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NORM ONE MULTIPLIERS ON SUBSPACES OF L^p

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ABSTRACT. We present a new elementary proof of the fact that a norm one multiplier ϕ on $L^p(T)$ satisfying $\phi(0) = \phi(k) = 1$ is k-periodic, and extend this result, when possible, to multipliers on translation invariant subspaces of L^p. A consequence of our work is that all such multipliers on $H^p(T)$ are the restriction of a norm one multiplier on $L^p(T)$.

0. **Introduction.** Let G be a compact abelian group and let Γ be its dual group. A function $\phi : \Gamma \longrightarrow \mathbb{C}$ is called a multiplier on a subspace S of $L^p(G)$ if the map M_ϕ defined on S by $\widehat{M_\phi f}(\chi) = \phi(\chi)\hat{f}(\chi)$ for $f \in S$, $\chi \in \Gamma$, maps S to $L^p(G)$. The class of all multipliers on S will be denoted $M(S)$ and the operator norm of the multiplier $\phi \in M(S)$ will be denoted by $\|\phi\|_{M(S)}$. If μ is a measure on G then $\hat{\mu} \in M(L^p)$ for $1 \leq p \leq \infty$, and indeed all elements of $M(L^1)$ and $M(L^\infty)$ are of this form. The reader is referred to [3, Ch. 16] for standard results on multipliers.

In this paper we are interested in studying an extreme face of the unit ball of $M(S)$, namely

$$W(S) : \{\phi \in M(S) : \|\phi\|_{M(S)} = 1 = \phi(1)\}.$$

(Here 1 is the identity element of Γ.) The space $W\big(L^p(G)\big)$ was introduced by Shapiro [5]. For $1 < p < \infty$ the space $W\big(L^p(G)\big)$ is known to contain multipliers which are not the Fourier Stieltjes transform of a measure [4]. Shapiro and subsequently Benyamini and Lin (in [1] and [2]) have shown a striking similarity between certain multipliers in $W\big(L^p(G)\big)$ and the multipliers arising from probability measures on G. For example, Benyamini and Lin show that all multipliers $\phi \in W\big(L^p(T)\big)$ for $1 \leq p \leq \infty, p \neq 2$, satisfying $\phi(k) = 1$ for some $k \neq 0$, are k-periodic sequences on $\mathbb{Z}$. The cases $p = 1$ and $p = \infty$ are easy as any such multiplier $\phi = \hat{\mu}$ where μ is a probability measure supported on the k-th roots of unity.

We present new elementary proofs of these results and extend them (when possible) to multipliers on translation invariant subspaces of L^p such as the classical Hardy spaces $H^p(T)$. A consequence of our results is that any $\phi \in W\big(H^p(T)\big)$ satisfying $\phi(k) = 1$ for some $k \neq 0$, is the restriction of a norm one multiplier on $L^p(T)$.

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1. **Multipliers on subgroups.** Motivated by properties of probability measures, Shapiro [5] proved that if $p \neq 2$, $\phi \in W(L^p(T))$ and $\phi(-1) = 1$, then $\phi \equiv 1$. Subsequently it was shown that if G was any lca group and $\phi \in W(L^p(G))$ for $p \neq 2$, then $\{\gamma \in \Gamma : \phi(\gamma) = 1\}$ was a subgroup of Γ. (See [2] and remark (a) at the end of [5]). (Of course the $p = 2$ case is different since any bounded sequence is an L^2-multiplier.) Deep results about norm one projections of $L^p(G)$ were used by Benyamini and Lin to give an elegant proof of this generalization.

Shapiro's method was to find an appropriate test function $f \in L^p(G)$ and show $M_\phi = f$. Our approach is a little different. We choose test functions f belonging to the translation invariant subspace which is the domain of the map M_ϕ and then use Taylor series expansions to estimate the p-norms of f and $M_\phi f$. We make repeated use of the fact that if $|x| \leq r < 1$ then

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!}+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)x^4}{4!}+R(x)$$

where $|R(x)| \leq C(\alpha, r)|x|^5$.

First a preliminary estimate:

LEMMA 1.1. *Let $1 \leq p < \infty$, $\chi \in \Gamma$ and $\chi^2 \neq 1$. If b is a real number and $|r| \leq 1$, then as $b \longrightarrow 0$*

$$\|1 + b\chi + rb\chi^{-1}\|_p = 1 + b^2\Big(\frac{1}{2}(1 + |r|^2) + \frac{1}{2}(\frac{p}{2} - 1)|1 + r|^2\Big) + 0(|b|^3).$$

PROOF. Let

$$X = X(b, r) \equiv \frac{2\operatorname{Re}\chi(\overline{rb} + b) + 2\operatorname{Re}\chi^2 b^2\bar{r}}{1 + |b|^2(1 + |r|^2)}.$$

