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EXACT VALUES FOR DEGREE SUMS OVER STRIPS OF YOUNG DIAGRAMS

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1. Introduction. If $\lambda = (\lambda_1, \dots, \lambda_m)$ where $\lambda_1, \dots, \lambda_m$ are nonnegative integers with $\lambda_1 \geq \cdots \geq \lambda_m$, then λ is a partition of $|\lambda| = \lambda_1 + \cdots + \lambda_m$, and we write $\lambda \vdash |\lambda|$. The non-zero λ_i's are the *parts* of λ, so λ_1 is the largest part, and $\ell(\lambda)$ is the number of parts of λ. Two partitions with the same parts, so they differ only in number of zeros, are the same. The set of all partitions, including the partition of 0 (with 0 parts) is denoted by $\mathcal{P}$. The *conjugate* of λ, denoted by $\tilde{\lambda}$, is the partition $(\mu_1, \dots, \mu_k)$, in which μ_i is the number of λ's that are $\geq i$, for $i = 1, \dots, k$, where $k = \lambda_1$.

Let $S_{\mathcal{N}_n}$ be the symmetric group on $\mathcal{N}_n = \{1, \dots, n\}$. The irreducible representations of $S_{\mathcal{N}_n}$ are indexed by the partitions λ of n; the *degrees* f^λ of these representations are given by the hook formula of Frame, Robinson and Thrall [**2**].

In this paper, formulas are derived for $i = 1, 2$ and various values of n and m, for the sums

$$S_m^{(i)}(n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq m}} (f^\lambda)^i$$

and

$$T_m^{(i)}(n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = m}} (f^\lambda)^i.$$

Clearly $S_1^{(i)}(n) = T_1^{(i)}(n) = 1$ for $i = 1, 2$. Formulas in terms of Catalan numbers are given for $S_2^{(2)}(n)$ in Knuth [**7**], for $T_2^{(1)}(n)$, $T_3^{(1)}(n)$ in Regev [**13**], and for $T_4^{(1)}(n)$, $T_5^{(1)}(n)$ in Gouyou-Beauchamps [**6**]. Determinantal forms of the exponential generating function in n for the $S_m^{(i)}(n)$ for $i = 1, 2$ and arbitrary m are given in Gessel [**3**] (See also Bender and Knuth [**1**], Gordon [**4**], Gordon and Houten [**5**]). The formulas for $S_m^{(i)}(n)$ that follow from these determinantal forms increase in complexity as m grows. In contrast, the formulas in the paper (Theorem 3.3, Corollary 3.4, Theorem 3.5) increase in complexity as n/m grows. Of course, the simplest result of our type is well-known (see, e.g., Stanley [**16**, Section 17]):

$$(1.1) \qquad S_n^{(2)}(n) = n! \text{ and } S_n^{(1)}(n) = \mathrm{Inv}(n)$$

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the number of involutions in $\mathcal{S}_{\mathcal{N}_n}$.

Regev [13], has obtained the asymptotic form in n for $S_m^{(i)}(n)$, where m is a fixed positive integer and i is a fixed real number. The introduction to Regev's paper gives a nice account of various contexts in which $S_m^{(2)}(n)$ occurs, including representation theory (Schur [15]), combinatorics (Schensted [14]), polynomial identities (Latyshev [8]), and the Procesi-Razmyslov theory of trace-identities (Procesi [10, 11], Razmyslov [12]). In particular, combinatorially, $S_m^{(2)}(n)((T_m^{(2)}(n))$ can be interpreted as the number of permutations of n distinct symbols whose longest increasing subsequence has length at most m (exactly m), and $S_m^{(1)}(n)((T_m^{(1)}(n))$ is the number of involutions on n distinct symbols whose longest increasing subsequence has length at most m (exactly m).

Stanley [17] refers to a private communication of D. Zeilberger showing that $\{T_m^{(1)}(n)\}$ $n \geqq 0$ is P-recursive for each fixed m. (A sequence is P-recursive if it satisfies a homogeneous linear recurrence equation of fixed order with polynomial coefficients.)

Formulas in this paper are derived by the manipulation of symmetric functions in the variables $x = (x_1, \ldots, x_n)$. (See Macdonald [9, Chapter I], for a complete treatment.) The *power-sum* symmetric functions $p_0(x), p_1(x), \ldots$ are defined by

$$p_i(x) = \sum_{j=1}^{n} x_j^i, \quad i \geq 1$$

and $p_0(x) = 1$. If $E(z,x) = \prod_{j=1}^{n}(1+x_jz)$, then the *elementary* symmetric functions $e_0(x)$, $e_1(x), \ldots$ are given by

$$e_i(x) = [z^i]E(z,x), \quad i \geq 0,$$

where $[z^i]$ denotes the coefficient of z^i in the expression to the right. The *complete* symmetric functions $h_0(x), h_1(x), \ldots$ are given by

$$h_i(x) = [z^i]E(-z,x)^{-1}, \quad i \geq 0.$$

Let $\delta = (n-1, n-2, \ldots, 1, 0)$. If $b = (b_1, \ldots, b_n)$ is an n-tuple of non-negative integers, then the *Schur* symmetric function $s_b(x)$ is given by

$$s_b(x) = a_\delta(x)^{-1} a_{b+\delta}(x),$$

where for $c = (c_1, \ldots, c_n)$, $a_c(x)$ is the alternant

$$a_c(x) = \det(x_i^{c_j})_{n \times n}$$

Thus $a_\delta(x)$ is the Vandermonde determinant, and we denote it by $V(x)$.

