Canad. Math. Bull. Vol. **60** (2), 2017 pp. 319–328
http://dx.doi.org/10.4153/CMB-2016-096-1
©Canadian Mathematical Society 2017

The Weakly Nilpotent Graph of a Commutative Ring

Soheila Khojasteh and Mohammad Javad Nikmehr

Abstract. Let R be a commutative ring with non-zero identity. In this paper, we introduce the weakly nilpotent graph of a commutative ring. The weakly nilpotent graph of R denoted by $\Gamma_w(R)$ is a graph with the vertex set R^* and two vertices x and y are adjacent if and only if $xy \in N(R)^*$, where $R^* = R \setminus \{0\}$ and $N(R)^*$ is the set of all non-zero nilpotent elements of R. In this article, we determine the diameter of weakly nilpotent graph of an Artinian ring. We prove that if $\Gamma_w(R)$ is a forest, then $\Gamma_w(R)$ is a union of a star and some isolated vertices. We study the clique number, the chromatic number, and the independence number of $\Gamma_w(R)$. Among other results, we show that for an Artinian ring R, $\overline{\Gamma_w(R)}$ is not a disjoint union of cycles or a unicyclic graph. For Artinan rings, we determine $\operatorname{diam}(\overline{\Gamma_w(R)})$. Finally, we characterize all commutative rings R for which $\overline{\Gamma_w(R)}$ is a cycle, where $\overline{\Gamma_w(R)}$ is the complement of the weakly nilpotent graph of R.

1 Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring; see, for example, [1, 4, 5, 7–13]. For an arbitrary commutative ring R, the *zero-divisor graph* $\Gamma(R)$, is a graph whose vertices are all non-zero zero-divisors of R and such that two distinct vertices x and y are adjacent if and only if $xy = 0$. In [7], Chen defined a kind of graph structure of a ring R whose vertices are all the elements of R, and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all nilpotent elements of R. Recently, $\Gamma_N(R)$ was defined with the vertex set $\mathcal{Z}_N(R)^*$, where $\mathcal{Z}_N(R) = \{x \in R \mid xy \in N(R), \text{ for some } y \in R^*\}$, $X^* = X \setminus \{0\}$ for any subset X of R and two distinct vertices are adjacent if and only if $xy \in N(R)$, or equivalently, $yx \in N(R)$; see [8, 9]. It is easy to see that the usual *zero-divisor graph* is a subgraph of $\Gamma_N(R)$. Motivated by previous studies, in this paper, we define the weakly nilpotent graph of a commutative ring. Our main goal is to study the connection between the algebraic properties of a ring and the graph theoretic properties of the graph associated with it.

Throughout this paper, R is a commutative ring with non-zero identity. The *weakly nilpotent graph* of R denoted by $\Gamma_w(R)$ is defined to be the undirected simple graph with the vertex set R^* and two vertices x and y are adjacent if and only if $xy \in N(R)^*$. Clearly, $\Gamma_w(R)$ is a subgraph of the nilpotent graph that was introduced by Chen [7]. We also think that the weakly nilpotent graph of a ring helps us to study the algebraic

Received by the editors January 6, 2014; revised May 5, 2015.
Published electronically January 26, 2017.
AMS subject classification: **05C15**, 16N40, 16P20.
Keywords: weakly nilpotent graph, zero-divisor graph, diameter, girth.

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

properties of rings using graph theoretical tools. Now, consider the complement of the weakly nilpotent graph of R, denoted by $\overline{\Gamma_w(R)}$. For any two distinct vertices x and y in R^*, x is adjacent to y if and only if $xy \notin N(R)^*$. Obviously, the usual zero divisor graph is a subgraph of $\overline{\Gamma_w(R)}$.

We denote the set of unit elements of R, the set of zero divisors of R, the set of nilpotent elements of R, and the Jacobson radical of R by $U(R)$, $Z(R)$, $N(R)$, and $J(R)$, respectively. If R has a unique maximal ideal $\mathfrak{m}$, then R is said to be a local ring and it is denoted by $(R, \mathfrak{m})$. A ring R is said to be a *reduced ring* if $N(R) = 0$. A subset S of a commutative ring R is called a *multiplicative closed subset* (m.c.s.) of R if $1 \in S$ and $x, y \in S$ implies that $xy \in S$. If S is an m.c.s. of R, then we denote by $S^{-1}R$ the ring of fractions of R.