With this notation

$$\|1 + b\chi + rb\chi^{-1}\|_p^p = \big(1 + b^2(1 + |r|^2)\big)^{\frac{p}{2}} \int (1 + X)^{\frac{p}{2}}.$$

If $|b|$ is sufficiently small a Taylor series expansion gives

$$\int (1 + X)^{\frac{p}{2}} = \int \Big(1 + \frac{p}{2}X + \frac{p}{2}\frac{(\frac{p}{2} - 1)}{2}X^2 + 0(\|X\|_\infty^3)\Big).$$

As $\int \chi^{\pm 1} = \int \chi^{\pm 2} = 0$ the latter integral simplifies to

$$1 + \frac{\frac{p}{2}(\frac{p}{2} - 1)b^2|r + 1|^2}{\big(1 + b^2(1 + |r|^2)\big)^2} + 0(|b|^3).$$

After taking a Taylor series expansion for $\big(1 + b^2(1 + |r|^2)\big)^{\frac{p}{2}}$ we see that

$$\|1 + b\chi + rb\chi^{-1}\|_p = \Big[1 + b^2 p\Big(\frac{1 + |r|^2}{2} + \frac{1}{2}(\frac{p}{2} - 1)|1 + r|^2\Big) + 0(|b|^3)\Big]^{\frac{1}{p}}$$

and one final Taylor series expansion completes the proof. ■

For $E \subseteq \Gamma$ let $L^p_E(G) = \{f \in L^p(G) : \hat{f}(\chi) = 0 \text{ if } \chi \notin E\}$. Of course $L^p_\Gamma(G) = L^p(G)$. It is well known that all translation invariant subspaces of L^p are of this form; for example $H^p(T) = L^p_{\mathbb{Z}+}(T)$.

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THEOREM 1.2. *Let $1 \leq p \leq \infty$, $p \neq 2$ and suppose $\phi \in M(L^p_E)$ is a multiplier of norm 1. Assume that χ, $\chi\psi$ and $\chi\psi^2$ (or $\chi\psi^{-1}$) belong to E, and $\phi(\chi) = \phi(\chi\psi) = 1$. Then $\phi(\chi\psi^2)$ (or $\phi(\chi\psi^{-1})$)= 1.*

PROOF. We may assume $\psi^2 \neq 1$ else there is nothing to prove and we consider the cases $p = \infty$ and $1 \leq p < \infty$ but $p \neq 2$ separately. Note that when $E \neq \Gamma$ the case $p = \infty$ does not follow by duality from the case $p = 1$.

Suppose $\phi(\chi\psi^2) = s \neq 1$. (The case $\phi(\chi\psi^{-1}) \neq 1$ is similar). Replacing ϕ if necessary by the norm one multiplier $\frac{1}{N}\sum_{n=1}^{N}\phi^n$, we may assume $|s|$ is arbitrarily small.

Let $f = \chi\psi + b\chi + rb\chi\psi^2 \in L^p_E$ for $|r| \leq 1$ and b real and small. Since

$$\frac{\|M_\phi f\|_p}{\|f\|_p} = \frac{\|\chi\psi + b\chi + rbs\chi\psi^2\|_p}{\|\chi\psi + b\chi + rb\chi\psi^2\|_p} \leq 1$$

with $|s|$ arbitrarily small, we may as well assume $s = 0$. (We could also reach this conclusion by replacing ϕ by a weak cluster point of the sequence $\frac{1}{N}\sum_1^N \phi^n$ in the weak operator topology, but we prefer to keep the proof entirely elementary.) When $1 \leq p < \infty$ Lemma 1.1 shows that

$$\frac{\|M_\phi f\|_p}{\|f\|_p} = \frac{1 + \frac{b^2 p}{4} + 0(|b|^3)}{1 + \frac{b^2}{2}(1 + |r|^2 + (\frac{p}{2} - 1)|1 + r|^2) + 0(|b|^3)}.$$

Since ϕ is a norm one multiplier, letting $b \longrightarrow 0$ we see that

$$2\,\mathrm{Re}\, r(\frac{p}{2} - 1) + \frac{p}{2}|r|^2 \geq 0.$$

When $p \neq 2$ we can clearly choose r with $|r| \leq 1$ but contradicting this inequality. Hence s must equal 1.

