-i}$, $\xi(k) = \{x_i^*\}$ where $x_i^* = x_{i+k}$, $\xi^+ = \{x_i \mid i > 0\}$. We let $\beta(\xi) = \lim_{n\to\infty} [0;\ x_1, x_2, \ldots, x_n] = [0; x_1, x_2, \ldots]$ where

$$[0; x_1, x_2, \ldots, x_n] = \frac{1}{x_1+}\frac{1}{x_2+}\cdots\frac{1}{+x_n}.$$

We let

$$M(\xi, k) = x_k + \beta(\overline{\xi(k)}) + \beta(\xi(k))$$

and define

$$M(\xi) = \sup_k M(\xi, k), \qquad L(\xi) = \limsup_{k\to\infty} M(\xi, k).$$

The range of $L(\xi)$ is known as the Lagrange spectrum and the range of $M(\xi)$ as the Markov spectrum. It is known that both are closed and that the Markov spectrum includes the Lagrange spectrum. It has been shown by P. Koganija [**5**] that the two coincide above $\sqrt{10}$. The spectra contain all values above $(14 + 7\sqrt{2})/4$, [**2**]. Below $2\sqrt{3}$, the Lagrange spectrum is rather sparse, although of the power of the continuum (see [**1**; **3**; **6**]).

We consider here the spectra for the range where the entries of ξ are restricted to 1 and 2, which insures that the points of the spectra are at most $2\sqrt{3}$.

We introduce the following notation: We denote the complement of a set A by cA. We let

$$\mathscr{E} = \{\xi | 1 \leqq x_i \leqq 2, \text{ for all } i\}, \quad \mathscr{E}^+ = \{\xi^+ | \xi \in \mathscr{E}\},$$
$$\mathscr{R} = \{\xi | \xi = \xi(s) \text{ for some } s\}, \quad \mathscr{R}^+ = \{\xi^+ | \xi \in \mathscr{R}\},$$
$$\mathscr{R}e = \{\xi | \xi(i)^+ \in \mathscr{R}^+, \overline{\xi(-i)}^+ \in \mathscr{R}^+, \text{ for some } i\}, \quad \mathscr{R}e^+ = \{\xi^+ | \xi \in \mathscr{R}e\}.$$

A bar will be used to indicate the repeating section.

We denote the range of M, L, β with the argument restricted to a set $\mathscr{A}$ by

$$M[\mathscr{A}] = \{M(\xi) : \xi \in \mathscr{A}\}, L[\mathscr{A}] = \{L(\xi) : \xi \in \mathscr{A}\}, \beta[\mathscr{A}] = \{\beta(\xi) : \xi \in \mathscr{A}\}.$$

We let

$$I(\xi, n) = \{\eta | \eta = \{y_i\}, y_i = x_i, |i| \leqq n\}.$$

We say that $\xi^j \to \xi$ if $\xi^j \in I(\xi, n)$ for $j > j(n)$. We note that $M(\xi, 0)$ is a continuous function of ξ.

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We say that (a, b) is an interval of the complement of $L[\mathscr{E}]$ $(M[\xi])$ if $a \in L[\mathscr{E}]$ $(M[\mathscr{E}])$, $b \in L[\mathscr{E}]$ $(M[\mathscr{E}])$ and $(a, b) \subset cL[\mathscr{E}]$ $(cM[\mathscr{E}])$.

In Section 2 we prove

THEOREM 1. *If (a, b) is an interval of the complement of $L[\mathscr{E}]$ or $M[\mathscr{E}]$, then a and b are each the sum of solutions of quadratic equations.*

In Section 3 we prove

THEOREM 2. (*a*) *$L[\mathscr{R}]$ is dense in $L[\mathscr{E}]$.* (*b*) *$M[\mathscr{R}]$ is dense in $M[\mathscr{E}]$.*

Also we prove other results concerning the role of the periodic and eventually periodic ξ. Not so much can be said in this direction as there are $\xi \in \mathscr{R}e$, for instance $(\ldots, \overline{1, 1}, 2, 2, \overline{1, 1}, \ldots)$ is such that $M(\xi)$ is clustered on both sides by $M[\mathscr{R}e]$.