The connection between the degree sums given above and symmetric functons is that

$$f^{\lambda} = f^{\tilde{\lambda}} = \left[\frac{p_1(x)^n}{n!}\right] s_{\lambda}(x)$$

when $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of n. (This coefficient is well-defined for n variables since the degree of $p_1(x)^n$ is $\leq n$.) Thus, define

$$\begin{aligned}\Psi(u,x) &= \sum_{m\geq 0} u^m \Psi_m(x)\\ &= \sum_{\lambda\in\mathcal{P}} \sum_{j\geq\lambda_1} u^j s_{\lambda}(x)\end{aligned}$$

$$\begin{aligned}\Phi(u,x,y) &= \sum_{m\geq 0} u^m \Phi_m(x,y)\\ &= \sum_{\lambda\in\mathcal{P}} \sum_{j\geq\lambda_1} u^j s_{\lambda}(x) s_{\lambda}(y),\end{aligned}$$

where $y = (y_1, \ldots, y_n)$, so we obtain (using the fact that $\lambda_1 = l(\tilde{\lambda})$) the following result.

PROPOSITION 1.1.
(a) $S_m^{(1)}(n) = \left[\frac{p_1(x)^n}{n!}\right] \Psi_m(x)$,
(b) $S_m^{(2)}(n) = \left[\frac{p_1(x)^n}{n!} \frac{p_1(y)^n}{n!}\right] \Phi_m(x,y)$.

Note that

$$\begin{aligned}\left\{(1-u)\Psi(u,x)\right\}\big|_{u=1} &= \sum_{\lambda\in\mathcal{P}} s_{\lambda}(x)\\ \left\{(1-u)\Phi(u,x,y)\right\}\big|_{u=1} &= \sum_{\lambda\in\mathcal{P}} s_{\lambda}(x) s_{\lambda}(y)\end{aligned}$$

and these summations are well-known as products (see, e.g. [**9**], p. 33, 45)

$$\begin{aligned}\psi(x) &= \sum_{\lambda\in\mathcal{P}} s_{\lambda}(x)\\ &= \prod_{i=1}^{n} (1-x_i)^{-1} \prod_{1\leq i<j\leq n} (1-x_i x_j)^{-1},\\ \phi(x,y) &= \sum_{\lambda\in\mathcal{P}} s_{\lambda}(x) s_{\lambda}(y)\\ &= \prod_{i,j=n} (1-x_i y_j)^{-1}.\end{aligned} \tag{1.2}$$

So that Proposition 1.1 can be used, expressions must first be developed for $\Psi_m(x)$ and $\Phi_m(x,y)$. The technical details of these developments are given in Section 2, and begin with a technique of Macdonald [**9**, p. 124]. The nature of these expressions is that they allow us to write

$$(1.3)\qquad \Psi_m(x) = \sum_{k=0}^{n} \Psi_{m,k}(x),$$
$$\Phi_m(x,y) = \sum_{k=0}^{n} \Phi_{m,k}(x,y),$$

where as k increases,

(i) $\Psi_{m,k}$ and $\Phi_{m,k}$ become increasingly complicated (explicit formulas are given in Theorems 2.3 and 2.4),

(ii) the minimum total degree in x of the monomials in $\Psi_{m,k}$ and $\Phi_{m,k}$ is $k(m+k)$.

The effect of (*i*) and (*ii*) is that we can evaluate $S_m^{(1)}(n)$ and $S_m^{(2)}(n)$ for

$$\frac{n-(j+1)^2}{j+1} < m \leq \frac{n-j^2}{j}$$

by considering only the terms in (1.3) corresponding to $k = 0, \ldots, j$.

In Section 3 we apply Proposition 1.1 to obtain the most compact of the explicit formulas for $S_m^{(i)}(n)$, corresponding to the first few values of j above. We also give a few expressions for the simplest values of $T_m^{(i)}(n)$, since some collapse in the formulas for the equivalent $S_m^{(i)}(n) - S_{m-1}^{(i)}(n)$ can be exploited.