Let G be a graph with vertex set $V(G)$. A path from x to y is a series of adjacent vertices $x - x_1 - x_2 - \cdots - x_n - y$. For $x, y \in V(G)$ with $x \neq y$, $d(x, y)$ denotes the length of the shortest path from x to y; if there is no such path, we will make the convention $d(x, y) = \infty$. The diameter of G is defined as

$$\operatorname{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}.$$

For any $x \in V(G), d(x)$ denotes the number of edges incident with x, called the degree of x. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated and all vertices other than the starting and ending vertex are distinct. We use C_n to denote the cycle with n vertices, where $n \geq 3$. We denote the complete graph with n vertices by K_n. If a graph G has a cycle, then the *girth* of G (notated $\operatorname{gr}(G)$) is defined as the length of the shortest cycle of G; otherwise $\operatorname{gr}(G) = \infty$. A *bipartite graph* is a graph whose vertices can be partitioned into two disjoint sets U and V such that every edge connects a vertex in U to one in V. A *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. We denote by $K_{m,n}$ the complete bipartite graph, with part sizes m and n. The *star graph* is denoted by $K_{1,n}$, for a positive integer n. We say that a graph G is *totally disconnected* if no two vertices of G are adjacent. The *disjoint union* of graphs G_1 and G_2, which is denoted by $G_1 \cup G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. A *unicyclic graph* is a connected graph with a unique cycle, or we can regard a unicyclic graph as a cycle attached with each vertex a (rooted) tree. An *independent set* is a subset of the vertices of a graph such that no vertices are adjacent. The number of vertices in a maximum independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. A *clique* of a graph is a complete subgraph and the number of vertices in a largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. By $\chi(G)$, we denote the *chromatic number* of G, *i.e.,* the minimum number of colors that can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors.

In Section 2, we determine the diameter of weakly nilpotent graph of an Artinian ring. We prove that if $\Gamma_w(R)$ is a forest, then $\Gamma_w(R)$ is a union of a star and isolated vertices. We study the clique number, the chromatic number, and the independence number of $\Gamma_w(R)$. Among other results, we show that for an Artinian ring R, $\Gamma_w(R)$ is not a disjoint union of cycles or a unicyclic graph. In Section 3, we determine

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

$\mathrm{diam}(\overline{\Gamma_w(R)})$, where R is an Artinian ring. We characterize all commutative rings R for which $\overline{\Gamma_w(R)}$ is a cycle.

2 The Weakly Nilpotent Graph of a Commutative Ring

In this section, we will focus on the weakly nilpotent graph of an Artinian ring. We prove that if $\Gamma_w(R)$ has no isolated vertex, then $\mathrm{diam}(\Gamma_w(R)) \le 4$, where R is an Artinian ring. It is shown that if $\Gamma_w(R)$ is a forest, then $\Gamma_w(R)$ is a union of a star and some isolated vertices. We start with the following remark.

Remark 2.1 A ring R is a reduced ring if and only if $\Gamma_w(R)$ is totally disconnected. In fact, if R is a reduced ring, then $N(R)^* = \varnothing$, and so $\Gamma_w(R)$ is totally disconnected. Conversely, if $\Gamma_w(R)$ is totally disconnected and $N(R)^* \ne \varnothing$, then every element of $N(R)^*$ is adjacent to 1, a contradiction. Therefore, R is a reduced ring.

Theorem 2.2 *Let R be a commutative Artinian ring. Then $\Gamma_w(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2$.*

Proof One side is clear. For the other side, assume that $\Gamma_w(R)$ is a complete graph. Hence, $|U(R)| = 1$. Since R is an Artinian ring, by [2, Lemma 1], $R \cong \mathbb{Z}_2^r$, for some positive integer r. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other components are 0. If $r \ge 2$, then e_1 is not adjacent to e_2, a contradiction. Therefore, $R \cong \mathbb{Z}_2$. ∎

Clearly, $\Gamma_w(\mathrm{F})$ is totally disconnected, where F is a field. Therefore, in the next theorem we assume that the maximal ideal is non-zero.

Theorem 2.3 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{m} \ne 0$. If $\mathfrak{m} = N(R)$, then*

$$\mathrm{diam}\big(\Gamma_w(R)\big) = 2.$$

Proof Obviously, every element of $N(R)^*$ is adjacent to each element of $U(R)$. This shows that $\Gamma_w(R)$ is a connected graph and $\mathrm{diam}(\Gamma_w(R)) \le 2$. If $U(R) = \{1\}$, then by [6, p. 10], $1 + \mathfrak{m} \subseteq U(R)$ and so $\mathfrak{m} = 0$, a contradiction. Therefore, $|U(R)| \ge 2$. Let $u, v \in U(R)$. Since u and v are not adjacent and $\mathrm{diam}(\Gamma_w(R)) \le 2$, $d(u,v) = 2$. Thus, $\mathrm{diam}(\Gamma_w(R)) = 2$. ∎

Theorem 2.4 *Let R be a commutative Artinian ring. If $\Gamma_w(R)$ has not any isolated vertex, then* $\mathrm{diam}(\Gamma_w(R)) \le 4$.

Proof By [6, Theorem 8.7], we know that $R \cong \prod_{i=1}^n R_i$, where $n \ge 1$ and $(R_i, \mathfrak{m}_i)$ is a local ring, for every i, $1 \le i \le n$. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other components are 0. If $\mathfrak{m}_1 = 0$, then e_1 is an isolated vertex, a contradiction. Therefore, $\mathfrak{m}_1 \ne 0$. Similarly, $\mathfrak{m}_i \ne 0$ for every i, $1 \le i \le n$. If $n = 1$, then by Theorem 2.3, $\mathrm{diam}(\Gamma_w(R)) = 2$. Therefore, we can assume that $n \ge 2$. Let $a = \sum_{i=1}^n a_i e_i$, $b = \sum_{i=1}^n b_i e_i \in V(\Gamma_w(R))$. We have the following three cases.