We conclude the introduction by stating some simple lemmas, of which the first two are well-known and the proofs will not be given.

LEMMA 1. *If $\eta \in I(\xi, n) \cap cI(\xi, n+1)$ then $\epsilon_l(n) < |\beta(\xi) - \beta(\eta)| < \epsilon_u(n)$, where*

$$\epsilon_l(n) = L(1 + \sqrt{2})^{-2n}, \text{ and } \epsilon_u(n) = K[(1 + \sqrt{5})/2]^{-2n}$$

where L and K are independent of ξ.

LEMMA 2. *For any $\xi \in \mathscr{E}$, $\eta \in \mathscr{E}$,*

$$[0; x_1, \ldots, x_{n-1}, \delta_n + \beta(\xi, n)] > [0; x_1, \ldots, x_{n-1}, \delta_{\tilde{n}} + \beta(\eta, n)],$$

where $\delta_n = 1$ if n is odd and $\delta_n = 2$ if n is even.

LEMMA 3. *If $\xi \in \mathscr{E}$, there is an $\eta \in \mathscr{E}$ for which $M(\xi) = M(\eta, 0)$.*

Proof. If $\sup_k M(\xi, k)$ is obtained for $k < \infty$, $M(\xi) = M(\xi(k), 0)$. If not we can find $N' = \{i'\} \subset N$, where $M(\xi) = \lim_{i\to\infty} M(\xi(i'))$ and $\lim_{i\to\infty} \xi(i') = \eta \in \mathscr{E}$. Then $M(\eta) \geqq M(\eta, 0) = M(\xi)$. For fixed s

$$M(\eta, s) = \lim_{i'\to\infty} M(\xi, s + i') \leqq \sup_k M(\xi, k) = M(\xi).$$

2. To prove Theorem 1 we first prove two lemmas.

LEMMA 4. *Suppose (a, b) to be an interval of the complement of $M[\mathscr{E}]$. Then if $s >$ inf $\{n|\epsilon_u(n) > (b - a)/2\}$, $M(\xi, 0) \geqq b$, and $\eta \in I(\xi, s)$, we have $M(\eta) \geqq b$.*

We call $S(a, b)$ the set of sequences $x_{-s}, x_{-x+1}, \ldots, x_0, x_1, \ldots, x_s$ corresponding to ξ with $M(\xi, 0) \geqq b$. It is clear that $M(\eta) \geqq b$ if

$$y_{i-s}, y_{i-s+1}, \ldots, y_{i+s} \in S(a, b)$$

for some i, and $M(\eta) \leqq a$ if $y_{i-s}, y_{i-s+1}, \ldots, y_{i+s} \notin S(a, b)$ for every i.

Proof. Suppose $\eta \in L(\xi, s)$, then

$$M(\eta, 0) = 2 + \beta(\bar{\eta}) + \beta(\eta) > 2 + \beta(\bar{\xi}) + \beta(\xi) - (b - a)$$
$$\geqq M(\xi, 0) - (b - a) = b - (b - a) = a.$$

Since by hypothesis $M(\eta) \notin (a, b)$, we must have $M(\eta) \geqq b$.

Definition. We say that $\xi \in \mathscr{E}$ is subject to a finite stationary restriction A_s of length s if there is a finite set of sequences of 1's and 2's, each of length s, $A_s = \{[a_1, \ldots, a_s]\}$, such that for every $j \geqq 0$, $[x_{j+1}, \ldots, x_{j+s}] \in A_s$. We denote such ξ by $\mathscr{E}/A_s$.

We will consider only A_s for which $\mathscr{E}/A_s$ is non empty. It is not difficult to see that $\mathscr{E}/A_s$ is a closed set.

LEMMA 5. *If* $\xi \in \mathscr{E}/A_s$ *and* $\beta(\xi) = \max\{\beta(\xi')|\xi' \in \mathscr{E}/A_s\}$ *or* $\min\{\beta(\xi')|\xi' \in \mathscr{E}/A_s\}$ *then,* $\xi^+ \in \mathscr{R}e^+$.