Note that, in principle, the symmetric group character summation

$$\sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq m}} \chi_\mu^\lambda \chi_\nu^\lambda$$

can be evaluated analogously to Proposition 1.1 by extracting appropriate power sum coefficients from $\Phi_m(x,y)$. For arbitrary partitions μ, ν, the resulting expressions seem awkward, and are not given.

2. Generating functions for the degree sums. The first stage in the derivation of formulas for $\Psi_m(x)$ and $\Phi_m(x,y)$ follows the method used by Macdonald [**9**, p. 124] for a sum of Hall-Littlewood symmetric functions analogous to $\Psi_m(x)$. The details are included here, in Theorem 2.1 below, because the details are simpler for these Schur function summations, we need a special form for the result, and for completeness. (Macdonald [p. 51,52] uses his form for the result to derive the generating functions for various classes of column-strict plane partitions in a box.)

The following notation is used in this section: if $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \mathcal{N}_n$ with $\alpha_1 < \cdots < \alpha_k$, then $\bar{\alpha} = \mathcal{N}_n - \alpha$, $x_\alpha = (x_{\alpha_1}, \ldots, x_{\alpha_k})$, $x_\alpha^i = x_{\alpha_1}^i, \ldots, x_{\alpha_k}^i$ and $I(\alpha) = \left|\{(i,j) \in \alpha \times \bar{\alpha} \text{ and } i < j\}\right|$.

THEOREM 2.1. *(a)* $\Psi_m(x) = \sum_{k=0}^n \Psi_{m,k}(x)$ *where*

$$\Psi_{m,k}(x) = \sum_{\substack{\alpha\subseteq N_n\\ |\alpha|=k}} (-1)^{I(\alpha)+\binom{k}{2}+k} x_\alpha^{n+m} \frac{V(x_\alpha)V(x_{\bar\alpha})}{V(x)} \psi(x_\alpha)\psi(x_{\bar\alpha})$$

(b) $\Phi_m(x,y) = \sum_{k=0}^n \Phi_{m,k}(x,y)$ *where*

$$\Phi_{m,k}(x,y) =$$
$$\sum_{\substack{\alpha,\beta\subseteq N_n\\ |\alpha|=|\beta|=k}} (-1)^{I(\alpha)+I(\beta)+k} x_\alpha^{n+m} y_\beta^{n+m} \frac{V(x_\alpha)V(x_{\bar\alpha})V(y_\beta)V(y_{\bar\beta})}{V(x)V(y)} \phi(x_\alpha,y_\beta)\phi(x_{\bar\alpha},y_{\bar\beta})$$

Proof. (a) Writing the Schur functions as a ratio of alternants, we get

$$\Psi(u,x) = \frac{1}{V(x)} \sum_{j\ge\lambda_1\ge\cdots\ge\lambda_n\ge 0} u^j \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) x_{\sigma_1}^{\lambda_1+n-1},\cdots,x_{\sigma_n}^{\lambda_n}.$$

Now let $d_0 = j-\lambda_1+1$, $d_1 = \lambda_1-\lambda_2+1,\ldots, d_{n-1} = \lambda_{n-1}-\lambda_n+1$, $d_n = \lambda_n$, so $d_0\ge 1,\ldots,d_{n-1}\ge 1$ and $d_n\ge 0$, and $\lambda_{n-1}+1 = d_{n-1}+d_n,\ldots,\lambda_1+n-1 = d_1+\cdots+d_n$, $j = d_1+\cdots+d_n-n$. Thus

$$\begin{aligned}
(2.1)\qquad \Psi(u,x) &= \frac{1}{u^nV(x)} \sum_{\substack{d_0,\ldots,d_{n-1}\ge 1\\ d_n\ge 0}} \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) u^{d_0+\cdots+d_n} x_{\sigma_1}^{d_1+\cdots+d_n}\cdots x_{\sigma_n}^{d_n}\\
&= \frac{1}{u^nV(x)} \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) \sum_{d_0\ge 1} u^{d_0}\cdots\\
&\qquad \cdots \sum_{d_{n-1}\ge 1} (ux_{\sigma_1}\cdots x_{\sigma_{n-1}})^{d_{n-1}} \sum_{d_n\ge 0} (ux_{\sigma_1}\cdots x_{\sigma_n})^{d_n}\\
&= \frac{1}{u^{n-1}(1-u)V(x)} \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) \frac{ux_{\sigma_1}}{1-ux_{\sigma_1}}\cdots\\
&\qquad \cdots \frac{ux_{\sigma_1}\cdots x_{\sigma_{n-1}}}{1-ux_{\sigma_1}\cdots x_{\sigma_{n-1}}} \frac{1}{1-ux_{\sigma_1}\cdots x_{\sigma_n}},
\end{aligned}$$

since the summations over the d_i's are all geometric series. In particular, note that

$$(2.2)\qquad \psi(x) = \frac{1}{V(x)} \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) \frac{x_{\sigma_1}}{1-x_{\sigma_1}}\cdots \frac{x_{\sigma_1}\cdots x_{\sigma_{n-1}}}{1-x_{\sigma_1}\cdots x_{\sigma_{n-1}}} \frac{1}{1-x_{\sigma_1}\cdots x_{\sigma_n}},$$