For the case $p = \infty$ set $r = -1$ and $b > 0$. Then

$$\begin{aligned}\|f\|_\infty^2 &= \sup\{\,|1 + b\big(\psi^{-1}(x) - \psi(x)\big)|^2 : x \in G\}\\ &= \sup\{\,|1 - 2bi\,\mathrm{Im}\,\psi(x)|^2 : x \in G\} \leq 1 + 4b^2,\end{aligned}$$

while

$$\|M_\phi f\|_\infty^2 \geq |M_\phi f(0)|^2 = |1 + b - bs|^2.$$

As before, if $s \neq 1$ we may assume $s = 0$, and since $(1 + b)^2 > 1 + 4b^2$ for b small we again obtain a contradiction. ∎

COROLLARY 1.3. *Let $1 \leq p \leq \infty$, $p \neq 2$. If E contains the arithmetic progression $\Lambda = \{\chi^{-m}, \ldots, \chi^{-1}, 1, \chi, \ldots, \chi^n\}$ for some $n, m, \in N$, and $\phi \in W(L^p_E)$ with $\phi(\chi) = 1$, then $\phi|_\Lambda = 1$.*

Next we generalize from arithmetic progressions to subgroups.

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THEOREM 1.4. *Let $1 \le p \le \infty, p \neq 2$ and suppose $\phi \in W(L_E^p(G))$. If $1, \chi, \psi, \chi\psi \in E$, none of χ, ψ or $\chi\psi$ are of order 2 and $\phi(\chi) = \phi(\psi) = 1$, then $\phi(\chi\psi) = 1$.*

REMARK. The condition $(\chi\psi)^2 \neq 1$ is unnecessary but without it several additional cases need to be considered. Our purpose here is not to give as complete a proof as possible, just to illustrate the technique.

PROOF. The case $p = \infty$ is easiest and does not require the order 2 condition. For $c < 0$ and $a = b = \sqrt{|c|}$, let $f = 1 + a\chi + b\psi + c\chi\psi$. As before, if $\phi(\chi\psi) = s \neq 1$ we can assume $s = 0$ so $M_\phi f = 1 + a\chi + b\psi$ and $\|M_\phi f\|_\infty = 1 + a + b$. Certainly

$$\|f\|_\infty \le \sup\{\,|1 + a\alpha + b\beta + c\alpha\beta| : |\alpha| = |\beta| = 1\}.$$

One can verify by routine calculations that for c sufficiently small $\|f\|_\infty$ is strictly less than $\|M_\phi f\|_\infty$, contradicting the fact that the norm of ϕ is 1.

Now assume $1 \le p < \infty, p \neq 2$. Without loss of generality we may assume none of the following products is 1; for if so then the fact that $\phi(\chi\psi) = 1$ is either obvious or follows immediately from Theorem 1.2:

$$\chi\psi,\ \chi\bar{\psi},\ \chi^2\psi,\ \psi^2\chi,\ \chi^2\bar{\psi},\ \psi^2\bar{\chi}.$$

Choose $\lambda = \lambda_p$ with λ^2 real so that

(1) if $\chi^3\psi = 1 = \psi^3\chi$ then $\lambda^2(p-2)p(\frac{p}{2} - 1) + \frac{p^2}{4} < 0$;

(2) if precisely one of $\chi^3\psi$ or $\psi^3\chi = 1$ then $(\lambda^2 p + \frac{p}{2} - 2)\frac{p}{2}(\frac{p}{2} - 1) + \frac{p^2}{2} < 0$; or

(3) if neither $\chi^3\psi$ nor $\psi^3\chi$ is 1 then $\frac{p^2}{2}\lambda^2(\frac{p}{2} - 1) + \frac{p^2}{4} < 0$.

(Note that as $p \neq 2$ these are always possible to do.)

In either case 1 or 3 we let $f = 1 + \lambda c(\chi + \psi) + c^2\chi\psi$ where c is a small real number. If $\chi^3\psi = 1$ but $\psi^3\chi \neq 1$ let $f = 1 + c\chi + \lambda^2 c\psi + c^2\chi\psi$ (case 2a) and if $\psi^3\chi = 1$ but $\chi^3\psi \neq 1$ let $f = 1 + \lambda^2 c\chi + c\psi + c^2\chi\psi$ (case 2b). For $a, b \in \mathbb{C}, d \in \mathbb{R}$ let

$$X(a,b,d) := \frac{1}{(1 + |a|^2 + |b|^2 + |d|^2)}\Big(2\operatorname{Re}\big(\chi(a + d\bar{b}) + \psi(b + d\bar{a}) + \chi\psi d + \chi\overline{\psi}\overline{ba}\big)\Big).$$

As usual we may assume $\phi(\chi\psi) = 0$, thus

$$\|f\|_p^p = (1 + |a|^2 + |b|^2 + |c|^4)^{p/2} \int \big(1 + X(a,b,c^2)\big)^{p/2}$$

and

$$\|M_\phi f\|_p^p = (1 + |a|^2 + |b|^2)^{p/2} \int \big(1 + X(a,b,0)\big)^{p/2}$$

where $a = b = \lambda c$ in (1) or (3), $a = c, b = \lambda^2 c$ in (2a) and $a = \lambda^2 c, b = c$ in (2b).

