INTEGRALS INVOLVING E-FUNCTIONS

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(Received 22 March, 1965)

1. In this paper two integrals involving E-functions are evaluated in terms of E-functions. The formulae to be established are:

$$\int_0^\pi (\sin\phi)^{\gamma-1}e^{-\delta\phi}E(p;\ \alpha_r:q;\ \beta_s:z(\sin\phi)^{-2n})\,d\phi$$
$$= e^{-\frac{1}{2}\pi\delta}(\pi/n)^{\frac{1}{2}}E(p+2n;\ \alpha_r:q+2n;\ \beta_s:z), \qquad (1)$$

where n is a positive integer, $\operatorname{Re}(\gamma-2n)>0$, $p\geqq q+1$, $\operatorname{Re}\alpha_r\geqq 0$ $(r=1,2,\ldots,p-1)$,

$$\operatorname{Re}(\beta_s-\alpha_s)\geqq 0 \quad (s=1,2,\ldots,q), \quad |\arg z|<\pi$$

and

$$\alpha_{p+l}=\frac{\gamma-1+l}{2n} \quad (l=1,2,\ldots,2n),$$

$$\beta_{q+l}=\frac{\gamma+i\delta-1+2l}{2n}, \quad \beta_{q+n+l}=\frac{\gamma-i\delta-1+2l}{2n} \quad (l=1,2,\ldots,n);$$

$$\int_0^\pi (\sin\phi)^{\gamma-1}e^{-\delta\phi}E(p;\ \alpha_r:q;\ \beta_s:z(\sin\phi)^{2n})\,d\phi$$

$$= e^{-\frac{1}{2}\pi\delta}(\pi/n)^{\frac{1}{2}}\left\{\prod_{r=1}^{n}\sin\left[\left(\frac{2n-\gamma-i\delta-2r}{2n}\right)\pi\right]\prod_{r=1}^{n}\sin\left[\left(\frac{2n-\gamma+i\delta-2r}{2n}\right)\pi\right]\right.$$

$$\times\prod_{s=1}^{2n}\operatorname{cosec}\left[\left(\frac{2n+1-\gamma-s}{2n}\right)\pi\right]E(p+2n;\ q+2n;\ \beta_s:z)$$

$$-\sum_{s=1}^{2n}\frac{\prod_{r=1}^{n}\sin\left[\frac{i\delta+2r-s}{2n}\pi\right]\prod_{r=1}^{n}\sin\left[\frac{-i\delta+2r-s}{2n}\pi\right]}{\sin\left[\frac{2n+1-\gamma-s}{2n}\pi\right]\prod_{t=1}^{2n}{}'\sin\left[\frac{t-s}{2n}\pi\right]}z^{(1-\gamma-s)/2n}$$

$$\left.\times E\left(\begin{matrix}p+2n;\ \alpha_r-\dfrac{1-\gamma-s}{2n} & :z\\ \dfrac{\gamma+s-1-2n}{2n},\ \beta_1-\dfrac{1-\gamma-s}{2n},\ \ldots * \ldots,\ \beta_{q+2n}-\dfrac{1-\gamma-s}{2n} & \end{matrix}\right)\right\}, \qquad (2)$$

https://doi.org/10.1017/S2040618500035395 Published online by Cambridge University Press

where n is a positive integer, $p \geqq q+1$, $\operatorname{Re} \alpha_r \geqq 0$ $(r = 1, 2, \ldots, p-1)$,

$$\operatorname{Re}(\beta_s - \alpha_s) > 0 \quad (s = 1, 2, \ldots, q), \quad |\arg z| < \pi, \operatorname{Re} \gamma > 0$$

and

$$\alpha_{p+l} = \frac{2n-\gamma-i\delta+1-2l}{2n}, \quad \alpha_{p+n+l} = \frac{2n-\gamma+i\delta+1-2l}{2n} \quad (l = 1, 2, \ldots, n),$$

$$\beta_{q+l} = \frac{2n-\gamma+1-l}{2n} \quad (l = 1, 2, \ldots, 2n),$$

the prime and the asterisk denoting that the factor $\sin\{(s-s)\pi/2n\}$ and the parameter $\beta_{q+s} - \beta_{q+s} + 1$ are omitted. The definitions and properties of MacRobert's E-function can be found in [**1**, pp. 348–352] and [**3**, pp. 203–206].