Note that since an alternant with an equal pair of entries is equal to zero, we could have also started with the expression

$$\Psi(u,x) = \frac{1}{V(x)} \sum_{\mu_0\ge\cdots\ge\mu_n\ge 0} u^{\mu_0-(n-1)} \sum_{\sigma\in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) x_{\sigma_1}^{\mu_1}\cdots x_{\sigma_n}^{\mu_n}$$

which, modifying the above argument, leads to

$$(2.3)\qquad \psi(x)=\frac{1}{V(x)}\sum_{\sigma\in S_{\mathcal{N}_n}}\operatorname{sgn}(\sigma)\frac{1}{1-x_{\sigma_1}}\cdots\frac{1}{1-x_{\sigma_1}\cdots x_{\sigma_n}}.$$

But, from (2.1), $\Psi(u,x)$ is a rational function with denominator $u^{n-1}V(x)\prod_{\alpha\subseteq\mathcal{N}_n}(1-ux_\alpha)$, so it has the partial fraction expansion

$$\Phi(u,x)=\sum_{\alpha\subseteq\mathcal{N}_n}\frac{A_\alpha}{u^{n-1}(1-ux_\alpha)}$$

where

$$A_\alpha=\{u^{n-1}(1-ux_\alpha)\Phi(u,x)\}\,|_{u=x_\alpha^{-1}}$$

Now let $|\alpha|=k$, $\omega=\omega_1\cdots\omega_k=\sigma_k\cdots\sigma_1$, $\rho=\rho_1\cdots\rho_{n-k}=\sigma_{k+1}\cdots\sigma_n$, so from (2.1),

$$A_\alpha=\frac{1}{1-x_\alpha^{-1}}\frac{1}{V(x)}\sum_{\substack{\omega\in S_\alpha\\ \rho\in S_{\bar\alpha}}}\operatorname{sgn}(\sigma)\frac{(x_{\omega_1}\cdots x_{\omega_{k-1}})^{-1}}{1-(x_{\omega_1}\cdots x_{\omega_{k-1}})^{-1}}\cdots$$
$$\cdots\frac{x_{\omega_1}{}^{-1}}{1-x_{\omega_1}{}^{-1}}\frac{x_{\rho_1}}{1-x_{\rho_1}}\cdots\frac{x_{\rho_1}\cdots x_{\rho_{n-k-1}}}{1-x_{\rho_1}\cdots x_{\rho_{n-k-1}}}\frac{1}{1-x_{\rho_1}\cdots x_{\rho_{n-k}}}.$$

But $\operatorname{sgn}(\sigma)=\operatorname{sgn}(\omega)\operatorname{sgn}(\rho)(-1)^{I(\alpha)+\binom{k}{2}}$, which gives

$$A_\alpha=\frac{x_\alpha^1(-1)^{I(\alpha)+\binom{k}{2}}}{V(x)}\sum_{\omega\in S_\alpha}\operatorname{sgn}(\omega)\frac{-1}{1-x_{\omega_1}}\cdots$$
$$\cdots\frac{-1}{1-x_{\omega_1}\cdots x_{\omega_{k-1}}}\frac{-1}{1-x_{\omega_1}\cdots x_{\omega_k}}$$
$$\times\sum_{\rho\in S_{\bar\alpha}}\operatorname{sgn}(\rho)\frac{x_{\rho_1}}{1-x_{\rho_1}}\cdots\frac{x_{\rho_1}\cdots x_{\rho_{n-k-1}}}{1-x_{\rho_1}\cdots x_{\rho_{n-k-1}}}\frac{1}{1-x_{\rho_1}\cdots x_{\rho_{n-k}}}$$

The summation in ω can be evaluated by (2.3) and the summation in ρ can be evaluated by (2.2). This yields

$$A_\alpha=\frac{(-1)^{I(\alpha)+\binom{k}{2}+k}}{V(x)}x_\alpha^1V(x_\alpha)V(x_{\bar\alpha})\psi(x_\alpha)\psi(x_{\bar\alpha}).$$

The result follows since

$$\Psi_m(x)=[u^m]\Psi(u,x)=\sum_{\alpha\subseteq\mathcal{N}_n}x_\alpha^{n+m-1}A_\alpha.$$