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Taylor series expansions show that

$$\|f\|_p^p - \|Mf\|_p^p = (1+|a|^2+|b|^2+|c|^4)^{\frac{p}{2}} \int \Big\{X(a,b,c^2) - X(a,b,0)$$
$$+ \frac{\frac{p}{2}(\frac{p}{2}-1)}{2}\big(X^2(a,b,c^2) - X^2(a,b,0)\big)$$
$$+ \frac{\frac{p}{2}(\frac{p}{2}-1)(\frac{p}{2}-2)}{3!}\big(X^3(a,b,c^2) - X^3(a,b,0)\big)$$
$$+ \frac{\frac{p}{2}(\frac{p}{2}-1)(\frac{p}{2}-2)(\frac{p}{2}-3)}{4!}\big(X^4(a,b,c^2) - X^4(a,b,0)\big)\Big\}$$
$$+ \frac{p}{2}c^4 \int \big(1+X(a,b,0)\big)^{\frac{p}{2}} + 0(|c|^5).$$

Our assumptions clearly imply that

$$\int X(a,b,c^2) = \int X(a,b,0) = 0$$

in each case, and that

$$\int X^4(a,b,c^2) - X^4(a,b,0)$$
$$= \Big(\frac{1}{(1+|a|^2+|b|^2+|c|^4)^4} - \frac{1}{(1+|a|^2+|b|^2)^4}\Big)0(|c|^4) + 0(|c|^5)$$
$$= 0(|c|^5).$$

Similarly it can be seen that

$$\int X^2(a,b,c^2) - X^2(a,b,0)$$
$$= 2\Big(\frac{1}{(1+|a|^2+|b|^2+|c|^4)^2} - \frac{1}{(1+|a|^2+|b|^2)^2}\Big)\big(|a|^2+|b|^2+|ab|^2+\varepsilon 2\,\mathrm{Re}(a\bar{b})^2\big)$$
$$+ \frac{2}{(1+|a|^2+|b|^2+|c|^4)^2}(2\,\mathrm{Re}\,ac^2b + |c^2b|^2 + 2\,\mathrm{Re}\,bc^2a + |c^2a^2| + |c|^4)$$

where $\varepsilon = 1$ if $(\chi\bar{\psi})^2 = 1$ and $\varepsilon = 0$ otherwise. This simplifies to

$$2c^4 + 8c^4\lambda^2 + 0(|c|^5)$$

in each of the cases.

The most complicated term to examine is $\int X^3(a,b,c^2) - X^3(a,b,0)$. This is where the differences occur depending on whether or not $\chi^3\psi$ and/or $\psi^3\chi$ is 1. Let $\varepsilon_1 = 1$ if $\chi^3\psi = 1$ and 0 otherwise and let $\varepsilon_2 = 1$ if $\psi^3\chi = 1$ and 0 otherwise. A careful analysis of all the terms appearing in X^3 shows that

$$\int X^3(a,b,c^2) - X^3(a,b,0)$$
$$= \Big(\frac{1}{(1+|a|^2+|b|^2+|c|^4)^3} - \frac{1}{(1+|a|^2+|b|^2)^3}\Big)0(|c|^3)$$
$$+ \frac{1}{(1+|a|^2+|b|^2+|c|^4)^3}(\varepsilon_1 6\,\mathrm{Re}\,a^2c^2 + 12\,\mathrm{Re}\,abc^2 + \varepsilon_2 6\,\mathrm{Re}\,b^2c^2)$$
$$= \varepsilon_1 6a^2c^2 + 12abc^2 + \varepsilon_2 6b^2c^2 + 0(|c|^6).$$

To finish off consider each case separately. In case 1 for example, $\varepsilon_1 = \varepsilon_2 = 1$, $a = b = \lambda c$ so

$$\|f\|_p^p - \|M_\phi f\|_p^p = c^4\Big(\frac{p^2}{2} + \lambda^2(p-2)p(\frac{p}{2}-1)\Big) + 0(|c|^5).$$

By (1) this is negative if $|c|$ is sufficiently small, contradicting the fact that $\|\phi\| = 1$. The other cases are similar. ∎

Since there are no elements of order 2 in $\mathbb{Z}$ the following corollaries are obvious.

COROLLARY 1.5. *If ϕ is any norm one multiplier on $L_E^p(T)$ with $\phi(1) = \phi(n) = \phi(k) = 1$, and $n + k \in E$, then $\phi(n+k) = 1$.*

COROLLARY 1.6. *If $\phi \in W\big(L^p(T)\big)$ then $\{n : \phi(n) = 1\}$ is a subgroup of $\mathbb{Z}$.*

3. **Periodicity on cosets.** Benyamini and Lin were able to generalize the results of the first section to show that if ϕ was a norm one multiplier on $L^p(G)$, then ϕ was constant on each coset of $\{\gamma \in \Gamma : \phi(\gamma) = 1\}$. This answered a question of Carleson (see [5]). We will show that this result does not generalize to norm one multipliers on L_E^p, although it is true for multipliers on $H^p(T)$.