Proof. We let $\xi = \{x_i\}$. Of the sequences $[x_{2ks+1}, \ldots, x_{2ks+s}]$, $0 \leqq k \leqq 2^s + 1$, two must be identical, since there are at most 2^s different ones. Hence, there must be i, and a $d > s$ for which $[x_{i+1}, \ldots, x_{i+s}] = [x_{i+2d+1}, \ldots, x_{i+2d+s}]$. Suppose

$$x_{i+s+1} = \delta_{i+s+1} = \delta_{i+2d+s+1} \neq x_{i+2d+s+1}.$$

Let $\xi' = \{x_i'\}$ where $x_k' = x_k$ for $k \leqq i + 2d + s$ and $x_k' = x_{k-2d}$ for $k > i + 2d + s$. It is easily checked that $\xi' \in \mathscr{E}/A_s$ since each $[x_{t+1}', \ldots, x_{t+s}']$ appears in ξ. But by Lemma 2, $\beta(\xi') > \beta(\xi)$ contrary to hypothesis.

If $x_{i+s+1} \neq \delta_{i+2d+s+1} = x_{i+2d+s+1}$, we let $\xi'' = \{x_i''\}$, with $x_k'' = x_k$ for $k \leqq i + s$ and $x_k'' = x_{k+2d}$ for $k > i + s$. Again $\xi'' \in \mathscr{E}/A_s$ and $\beta(\xi'') > \beta(\xi)$ contrary to hypothesis. Hence we must have $x_{i+s+1} = x_{i+2d+s+1}$, and hence, by induction, $x_{i+j} = x_{i+2d+j}$ for all $j > 0$. Hence $\xi^+ \in \mathscr{R}e^+$.

By a similar argument we can show that if

$$\xi \in \mathscr{E}/A_s \quad \text{and} \quad \beta(\xi) = \min\{\beta(\xi')|\xi' \in \mathscr{E}/A_s\}$$

then $\xi^+ \in \mathscr{R}e^+$.

Proof of Theorem 1. It is sufficient to show that there is a $\xi' \in \mathscr{R}e$ and a $\xi \in \mathscr{R}e$ for which $M(\xi) = M(\xi, 0) = a$ and $M(\xi') = M(\xi', 0) = b$.

By Lemma 3, there is a $\xi \in \mathscr{E}$ for which $M(\xi) = M(\xi, 0) = a$. We let A be those sequences of 1's and 2's of length $2s + 1$ not appearing in $S(a, b)$. Then ξ and $\bar{\xi}$ are clearly subject to the finite stationary restrictions A of length $2s + 1$ and $\beta(\bar{\xi})$, $\beta(\xi)$ must be maxima of the continued fractions subject to A_{2s+1}. By applying Lemma 5, we obtain $\xi \in \mathscr{R}e$.

By Lemma 3, there is a $\xi \in \mathscr{E}$ for which $M(\xi) = M(\xi, 0) = b$. From the definition, $M(\xi, t) \leqq b$ for all t. We consider first

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Case 1. $M(\xi, t) = b$ for only a finite number of t: Then there is a k such that for $|t| > k$, $M(\xi, t) < b$. Then we have $M(\xi, 0) = 2 + \beta(\bar{\xi}) + \beta(\xi)$ where

$$\beta(\bar{\xi}) = [0; x_{-1}, x_{-2}, \ldots, x_{-2k} + \beta(\overline{\xi(-2k)})],$$
$$\beta(\xi) = [0; x_1, x_2, \ldots, x_{2k} + \beta(\xi, 2k)],$$

and $\overline{\xi(-2k)}^+$ and $\xi(2k)^+$ are subject to the finite stationary restriction A, mentioned above, and $\overline{\beta(\xi, -2k)}$, $\beta(\xi, 2k)$ must be the maxima of the continued fractions subject to this restriction. By applying Lemma 5, we have $\xi \in \mathscr{R}e$.