(b) Modifying the argument in (a), we obtain

$$\Phi(u,x,y) = \frac{1}{V(x)V(y)} \sum_{\lambda_0 \geq \cdots \geq \lambda_n \geq 0} u^{\lambda_0} \sum_{\substack{\sigma \in S_{\mathcal{N}_n} \\ \rho \in S_{\mathcal{N}_n}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)(x_{\sigma_1}y_{\rho_1})^{\lambda_1+n-1} \cdots \cdots (x_{\sigma_n}y_{\rho_n})^{\lambda_n}$$

$$= \frac{1}{u^{n-1}(1-u)V(x)V(y)} \sum_{\substack{\sigma \in S_{\mathcal{N}_n} \\ \rho \in S_{\mathcal{N}_n}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho) \frac{ux_{\sigma_1}y_{\rho_1}}{1-ux_{\sigma_1}y_{\rho_1}} \cdots \cdots \frac{ux_{\sigma_1}y_{\rho_1}\cdots x_{\sigma_{n-1}}y_{\rho_{n-1}}}{1-ux_{\sigma_1}y_{\rho_1}\cdots x_{\sigma_{n-1}}y_{\rho_{n-1}}} \frac{1}{1-ux_{\sigma_1}y_{\rho_1}\cdots x_{\sigma_n}y_{\rho_n}}$$

The result follows by a partial fraction expansion similar to that in (a), with denominators $1-ux_\alpha y_\beta$, for $\alpha, \beta \subseteq \mathcal{N}_n$, with $|\alpha| = |\beta|$.

If Proposition 1.1 is to be applied to Theorem 2.1, the division by $V(x)$ and $V(y)$ must first be carried out. This can be done by using the following result, in which expressions like those on the RHS of Theorem 2.1 are seen to arise in the Laplace expansion of the numerator alternant of a Schur function. (Recall that Schur functions are defined for an arbitrary vector of non-negative integers.)

PROPOSITION 2.2. *Let* $F(z_1,\ldots,z_k) = \sum_{a_1,\ldots,a_k \geq 0} c(a_1,\ldots,a_k) z_1^{a_1} \cdots z_k^{a_1}$ *be a symmetric function in* $z_1,\ldots,z_k$. *Then*

$$\sum_{\substack{\alpha \subseteq \mathcal{N}_n \\ |\alpha|=k}} (-1)^{I(\alpha)} \frac{V(x_{\bar\alpha})V(x_\alpha)}{V(x)} x_\alpha^{n-k} F(x_\alpha) = \sum_{a_1,\ldots,a_k \geqq 0} c(a_1,\ldots,a_k) s_{(a_1,\ldots,a_k)}(x)$$

Proof. Let $z = (z_1,\ldots,z_k)$. It is sufficient to prove this result when $F(z)$ is a monomial symmetric function. Thus, let

$$F(z) = \sum_{\rho \in S_{\mathcal{N}_k}} z_1^{\lambda_{\rho_1}} \cdots z_k^{\lambda_{\rho_k}}$$

where $\lambda_1,\ldots,\lambda_k \geq 0$. (This differs from the monomial symmetric function $m_\lambda(z)$ only by a multinomial coefficient scalar.) We now calculate

$$z^{n-k}V(z)F(z) = \sum_{\sigma \in S_{\mathcal{N}_n}} \operatorname{sgn}(\sigma) z_{\sigma_1}^{n-1} \cdots z_{\sigma_k}^{n-k} \sum_{\rho \in S_{\mathcal{N}_k}} z_1^{\lambda_{\rho_1}} \cdots z_k^{\lambda_{\rho_k}}$$

$$= \sum_{\sigma \in S_{\mathcal{N}_n}} \sum_{\rho \in S_{\mathcal{N}_k}} \operatorname{sgn}(\sigma) z_{\sigma_1}^{\lambda_{(\rho\sigma)_1}+n-1} \cdots z_{\sigma_k}^{\lambda_{(\rho\sigma)_k}+n-k}.$$

Now let $\omega = \rho\sigma$, so

$$z^{n-k}V(z)F(z) = \sum_{\omega \in S_{\mathcal{N}_k}} \sum_{\sigma \in S_{\mathcal{N}_k}} \operatorname{sgn}(\sigma) z_{\sigma_1}^{\lambda_{\omega_1}+n-1} \cdots z_{\sigma_k}^{\lambda_{\omega_k}+n-k}$$
$$= \sum_{\omega \in S_{\mathcal{N}_k}} \det\left(z_i^{\lambda_{\omega_j}+n-j}\right)_{k\times k}.$$

With $z = x_\alpha$, this gives

$$\sum_{\substack{\alpha\subseteq\mathcal{N}_n \\ |\alpha|=k}} (-1)^{I(\alpha)} \frac{V(x_{\bar\alpha})V(x_\alpha)}{V(x)} x_\alpha^{n-k} F(x_\alpha)$$
$$= \frac{1}{V(x)} \sum_{\substack{\alpha\subseteq\mathcal{N}_n \\ |\alpha|=k}} (-1)^{I(\alpha)} \det\left(x_{\bar\alpha_i}^{n-k-j}\right)_{(n-k)\times(n-k)} \sum_{\omega\in S_{\mathcal{N}_k}} \det\left(x_{\alpha_i}^{\lambda_{\omega_j}+n-j}\right)_{k\times k}$$
$$= \frac{1}{V(x)} \sum_{\omega\in S_{\mathcal{N}_k}} a_{(\lambda_{\omega_1}+n-1,\cdots,\lambda_{\omega_k}+n-k,n-k-1,\cdots,0)}(x),$$