The following formulae are required in the proofs:

$$E(p;\ \alpha_r : q;\ \rho_s : z) = \frac{1}{2\pi i} \int \Gamma(\zeta) \frac{\prod_{r=1}^{p} \Gamma(\alpha_r - \zeta)}{\prod_{t=1}^{q} (\rho_t - \zeta)} z^{\zeta}\, d\zeta, \tag{3}$$

where $|\arg z| < \frac{1}{2}(p-q+1)\pi$, and where the contour of integration is of Barnes type with loops, if necessary, to separate the pole at the origin from the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ [**1**, p. 374]. Also the proof depends on the expression of the generalized E-function in terms of E-functions [**1**, p. 419]

$$E\left(\begin{matrix} p;\ \alpha_r \\ q;\ \rho_s \end{matrix}\middle| \begin{matrix} m;\ \rho_{q+s} : z \\ l;\ \alpha_{p+r} \end{matrix}\right) = \frac{1}{2\pi i} \int \Gamma(\zeta) \frac{\prod_{r=1}^{p} \Gamma(\alpha_r - \zeta) \prod_{s=1}^{m} \Gamma(\zeta - \rho_{q+s} + 1)}{\prod_{s=1}^{q} \Gamma(\rho_s - \zeta) \prod_{r=1}^{l} \Gamma(\zeta - \alpha_{p+r} + 1)} z^{\zeta}\, d\zeta$$

$$= \pi^{m-l} \prod_{r=1}^{l} \sin(\alpha_{p+r}\pi) \prod_{s=1}^{m} \operatorname{cosec}(\rho_{q+s}\pi) E(p+l;\ \alpha_r : p+m;\ \rho_s : \omega z)$$

$$- \sum_{s=1}^{m} \pi^{m-l} \frac{\prod_{r=1}^{l} \sin[(\rho_{q+s} - \alpha_{p+r})\pi]}{\sin(\rho_{q+s}\pi) \prod_{t=1}^{m}{}' \sin[(\rho_{q+s} - \rho_{q+t})\pi]} z^{\rho_{q+s}-1}$$

$$\times E\left(\begin{matrix} p+l;\ \alpha_r - \rho_{q+s} + 1 & : \omega z \\ 2-\rho_{q+s}, \rho_1 - \rho_{q+s} + 1, \ldots * \ldots \rho_{q+m} - \rho_{q+s} + 1 & \end{matrix}\right), \tag{4}$$

where l and m are positive integers and the contour passes up the η-axis from $-\infty$ to ∞, with loops, if necessary, to ensure that the poles of the integrand at the origin and at

$$\rho_{q+1} - 1, \rho_{q+2} - 1, \ldots, \rho_{q+m} - 1$$

lie to the left, and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the right of the contour. When necessary the contour is bent to the left or right at both ends until it is parallel to the ξ-axis. The prime and the asterisk denote the omission of the factor $\sin[(\rho_{q+s}-\rho_{q+s})\pi]$ and of the parameter $\rho_{q+s}-\rho_{q+s}+1$, and $\omega = 1$ or $e^{\pm i\pi}$ according as $l+m$ is even or odd.

If $\mathrm{Re}\,\gamma > 0$ [**2**, p. 159],

$$\int_0^\pi (\sin\phi)^{\gamma-1}e^{-\delta\phi}\,d\phi = \frac{\pi\Gamma(\gamma)e^{-\frac{1}{2}\pi\delta}}{2^{\gamma-1}\Gamma\left(\frac{\gamma+i\delta+1}{2}\right)\Gamma\left(\frac{\gamma-i\delta+1}{2}\right)}. \tag{5}$$

Finally, if m is a positive integer [**3**, p. 4],

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)}m^{mz-\frac{1}{2}}\prod_{l=1}^{m}\Gamma\left(z+\frac{l-1}{m}\right). \tag{6}$$