Case 2. We suppose $M(\xi) = M(\xi, 0) = M(\xi, t) = b$ for an infinite set $T = \{t(i)\} \subset N$, $i \in N$, $t(0) = 0$, $t(i)$ increasing: We consider first Case 2a: for some k, $t(i) - t(i-1) < k$ for all $i \in N$. If k is less than the s of Lemma 4 we take A to be the $S(a, b)$ of Lemma 4. If k is greater than s, we take A to be the set of sequences $\{[x_1, \ldots, x_{2k+1}]\}$ of 1 and 2 with

$$[x_{i+1}, x_{i+2}, \ldots, x_{i+2s+1}] \in S(a, b)$$

for some i in the range $0 \leqq i \leqq 2k - 2s$. Then ξ^+, $\bar{\xi}^+$ are sequences subject to the finite stationary restriction A, and $\beta(\xi)$, $\beta(\bar{\xi})$ are minimum continued fractions subject to these restrictions. Hence, applying Lemma 5, we have $\xi \in \mathscr{R}e$.

The remainder of the proof consists of showing that we can find a ξ which fits either Case 1 or Case 2a. We consider Case 2b: there is a subsequence $N' = \{i'\} \subset N$, and a sequence $s(i')$ increasing to infinity and a K such that

$$t(i' + v) - t(i' + v - 1) < K, \quad \text{for} \quad 1 \leqq v \leqq 2s(i').$$

In this case there is a further subsequence $N'' = \{i''\} \subset N'$ along which $\xi(t(i'')) \to \xi'$. We find, by a simple limit argument that $b = M(\xi', 0) = M(\xi', t'(i))$ for a set of $t'(i)$ with $t'(i+1) - t'(i) < K$. So this case reduces to Case 2a.

Case 2c: $T = \{t(i)\}$ is infinite, and the situation of Case 2*b* does not arise. We must have a sequence $N' = \{i'\} \subset N$, a bounded sequence $s(i')$ and a constant L such that:

$$t(i') - t(i' - 1) \to \infty, \; t(i' + s(i')) - t(i') < L,$$
$$t(i' + s(i') + 1) - t(i' + s(i')) \to \infty.$$

Then for some further subsequence $N'' = \{i''\} \subset N'$, $\xi(t(i'')) \to \xi^*$. We find, by a simple limit argument that $b = M(\xi^*, 0) = M(\xi^*, t)$ for only a finite set of t. So this case reduces to Case 1.

3. We begin with the proof of Theorem 2.

Proof of Theorem 2(a). We choose $N' = \{i'\}$ so that $\xi(i') \to \xi'$. Then $L(\xi') = \lim_{i' \in N'} M(\xi, i')$. We take n and m from this sequence so large that

$$|M(\xi, n) - M(\xi')| + |M(\xi, m) - M(\xi')| < \epsilon/2$$

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and so far apart that $m - n + 1 = v > t$, where the t is so large that in Lemma 1, $\epsilon_u(t) < \epsilon/4$. We let $y_{sv} = 2$ and $y_{sv+j} = x_{n+j}$, for $0 \leqq j < v$, $-\infty < s < \infty$ and set $\eta = (\ldots, -y, y_0, y_1, \ldots)$. Now for $0 \leqq j \leqq v$

$$\eta(j) \in I(\xi(n+j), v)$$

and hence

$$M(\eta(sv+j)) = M(\eta(j)) < M(\xi, n+j) + \epsilon/2 < M(\xi) + \epsilon.$$

On the other hand

$$M(\eta, 0) = 2 + \beta(\eta) + \beta(\bar{\eta}) > M(\xi') - \epsilon.$$

For periodic η, $M(\eta) = L(\eta)$ so the result holds.

Proof of Theorem 2(b). Koganija [**5**] has shown that above $\sqrt{10}$, $M[\mathscr{E}]$ and $L[\mathscr{E}]$ coincide, so Theorem 2(a) suffices in this case. She has also shown that $(4\sqrt{30}/7, \sqrt{10})$ contains no points of either spectrum. In [**3**] it is shown that the fractional dimension of the part of $M[\mathscr{E}]$ below $4\sqrt{30}/7$ is of fractional dimension less than one, which implies that it is also of measure zero. Hence the complement of $M[\mathscr{E}]$ is dense there, so the end points of the intervals composing the complement are dense. Hence we may apply Theorem 1 in this interval to complete the proof.