(by the Laplace expansion on the first k columns)

$$= \sum_{\omega\in S_{\mathcal{N}_k}} s_{(\lambda_{\omega_1},\ldots,\lambda_{\omega_k})}(x).$$

Thus the result is true for monomial symmetric functions $F(z)$, and by linearity, for all symmetric functions $F(z)$.

The special case $k = n$ of Proposition 2.2 is especially striking. It says that

$$\sum_a c(a)x^a = \sum_a c(a)s_a(x)$$

when $\sum_a c(a)x^a$ is a symmetric function in x. This also can be obtained as a special case of Macdonald ([**5**], p. 32, Ex.12).

We now apply Proposition 2.2 to Theorem 2.1(a) to give an explicit form for $\Psi_{m,k}$. We use $z_{(k)}$ to denote $(z_1, \ldots, z_k)$.

THEOREM 2.3. *For* $k = 0, \ldots, n$ *let*

$$\sum_{t_{(k)}} b_k(t_{(k)}) z_{(k)}^{t_{(k)}} = (-1)^{\binom{k}{2}+k} \psi(x) \frac{\prod_{i=1}^k \{z_i^{m+k} E(-z_i, x)\}}{\prod_{i=1}^k (1-z_i^2) \prod_{1\le i<j\le k}(1-z_iz_j)^2}.$$

Then

$$\Psi_{m,k}(x) = \sum_{t_{(k)}} b_k(t_{(k)}) s_{t_{(k)}}(x).$$

Proof. Proposition 2.2 can be applied to Theorem 2.1 (a) with

$$F(x_\alpha) = (-1)^{\binom{k}{2}+k} x_\alpha^{m+k} \psi(x_\alpha)\psi(x_{\bar\alpha}).$$

But

$$\psi(x_\alpha)\psi(x_{\bar\alpha}) = \psi(x) \prod_{\substack{i\in\alpha \\ j\in\bar\alpha}} (1 - x_i x_j) = \psi(x) \frac{\prod_{i\in\alpha} E(-x_i, x)}{\prod_{i,j\in\alpha}(1 - x_i x_j)}$$

and the result follows.

This result allows us to write $\Psi_{m,k}(x)$ as a summation over Schur functions indexed by k-tuples, whose total degree in x are at least $k(m+k)$. To calculate the coefficient of $\frac{p_1(x)^n}{n!}$, we use the fact that the degree formula is valid for our definition of Schur functions (see, e.g., Macdonald, p. 25,64).

To obtain an explicit form for $\Phi_{m,k}$, we require two applications of Proposition 2.2 to Theorem 2.1(b), one in x and the other in y. The result is stated without proof. We use $w_{(k)}$ to denote $(w_1, \ldots, w_k)$.

THEOREM 2.4. *For $k = 0, \ldots, n$, let*

$$\sum_{t_{(k)}\mu_{(k)}} c_k(t_{(k)}, r_{(k)}) z_{(k)}^{t_{(k)}} w_{(k)}^{t_{(k)}} = (-1)^k \frac{\prod_{i=1}^k \{(z_i w_i)^{m+k} E(-z_i, y) E(-w_i, x)\}}{\prod_{i,j=1}^k (1 - z_i w_j)^2}$$

Then

$$\Phi_{m,k}(x, y) = \sum_{t_{(k)}\mu_{(k)}} c_k(t_{(k)}, r_{(k)}) s_{t_{(k)}}(x) s_{r_{(k)}}(y).$$

3. Formulas for the degree sums. From Theorems 2.3 and 2.4 we deduce the following results for the generating functions Ψ, Φ.

THEOREM 3.1.

$$\Psi_m(x) = \sum_{k=0}^{n} \Psi_{m,k}(x)$$

where every monmial in the expansion of $\Psi_{m,k}(x)$ has total degree at least $k(m+k)$. Moreover

(a) $\Psi_{m,0}(x) = \psi(x)$,

(b) $\Psi_{m,1}(x) = \psi(x)\sum_{i,\ell\geq 0}(-1)^{i+1}e_i(x)h_{m+i+2\ell+1}(x)$,
(c) $\Psi_{m,2}(x) = \psi(x)\sum_{i,j,\ell,r,t\geq 0}(-1)^{i+j+1}(t+1)e_i(x)e_j(x)s_{(m+2+i+2\ell+t,m+2+j+2r+t)}(x)$.