2. Proofs. To prove (1) substitute the contour integral (3) for the E-function in the integrand of (1) and change the order of integration; the integral then becomes

$$\frac{1}{2\pi i}\int\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{s=1}^{q}\Gamma(\beta_t-\zeta)}\Gamma(\zeta)z^\zeta\,d\zeta\int_0^\pi(\sin\phi)^{\gamma-2n\zeta-1}e^{-\delta\phi}\,d\phi.$$

On using (5) and (6) this becomes

$$e^{-\frac{1}{2}\pi\delta}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)\prod_{l=1}^{2n}\Gamma\left(\frac{\gamma-1+l}{2n}-\zeta\right)z^\zeta\,d\zeta}{\prod_{s=1}^{q}\Gamma(\beta_s-\zeta)\prod_{l=1}^{n}\Gamma\left(\frac{\gamma+i\delta-1+2l}{2n}-\zeta\right)\prod_{l=1}^{n}\Gamma\left(\frac{\gamma-i\delta-1+2l}{2n}-\zeta\right)}.$$

The result now follows from (3).

Formula (2) can be derived in the same way. After using (5) and (6) we get

$$e^{-\frac{1}{2}\pi\delta}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)\prod_{s=1}^{2n}\Gamma\left(\zeta-\frac{2n+1-\gamma-s}{2n}+1\right)z^\zeta\,d\zeta}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)\prod_{r=1}^{n}\Gamma\left(\zeta-\frac{2n-\gamma-i\delta+1-2r}{2n}+1\right)\prod_{r=1}^{n}\Gamma\left(\zeta-\frac{2n-\gamma+i\delta+1-2r}{2n}+1\right)}$$

and the result follows from (4).

3. Applications. When $p = 1$, $q = 0$, the E-function reduces to

$$E(\alpha :: z) = \Gamma(\alpha)\Gamma(1+z^{-1})^{-\alpha}, \tag{7}$$

and when $p = q = 0$,

$$E(:: z) = e^{-1/z}. \tag{8}$$

When $p = 2$, $q = 0$,

$$\cos n\pi\, E(\tfrac{1}{2}+n, \tfrac{1}{2}-n :: 2z) = \sqrt{(2\pi z)}e^zK_n(z), \tag{9}$$

and

$$E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m :: z) = \Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m)z^{-k}e^{\frac{1}{2}z}W_{k,m}(z), \tag{10}$$

where $W_{k,m}(z)$ is the Whittaker function.

https://doi.org/10.1017/S2040618500035395 Published online by Cambridge University Press

Formula (1) together with (7), (8), (9) and (10) now give

$$\int_0^\pi (\sin\phi)^{\gamma-1} e^{-\delta\phi}[1+z^{-1}\sin^{2n}\phi]^{-\alpha}\,d\phi = \frac{(\pi/n)^{\frac{1}{2}}}{e^{\frac{1}{2}\pi\delta}\Gamma(\alpha)} E(1+2n;\ \alpha_r:2n;\ \beta_s:z), \tag{11}$$

$$\int_0^\pi (\sin\phi)^{\gamma-1} e^{-(\delta\phi+z^{-1}\sin^{2n}\phi)}\,d\phi = e^{-\frac{1}{2}\pi\delta}\left(\frac{\pi}{n}\right)^{\frac{1}{2}} E(2n;\ \alpha_r:q;\ \beta_s:z), \tag{12}$$

$$\int_0^\pi (\sin\phi)^{\gamma-n-1} e^{-\delta\phi+\frac{1}{2}z(\sin\phi)^{-2n}} K_m[\tfrac{1}{2}z(\sin\phi)^{-2n}]\,d\phi$$

$$= \frac{\cos m\pi}{(nz)^{\frac{1}{2}}} e^{-\frac{1}{2}\pi\delta} E(2+2n;\ \alpha_r:2n;\ \beta_s:z), \tag{13}$$

and

$$\int_0^\pi (\sin\phi)^{\gamma+2nk-1} e^{-\delta\phi+\frac{1}{2}z(\sin\phi)^{-2n}} W_{k,m}[z(\sin\phi)^{-2n}]\,d\phi$$

$$= \frac{z^k e^{-\frac{1}{2}\pi\delta}(\pi/n)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-k+m)} E(2+2n;\ \alpha_r:2n;\ \beta_s:z). \tag{14}$$

Similar results may be obtained from (2) by using (7), (8), (9) and (10).

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