EXAMPLE 2.1. Let $E = \{0,1,4,5\}$ and let $\phi : \mathbb{Z} \longrightarrow \mathbb{C}$ satisfy $\phi(0) = 1 = \phi(4)$, $\phi(1) = s$, $\phi(5) = t$. We will show that there exists an $\varepsilon > 0$ such that if $|s|, |t| \leq \varepsilon$ then ϕ is a norm one multiplier on $L_E^4(T)$.

PROOF. Let $f = d + ae^{ix} + be^{i4x} + ce^{i5x}$. It is routine to verify that

$$\begin{aligned}\|f\|_4^4 - \|M_\phi f\|_4^4 \leq |a|^4(1-|s|^4) &+ |c|^4(1-|t|^4)\\ &+ 4(1-|s|^2)(|ad|^2 + |ab|^2) + 4(1-|t|^2)(|bc|^2 + |cd|^2)\\ &+ 4(1-|st|^2)|ac|^2 + 8\,\mathrm{Re}\, a\bar{d}b\bar{c}(1-s\bar{t}).\end{aligned}$$

If $d = 0$ then clearly $\|f\|_4^4 \geq \|M_\phi f\|_4^4$ for all choices of s, t provided $|s|, |t| \leq 1$. Thus assume $d = 1$. Now

$$8|\,\mathrm{Re}\, ab\bar{c}(1-s\bar{t})| \leq 2|1-s\bar{t}|(|ab|^2 + |c|^2 + |a|^2 + |bc|^2).$$

Hence if s, t are chosen so that

$$2|1-s\bar{t}| \leq \min\big(4(1-|t|^2), 4(1-|s|^2)\big)$$

then $\|f\|_4^4 \geq \|M_\phi f\|_4^4$, hence ϕ is norm one. ∎

It is well known that there are multipliers on $H^1(T)$ which are not multipliers on $L^1(T)$. Our next example shows there are multipliers in $W\big(H^1(T)\big)$ which are not in $W\big(L^1(T)\big)$.

EXAMPLE 2.2. Let $\phi(0) = 1$, $0 < \phi(1) = a < \frac{1}{2}$ and $\phi(n) = 0$ for all other integers. The norm of ϕ as a multiplier on $L^1(T)$ is equal to $\|1 + ae^{it}\|_1$ which by Lemma 1.1 is greater than one if a is small enough. Thus $\phi \notin W\big(L^1(T)\big)$.

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Let $f \in H^1(T)$, say $f(t) = b + ce^{it} + g(t)$, where $g \in H^1(T)$, $\hat{g}(0) = \hat{g}(1) = 0$. If $b = 0$ then clearly $\|M_\phi f\|_1 \leq \|f\|_1$, so assume $b = 1$. If $|c| \geq 2$ then

$$\|M_\phi f\|_1 \leq 1 + |ac| \leq |c| \leq \|f\|_1.$$

Thus assume $|c| < 2$. Let $F(t) = 1 + \frac{1}{2}(e^{it} + e^{-it})$. As $\|F\|_1 = 1$, $\|F * f\|_1 \leq \|f\|_1$. But

$$\|F * f\|_1 = \|1 + \frac{c}{2}e^{it}\|_1 = \|(1 + \frac{c}{2}e^{it})^{1/2}\|_2^2.$$

Since $|c/2| < 1$ we can compute a Taylor series expansion for $(1 + \frac{c}{2}e^{it})^{1/2}$ to obtain the inequality

$$\|F * f\|_1 \geq 1 + \frac{|c|^2}{16}.$$

From Lemma 1.1

$$\|M_\phi f\|_1 = \|1 + ace^{it}\|_1 \leq 1 + \frac{|ac|^2}{4} + 0(|ac|^3),$$

so for a sufficiently small $\|M_\phi f\|_1 \geq \|f\|_1$ proving that $\phi \in W\big(H^1(T)\big)$. ∎

This example shows that properties of multipliers in $W\big(H^1(T)\big)$ do not follow automatically from the corresponding results for $W\big(L^1(T)\big)$; however it is possible to modify [1] to prove

THEOREM 2.3. *Let* $1 \leq p \leq \infty$, $p \neq 2$. *Suppose* ϕ *is a norm one multiplier on* $H^p(T)$ *with* $\phi(0) = \phi(k) = 1$ *for some* $k \neq 0$. *Then if* m *and* n *are positive integers and* $m \equiv n \bmod k$ *then* $\phi(m) = \phi(n)$.