Proof. In Theorem 2.3, let B_k denote the generating function for the b_k's. Then
(a) $B_0 = \psi(x)$,
(b) $B_1 = (-1)^1\psi(x)\frac{z_1^{m+1}E(-z_1,x)}{1-z_1^2}$
$= \psi(x)\sum_{i,\ell\geq 0}(-1)^{i+1}e_i(x)z^{m+i+2\ell+1}$,
(c) $B_3 = (-1)^3\psi(x)\frac{z_1^{m+1}z_2^{m+2}E(-z_1,x)E(-z_2,x)}{(1-z_1^2)(1-z_2^2)(1-z_1z_2)^2}$
$= \psi(x)\sum_{i,j,\ell,r,t\geq 0}(-1)^{i+j+3}(t+1)e_i(x)e_j(x)z_1^{m+2+i+2\ell+t}z_2^{m+2+j+2r+t}$.
and the results follow from Theorem 2.3.

The expression which we can derive for an arbitrary $\Psi_{m,k}(x,y)$ from Theorem 2.3, in general involves $\binom{k}{2}+2k$ summation variables, and Schur functions with k non-negative indices.

THEOREM 3.2.

$$\Phi_m(x,y) = \sum_{k=0}^{n}\Phi_{m,k}(x,y)$$

where every monomial in the expansion of $\Phi_{m,k}(x,y)$ *has total degree at least* $k(m+k)$ *in the* x_i*'s and at least* $k(m+k)$ *in the* y_i*'s. Moreover*
(a) $\Phi_{m,0}(x,y) = \phi(x,y)$,
(b) $\Phi_{m,1}(x,y) = \phi(x,y)\sum_{i,j,t\geq 0}(-1)^{i+j+1}(t+1)e_i(x)h_{m+j+t+1}(x)e_j(y)h_{m+i+t+1}(y)$.

Proof. In Theorem 2.4, let C_k denote the generating function for the c_k's. Then
(a) $C_0 = \phi(x,y)$,
(b) $C_1 = (-1)^1\phi(x,y)\frac{z_1^{m+1}w_1^{m+1}E(-z_1,y)E(-w_1,x)}{(1-z_1w_1)^2}$
$= \phi(x,y)\sum_{i,j,t\geq 0}(-1)^{i+j+1}(t+1)e_i(x)e_j(y)z_1^{m+i+t+1}w_1^{m+i+t+1}$
and the results follow from Theorem 2.4.

The expression which we can derive for an arbitrary $\Phi_{m,k}(x,y)$ from Theorem 2.4, in general involves k^2+2k summation variables, and pairs of Schur functions in x and y, with k non-negative indices each.

Now, Proposition 1.1 is applied to the above results to obtain formulas for certain values of $S_m^{(i)}(n)$. We implicitly use the fact that, when $p_1(x) = x, p_i(x) = 0, i\geq 2$,

$$e_j(x) = h_i(x) = \frac{x^i}{j!}.$$

The symbol $(n)_i$ denotes the product $n(n-1)\ldots(n-i+1)$ when i is a positive integer, and $(n)_0 = 1$.

THEOREM 3.3. *For* $\frac{n-3}{2} \leq m \leq n-1$,

(a) $S_m^{(1)}(n) = \text{Inv}(n) - \sum_{\substack{i,j,l\geq 0 \\ 2i+2l+j=n-m-1}} \frac{(-1)^i}{i!j!}(n)_{i+j}\,\text{Inv}(j)$

(b) $S_m^{(2)}(n) = n! - \sum_{\substack{i,j,l\geq 0 \\ i+l+j\leq n-m-1}} \frac{(-1)^{i+j}}{l!\,i!\,j!}(n-m-i-j-l)\,(n)_{i+l}\,(n)_{j+l}$

Proof. (a) By definition

$$S_m^{(1)}(n) = \left[\frac{p_1(x)^n}{n!}\right]\Psi_m(x) = \left[\frac{p_1(x)^n}{n!}\right]\big(\Psi_{m,0}(x) + \Psi_{m,1}(x)\big),$$

since, by Theorem 3.1, every monomial in $\Psi_m(x) - \Psi_{m,0}(x) - \Psi_{m,1}(x)$ has total degree greater than n for m in the given range. We now use the expression for $\Psi_{m,1}(x)$ given by Theorem 3.1(a), and set $p_1(x) = x$, $p_i(x) = 0$ for $i \geq 2$, to get

$$\left[\frac{p_1(x)^n}{n!}\right]\Psi_{m,1}(x) = \left[\frac{x^n}{n!}\right]\psi(x)\sum_{i,l\geq 0}(-1)^{i+l}\frac{x^i}{i!}\frac{x^{m+i+l}}{(m+i+l)!}.$$

But, when $p_1(x) = x$, $p_i(x) = 0$ for $i \geq 2$,

$$\psi(x) = \sum_{j\geq 0}\text{Inv}(j)\frac{x^j}{j!},$$

from (1.1), and the result follows.