Definition. We say that $L(\xi) = M(\xi) = M(\xi, 0)$ has a maximum (minimum) at ξ if there is an $\epsilon(\xi)$ and an $I(\xi, n)$, if $\eta \in I(\xi, n)$ and, we have either $L(\xi) \geqq M(\eta)$ or $M(\eta) > L(\xi) + \epsilon(\xi)$. ($L(\xi) \leqq M(\eta)$ or $M(\eta) < L(\xi) - \epsilon(\xi)$).

If $L(\xi)$ has both a local maximum and a local minimum at ξ we say that $L(\xi)$ is locally isolated there.

THEOREM 3. *If ξ is periodic, and $L(\xi) = M(\xi, 0)$, then $L(\xi)$ has a local maximum at ξ.*

Proof. Since $L[\mathscr{R}]$ is dense in $L[\mathscr{E}]$ it is sufficient to consider periodic η, with $x_i = y_i$ where $|i| < 2as + v - 1$, where $v < 2s$, s the period of ξ, and a will be specified later. Without loss of generality we may suppose $\beta(\eta) > \beta(\xi)$. Let $k = 2as + v = \min\{i : y_i \neq x_i, i > 0\}$. Then we must have

$$y_{2as+v} = \delta_{2as+v} = \delta_v, \quad x_{2as+v} = 3 - \delta_{2as+v} = 3 - \delta_v,$$
$$M(\xi, 2as) = M(\xi) = L(\xi) = 2 + \beta(\xi) + \beta(\xi),$$
$$M(\eta, 2as) = 2 + \beta(\eta(2as)) + \beta(\overline{\eta(2as)}).$$

Now $\bar{\eta}(2as)$ agrees with $\bar{\xi}$ in $2as$ more places than did $\bar{\eta}$, so we may make $|\beta(\bar{\xi}) - \beta(\overline{\eta(2as)})|$ as small as we please. However,

$$\beta(\eta(2as)) = [0; x_1, x_2, \ldots, x_{v-1}, \delta_v, \ldots]$$
$$\beta(\xi) = [0; x_1, x_2, \ldots, x_{v-1}, 3 - \delta_v, \ldots].$$

So by Lemma 1, $\beta(\eta(2as)) - \beta(\xi) > 2\epsilon_1$, where ϵ_1 can be so chosen as to depend only on s. By choosing a, then, which determines the neighborhood, so that $|\beta(\xi) - \beta(\eta(2as))| < \epsilon_1$ we insure that $L(\eta) > L(\xi) + \epsilon_1$.

THEOREM 4. *If ξ is of odd period, $L(\xi)$ is locally isolated.*

Proof. From the above theorem, we need only show that for η periodic, $\eta \in I(\xi, n)$, for sufficiently large n, $M(\eta) > M(\xi)$. The period of ξ we take as $2s + 1$,

$$n = (2a + 1)(2s + 1) + v, \quad \text{with} \quad v < 2s + 1, n = \inf \{i | x_i \neq y_i, i > 0\}$$

where a is to be determined later. We must have

$$x_{(2a+1)(2s+1)+v} = \delta_{(2a+1)(2s+1)+v} = 3 - \delta_v,$$
$$y_{(2a+1)(2s+1)+v} = 3 - \delta_{(2a+1)(2s+1)+v} = \delta_v.$$

Without loss of generality, $\beta(\xi, 0) < \beta(\eta, 0)$. Now

$$M(\xi, (2a + 1)(2s + 1)) = M(\xi, 0) = 2 + \beta(\xi) + \beta(\bar{\xi}),$$
$$M(\eta, (2a + 1)(2s + 1)) =$$
$$2 + \beta(\eta((2a + 1)(2s + 1))) + \beta(\overline{\eta((2a + 1)(2s + 1))}).$$

As before, by choosing a large enough, we will insure that

$$|\beta(\bar{\xi}) - \beta(\overline{\eta((2a + 1)(2s + 1))})| < \epsilon, \text{ but since } \delta_{(2a+1)(2s+1)+v} = 3 - \delta_v,$$
$$\beta(\eta(2a + 1 (2s + 1))) = [0; x_1, \ldots, x_{v-1}, \delta_v, \ldots],$$
$$\beta(\xi) = [0; x_1, \ldots, x_{v-1}, 3 - \delta_v, \ldots]$$

and we have, by Lemma 1,

$$\beta(\eta, (2a + 1)(2s + 1)) - \beta(\xi, 0) > 2\epsilon$$

where ϵ depends only on s.