PROOF. The cases $1 \leq p < 2$, $2 < p < \infty$ and $p = \infty$ are treated separately.

(a) $2 < p < \infty$: Assume $\phi(m) \neq \phi(n)$ for some $m \equiv n \bmod k$. Let $f(t) = e^{imt} - e^{int}$. Note that f is $\frac{2\pi}{k}$ periodic, and as f is continuous and $f(0) = 0$ an application of the mean value theorem shows that there is a neighbourhood I_ε of 0 such that $|I_\varepsilon| = C\varepsilon$ (for $C = \frac{1}{|n-m|}$) and $|f| \leq \varepsilon$ on I_ε. Since $M_\phi(f)(0) = \phi(m) - \phi(n) \neq 0$ there is an interval I and constant C_0 such that $|M_\phi(f)| \geq C_0 > 0$ on I. Without loss of generality $I_\varepsilon \subseteq I$ and $|I_\varepsilon| \leq \frac{2\pi}{k}$. Let $J_\varepsilon = \cup_{j=0}^{k-1}(I_\varepsilon + \frac{2\pi j}{k})$. By periodicity $|f| \leq \varepsilon$ on J_ε.

Choose $0 < r < p - 2$ and $s > 1 + 2/r$. Choose a polynomial $g_1 = g_1(\varepsilon)$ such that

(i) $1 - \varepsilon^s \leq |g_1| \leq 1$ on kI_ε,

(ii) $\|g_1|_{(kI_\varepsilon)^c}\|_p \leq \varepsilon^s$, and

(iii) $\|g_1\|_\infty \leq 1$.

Let $g_2(t) = g_1(kt)$ so $\hat{g}_2$ is supported on $k\mathbb{Z}$. Furthermore notice that

(i′) $1 - \varepsilon^s \leq |g_2| \leq 1$ on J_ε and

(ii′) $\|g_2|_{J_\varepsilon c}\|_p \leq \varepsilon^s$.

Since $g_2 f 1_{J_\varepsilon} \in L^1(T)$, it follows from the Riemann Lebesgue lemma that we can choose $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\left|\int_{J_\varepsilon} e^{iNkt} g_2 \bar{f}\, dt\right| + \left|\int_{J_\varepsilon} e^{-iNkt} \bar{g}_2 f\, dt\right| \leq \varepsilon^s.$$

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If in addition we choose N large enough we can assume $g \equiv e^{iNkt} g_2 \in H^p(T)$. Notice that g has properties (i′) and (ii′), and supp $\hat{g} \subseteq kZ$, so g is $\frac{2\pi}{k}$ periodic.

Now we make some estimates. By (ii′) it follows that

$$\int_{J_\varepsilon^c} |g + \varepsilon^{1/r} f|^p \leq 2^p(\varepsilon^{sp} + \varepsilon^{p/r} 2^p).$$

Also

$$\int_{J_\varepsilon} |g + \varepsilon^{1/r} f|^p = \int_{J_\varepsilon} (1 + \varepsilon^{2/r}|f|^2 + |g|^2 - 1 + 2\operatorname{Re} g\bar{f}\varepsilon^{1/r})^{\frac{p}{2}}.$$

Since $|f| \leq \varepsilon$ on J_ε and $\big||g|^2 - 1\big| \leq 1 - (1-\varepsilon)^{2s}$ on J_ε, we can use our usual Taylor series expansion (provided ε is small enough) to obtain

$$\int_{J_\varepsilon} |g + \varepsilon^{1/r} f|^p = \int_{J_\varepsilon} (1 + \frac{p}{2}(\varepsilon^{2/r}|f|^2 + |g|^2 - 1 + 2\operatorname{Re} g\bar{f}\varepsilon^{1/r}) + 0\big(\max(\varepsilon^{2/r+2}, \varepsilon^{2s})\big)\Big)$$

Recalling further the definition of g we see that

$$\left|\int_{J_\varepsilon} \operatorname{Re} g\bar{f}\right| \leq \varepsilon^s,$$

thus combining these results we get that

$$\begin{aligned} \|g + \varepsilon^{1/r} f\|_p^p &\leq |J_\varepsilon|\Big(1 + 0\big(\max(\varepsilon^{2/r+2}, \varepsilon^s)\big)\Big) + 0\big(\max(\varepsilon^{s+1/r}, \varepsilon^{sp}, \varepsilon^{p/r})\big) \\ &= k|I_\varepsilon|\Big(1 + 0\big(\max(\varepsilon^{2/r+2}, \varepsilon^s)\big)\Big) + 0\big(\max(\varepsilon^{s+1/r}, \varepsilon^{p/r})\big). \end{aligned}$$