(b) By definition,

$$S_m^{(2)}(n) = \left[\frac{p_1(x)^n}{n!}\frac{p_1(y)^n}{n!}\right]\Phi_m(x,y) = \left[\frac{p_1(x)^n}{n!}\frac{p_1(y)^n}{n!}\right]\big(\Phi_{m,0}(x,y) + \Phi_{m,1}(x,y)\big),$$

since, by Theorem 3.2, every monomial in $\Phi_m(x,y) - \Phi_{m,0}(x,y) - \Phi_{m,1}(x,y)$ has total degree greater than n in the x_i's and greater than n in the y_i's for m in the given range. We now use the expression for $\Phi_{m,1}(x,y)$ given by Theorem 3.2(b), and set $p_1(x) = x$, $p_1(y) = y$, $p_i(x) = p_j(y) = 0$ for $i \geq 2$, to get

$$\left[\frac{p_1(x)^n}{n!}\ \frac{p_1(y)^n}{n!}\right]\Phi_{m,1}(x,y) = \left[\frac{x^n}{n!}\frac{y^n}{n!}\right]\phi(x,y)\sum_{i,j,t\geq 0}(-1)^{i+j}\frac{x^i}{i!}\frac{x^{m+j+t}}{(m+j+t)!}\frac{y^j}{j!}\frac{y^{m+i+t}}{(m+i+t)!}.$$

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But under the above power sum substitutions,

$$\phi(x,y) = \sum_{l\geq 0} \frac{x^l y^l}{l!},$$

from (1.1), and the result follows.

The corresponding values for $T_m^{(i)}(n)$ are now given (without proof), since some collapse occurs in evaluating

$$S_m^{(i)}(n) - S_{m-1}^{(i)}(n).$$

COROLLARY 3.4. *For* $\frac{n-1}{2} \leq m \leq n$,

(a) $T_m^{(1)}(n) = (-1)^{n-m} \sum_{\substack{i,j\geq 0\\ 2i+j\leq n-m}} \frac{(-1)^{i+j}}{i!j!}(n)_{i+j}\,\mathrm{Inv}(j)$,

(b) $T_m^{(2)}(n) = \sum_{\substack{i,j,l\geq 0\\ i+j+l\leq n-m}} \frac{(-1)^{i+j}}{i!j!l!}(n)_{i+l}(n)_{j+l}$

For example, the degrees of the partitions of 9 with 4 parts are 84, 168, 216, 189, 56, so $T_4^{(1)}(9) = 929$ and $T_4^{(2)}(9) = 17,557$, in agreement with the above formulas.

The simple nature of the summations in Theorem 3.3 and Corollary 3.4 suggests that it might be possible to derive these by simpler means than those used here, presumably by an inclusion-exclusion argument, since they are alternating sums. Especially noteworthy is the fact that the summands in Corollary 3.4 are independent of m.

Finally, a single more complicated result is given, in which $\left[\frac{p_1(x)^n}{n!}\right]$ must be applied to an expression involving Schur functions indexed by pairs of non-negative integers (see the comment following Theorem 2.3).

THEOREM 3.5. *For* $\frac{n-8}{3} \leq m \leq \frac{n-4}{2}$,

$$S_m^{(1)}(n) = \mathrm{Inv}(n) - \sum_{\substack{i,j,l\geq 0\\ 2i+2l+j=n-m-1}} \frac{(-1)^i}{i!\,j!}(n)_{i+j}\,\mathrm{Inv}(j)$$
$$- n! \sum_{\substack{i,j,l,r,t,u\geq 0\\ 2(i+l+j+r+t)+u=n-2m-4}} \frac{(-1)^{i+j}(t+1)(i-j+2(l-r)+1)\,\mathrm{Inv}(u)}{i!\,j!\,u!\,(m+3+i+2l+t)!\,(m+2+j+2r+t)!}$$

Proof. Following the proof of Theorem 3.3(a), we get

$$S_m^{(1)}(n) = \left[\frac{p_1(x)^n}{n!}\right]\left(\Psi_{m,0}(x) + \Psi_{m,1}(x) + \Psi_{m,2}(x)\right).$$

The value of this coefficient in $\Psi_{m,0}(x,y)+\Psi_{m,1}(x,y)$ has been obtained in Theorem 3.3(a). We must add to this the value of

$$\left[\frac{p_1(x)^n}{n!}\right]\Psi_{m,2}(x),$$

which can be determined in a straightforward manner from Theorem 3.1(c), since

$$s_{(j,l)}(x) = \frac{(j+1-l)}{(j+1)!l!}x^{j+l}$$

under the substitution $p_1(x) = x$, $p_i(x) = 0$ for $i \geq 2$.

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