Hence $M(\eta) > M(\xi) + \epsilon$ and the theorem is proved.

THEOREM 5. *If ξ is periodic with period $2s$, and if for some $k < 2s$, $\bar{\xi}_k = \xi$, then $L(\xi) = M(\xi, 0)$ has a locally isolated value at ξ.*

The proof depends on the fact that $x_{2k} = 2$, $\beta(\overline{\xi(k)}) = \beta(\xi)$, $\beta(\xi(k)) = \beta(\bar{\xi})$, and depends as did the proof of Theorem 5 on the differing parity of subscripts. It is so similar to the proof of Theorem 5 that we will omit it.

We note that if a value of $M(\xi) = 2 + \beta(\xi) + \beta(\bar{\xi})$ is locally isolated at ξ, but not isolated, then there must be a $\xi' \neq \bar{\xi}$ such that $M(\xi) = M(\xi') = 2 + \beta(\xi') + \beta(\bar{\xi}')$ where $\beta(\xi') \neq \beta(\xi)$. We remark that Theorem 2.1 of [**3**] implies rather easily the following.

COROLLARY. If $\alpha \in \beta[\mathscr{E}]$, *then the only pair* $(\beta, \gamma) \in \beta[\mathscr{E}] \times \beta[\mathscr{E}]$ with $\beta + \gamma = 2\alpha$ *is* (α, α).

Together with Theorem 4 this implies the following

THEOREM 6. *If ξ is of odd period, and if $\xi = \bar{\xi}$, and if $L(\xi) = M(\xi, 0) = 2 + \beta(\xi)$, then $L(\xi)$ is isolated in the Markov spectrum.*

For example, if $\xi = \{\overline{2, 1, 1}\}$, or $\xi = \{\overline{2, 1, 1, 1, 1}\}$ then $M(\xi) = L(\xi)$ is isolated in both spectra.

Remark. We see from Theorem 6 that if $c \in L[\mathscr{E}]$ is an isolated point, there is a $\xi \in \mathscr{R}$ for which $M(\xi) = c$ and hence [**7**], c involves a single quadratic irrational. In case $(a, b) \subset cL[\mathscr{E}]$, with b a cluster point of $L[\mathscr{E}]$, we have by Theorem 3, that there is a $\xi \in \mathscr{R}e$ with $M(\xi, 0) = b$. If $M(\xi, t(i)) = b$ for an infinite number of i then there is an $N' = \{i'\}$ for which $\xi(t(i')) \to \xi' \in \mathscr{R}$, but $M(\xi)$ is locally isolated, so $\xi(t(i')) = \xi'$ for large i, and hence $\xi = \xi'$. Hence there must be a $\xi \in \mathscr{R}e \cap c\mathscr{R}$ if $M(\xi) = b$.

REFERENCES

1. C. J. Hightower, *The minima of indefinite binary quadratic forms*, J. Number Theory *2* (1970), 364–377.

2. J. R. Kinney and T. S. Pitcher, *The Hausdorff-Besicovich dimension of level sets of Perron's modular functions*, Trans. Amer. Math. Soc. *124* (1966), 122–130.

3. ——— *On the lower range of Perron's modular function*, Can. J. Math. *21* (1969), 808–816.

4. P. Kogonija, *On the connection between the spectra of Lagrange and Markov. II*, Tbiliss. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk *102* (1964), 95–104.

5. ——— *On the connection between the spectra of Lagrange and Markov. III*, Tbiliss. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk *102* (1964), 105–113.

6. ——— *On the connection between the spectra of Lagrange and Markov. IV*, Akad. Nauk. Gruzin. S.S.R. Trudy Tbiliss Mat. Inst. Razmadze *29* (1963), 15–35 (1964).

7. O. Perron, *Über die Approximation Irrationale Zahlen durch Rationals*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. *12* (1921), 3–17.

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