Next we estimate $\|M_\phi(g + \varepsilon^{1/r} f)\|_p^p$. By Corollary 1.3 we see that $\phi(z) = 1$ for $z \in k\mathbb{Z}^+$, so since $\hat{g}$ is supported on $k\mathbb{Z}^+$, $M_\phi(g) = g$. Thus

$$\|M_\phi(g + \varepsilon^{1/r} f)\|_p^p \geq \int_{J_\varepsilon} |g + \varepsilon^{1/r} M_\phi(f)|^p.$$

The definition of f ensures that

$$M_\phi f(t + \frac{2\pi j}{k}) = M_\phi f(t) \exp 2\pi \frac{inj}{k}.$$

Thus

$$\begin{aligned} \|M_\phi(g + \varepsilon^{1/r} f)\|_p^p &\geq \sum_{j=0}^{k-1} \int_{I_\varepsilon} |\, g + \varepsilon^{1/r} M_\phi f(t) \exp 2\pi \frac{inj}{k}|^p \\ &\geq (1 - \varepsilon^s)^p \sum_{j=0}^{k-1} \int_{I_\varepsilon} |1 + \varepsilon^{1/r} \frac{M_\phi f(t)}{g(t)} \exp 2\pi \frac{inj}{k}|^p. \end{aligned}$$

Hölder's inequality and orthogonality show that a lower bound for the sum is $k(1+C_0\varepsilon^{2/r})$ (cf. [1, p. 43]), thus

$$\|M_\phi(g + \varepsilon^{1/r} f)\|_p^p \geq (1 - \varepsilon^s)^p |I_\varepsilon| k(1 + C_0\varepsilon^{2/r}).$$

Upon considering the ratio

$$\frac{\|M_\phi(g + \varepsilon^{1/r} f)\|_p^p}{\|g + \varepsilon^{1/r} f\|_p^p} \leq 1$$

and recalling that $|I_\varepsilon| \leq C\varepsilon$ we see that we must have

$$\left(1 - p\varepsilon^s + 0(\varepsilon^{2s})\right)(1 + C_0\varepsilon^{2/r}) \leq 1 + 0\left(\max(\varepsilon^{2/r+2}, \varepsilon^s, \varepsilon^{s+1/r-1}, \varepsilon^{sp-1}, \varepsilon^{p/r-1})\right).$$

Since $2/r < s-1 < s$ the left hand side is at least $1+0(\varepsilon^{2/r})$. Also $sp-1 > 2s-1 > 2/r$ and $p/r-1 > (r+2)/r-1 = 2/r$, so for ε sufficiently small the right hand side is less than the left, providing the contradiction.

(b) $1 \leq p < 2$: Again assume $\phi(m) \neq \phi(n)$ for some $m \equiv n \bmod k$. Construct f, I_ε and J_ε as before and choose $0 < r < 2 - p$ and $s > 1 + p/r$. Choose a polynomial $g \in H^p(T)$ such that $\|g\|_\infty \leq 1$, $\|g|_{J_\varepsilon}\|_p \leq \varepsilon^s$, $|g| \geq 1 - \varepsilon^s$ on J_ε^c, $\hat{g}$ is supported on $k\mathbb{Z}$ and

$$\left|\int_{J_\varepsilon^c} \operatorname{Re} g\bar{f}\right| + \left|\int_{J_\varepsilon^c} \operatorname{Re} g\overline{M_\phi f}\right| \leq \varepsilon^s.$$

Again simple estimates show

$$\int_{J_\varepsilon} |g + \varepsilon^{1/r}f|^p \leq 2^p(\varepsilon^{ps} + \varepsilon^{p/r+p}|J_\varepsilon|) = 0\left(\max(\varepsilon^{ps}, \varepsilon^{p/r+p+1})\right)$$

and

$$\int_{J_{\varepsilon^c}} |g + \varepsilon^{1/r}f|^p \leq |J_\varepsilon^c| + 0\left(\max(\varepsilon^s, \varepsilon^{2/r})\right)$$

This time Hölder's inequality will not help in finding a lower bound for $\|M_\phi(g + \varepsilon^{1/r}f)\|_p$. Instead we observe that since $s > 1/r + 1/p$

$$\begin{aligned}
\int_{J_\varepsilon} |M_\phi(g + \varepsilon^{1/r}f)|^p &\geq \left[\left(\int_{J_\varepsilon} |\varepsilon^{1/r}M_\phi f|^p\right)^{\frac{1}{p}} - \left(\int_{J_\varepsilon} |g|^p\right)^{\frac{1}{p}}\right]^p \\
&\geq \left[|J_\varepsilon|^{\frac{1}{p}}\varepsilon^{1/r}C_0 - \varepsilon^s\right]^p \\
&\geq kCC_0^p\varepsilon^{1+p/r}\left(1 - \frac{\varepsilon^{s-1/p-1/r}}{(kC)^{\frac{1}{p}}C_0}\right)^p \\
&\geq C_1\varepsilon^{1+p/r}
\end{aligned}$$

for some constant $C_1 > 0$. (Assume ε is very small.)

Arguments similar to those used in the $2 < p < \infty$ case of the proof for estimating $\int_{J_\varepsilon} |g + \varepsilon^{1/r}f|^p$, show that

$$\int_{J_\varepsilon^c} |M_\phi(g + \varepsilon^{1/r}f)|^p \geq |J_\varepsilon^c| + 0\left(\max(\varepsilon^s, \varepsilon^{2/r})\right).$$

Thus

$$\|M_\phi(g + \varepsilon^{1/r}f)\|_p^p \geq |J_\varepsilon^c| + C_1\varepsilon^{1+p/r} + 0\left(\max(\varepsilon^s, \varepsilon^{2/r})\right),$$

while

$$\|g + \varepsilon^{1/r}f\|_p^p \leq |J_\varepsilon^c| + 0\left(\max(\varepsilon^s, \varepsilon^{2/r}, \varepsilon^{p/r+p+1})\right).$$

But $1 + p/r < \max(s, 2/r, p/r + p + 1)$, so we cannot have $\|M_\phi(g + \varepsilon^{1/r}f)\|_p^p \leq \|g + \varepsilon^{1/r}f\|_p^p$ for ε sufficiently small, again giving a contradiction.

(c) $p = \infty$: Once again assume $\phi(n) \neq \phi(m)$ for some $n \equiv m \bmod k$ and construct f, I_ε and J_ε as before. Choose an $H^\infty(T)$ function g with $\operatorname{supp}\hat{g} \subseteq k\mathbb{Z}$, $|g(0)| \geq 1 - \varepsilon$,

$\|g\|_\infty \le 1$ and $|g| \le \varepsilon$ on J_ε^c. Suppose $|M_\phi f(0)| \ge C_0 > 0$ and let $|\lambda| = 2\varepsilon / C_0$ where $\operatorname{sgn} \lambda M_\phi f(0) = \operatorname{sgn} g(0)$. Then, for ε small,

$$\|g + \lambda f\|_\infty = \max_t |g + \lambda f(t)| \le 1 + \frac{2\varepsilon^2}{C_0}$$

while

$$\|M_\phi(g + \lambda f)\|_\infty \ge |g(0) + \lambda M_\phi f(0)|$$
$$\ge (1 - \varepsilon) + \frac{2\varepsilon C_0}{C_0} = 1 + \varepsilon.$$

Again when ε is small this contradicts the fact that $\|\phi\| \le 1$. ∎

Our final result, an application of the previous theorem, should be contrasted with Example 2.2.

COROLLARY 2.4. *Let $1 \le p \le \infty$. Suppose ϕ is a norm one multiplier on $H^p(T)$ with $\phi(0) = \phi(k) = 1$ for some $k \neq 0$. Then ϕ is the restriction of a norm one multiplier on $L^p(T)$.*

PROOF. The case $p = 2$ is obvious so assume $p \neq 2$. By the previous theorem ϕ is a k-periodic sequence on $\mathbb{Z}^+$. Let μ be the measure on T given by

$$\mu = \sum_{j=0}^{k-1} \phi(j) \frac{e^{ijt}}{k} (\delta_{x_0} + \cdots + \delta_{x_{k-1}})$$

where $x_j = 2\pi j / k$.

Since $\phi(n) = \hat{\mu}(n)$ for $n \in \mathbb{Z}^+$, ϕ is the restriction of the L^p multiplier $\hat{\mu}$ to $H^p(T)$. Clearly the multiplier norm of $\hat{\mu}$ is at least one.

Let f be a trigonometric polynomial and assume $g(t) = f(t)e^{iNkt} \in H^p(T)$.

Since $\hat{g}(n) = \hat{f}(n - Nk)$, and ϕ and $\hat{\mu}$ are k-periodic, if $n \in \mathbb{Z}^+$

$$\widehat{M_\phi g}(n) = \phi(n)\hat{g}(n) = \hat{\mu}(n - Nk)\hat{f}(n - Nk) = \widehat{M_{\hat{\mu}} f}(n - Nk).$$

If n is a negative integer $\widehat{M_\phi g}(n) = 0 = \widehat{M_{\hat{\mu}} f}(n - Nk)$. Thus $M_\phi g = e^{iNkt} M_{\hat{\mu}} f$ and $\|M_{\hat{\mu}} f\|_p = \|M_\phi g\|_p \le \|g\|_p = \|f\|_p$. Since the trigonometric polynomials are dense in L^p, $\hat{\mu}$ is a norm one multiplier on $L^p(T)$. ∎

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