Case 1. $a, b \in U(R)$. Then we have $a - x - b$, where $x \in N(R)^*$. Hence, $d(a,b) = 2$.

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

Case 2. $a \in U(R)$ and $b \notin U(R)$. If $b_i \in \mathfrak{m}_i$, for every $i, 1 \le i \le n$, then a is adjacent to b and $d(a,b) = 1$. Otherwise, suppose that $J = \{ i \mid 1 \le i \le n, \ b_i \in U(R_i) \}$. Let $x = \sum_{i=1}^{n} r_i e_i$ and $y = \sum_{i \notin J} e_i + \sum_{i \in J} r_i e_i$, for $r_i \in \mathfrak{m}_i^*$. Then $a - x - y - b$ is a path between a and b. Therefore, $d(a,b) \le 3$.

Case 3. $a, b \notin U(R)$. If $a_i, b_i \in \mathfrak{m}_i$, for every $i, 1 \le i \le n$, then $\sum_{i=1}^{n} e_i$ is adjacent to a and b. Therefore, $d(a,b) \le 2$. Otherwise, let $I = \{ i \mid 1 \le i \le n, \ a_i \in U(R_i) \}$ and $J = \{ i \mid 1 \le i \le n, b_i \in U(R_i) \}$. If $x = \sum_{i \notin I} e_i + \sum_{i \in I} r_i e_i$, $y = \sum_{i=1}^{n} r_i e_i$ and $w = \sum_{i \notin J} e_i + \sum_{i \in J} r_i e_i$, for $r_i \in \mathfrak{m}_i^*$, then $a - x - y - w - b$ is path between a and b. Therefore, $d(a,b) \le 4$. Thus, $\operatorname{diam}(\Gamma_w(R)) \le 4$. ■

Remark 2.5 In view of proof of Theorem 2.4, we find that if $R = \prod_{i=1}^{n} R_i$, $(R_i, \mathfrak{m}_i)$ is a local ring, for every $i, 1 \le i \le n$, and $\mathfrak{m}_j = 0$, for some $j, 1 \le j \le n$, then $\Gamma_w(R)$ has at least one isolated vertex.

By Remark 2.1, we know that $\Gamma_w(\mathbb{Z}_p) = \overline{K}_{p-1}$, where p is a prime number. In the following example we determine when $\Gamma_w(\mathbb{Z}_n)$ is a connected graph and compute $\operatorname{diam}(\Gamma_w(\mathbb{Z}_n))$.

Example 2.6 Let $n = p_1^{k_1} \times \cdots \times p_s^{k_s}$, where p_i is a prime number and k_i is a positive integer number. Then $\Gamma_w(\mathbb{Z}_n)$ is a connected graph if and only if $k_i \ge 2$, for every i, $1 \le i \le s$. Moreover, if $\Gamma_w(\mathbb{Z}_n)$ is a connected graph, then $\operatorname{diam}(\Gamma_w(\mathbb{Z}_n)) = 2$.

Proof First suppose that $\Gamma_w(\mathbb{Z}_n)$ is a connected graph. If $k_1 = 1$, then by Remark 2.1, $s \ge 2$. We show that $a = p_2^{k_2} \times \cdots \times p_s^{k_s}$ is an isolated vertex. To see this we note that if a is adjacent to b, for some $b \in \mathbb{Z}_n^*$, then since $ab \ne 0$ and $a \notin p_1\mathbb{Z}$, we conclude that $b \notin p_1\mathbb{Z}$. On the other hand, since $ab \in N(R)^*$, $b \in p_1\mathbb{Z}$, a contradiction. Therefore, $k_1 \ge 2$. Similarly, $k_i \ge 2$, for every $i, 1 \le i \le s$.

Now, suppose that $k_i \ge 2$ for every $i, 1 \le i \le s$. We show that $\operatorname{diam}(\Gamma_w(\mathbb{Z}_n)) = 2$. Let $a, b \in \mathbb{Z}_n^*$. There are three cases.

Case 1. $a, b \in U(\mathbb{Z}_n)$. Then a is not adjacent to b, and we have $a - p_1 \times \cdots \times p_s - b$. Therefore, $d(a,b) = 2$.

Case 2. $a \in U(\mathbb{Z}_n)$ and $b \notin U(\mathbb{Z}_n)$. We can assume that $b = up_1^{t_1} \times \cdots \times p_r^{t_r}$, where $1 \le r \le s$ and $u \notin p_i\mathbb{Z}$, for every $i, 1 \le i \le s$ and t_i is a positive integer, for every i, $1 \le i \le r$. If $r = s$, then a is adjacent to b. If $r \ne s$, then since $k_s \ge 2$, $bp_1 \times \cdots \times p_s \ne 0$. Now, a and b are adjacent to $p_1 \times \cdots \times p_s$. Thus, $d(a,b) \le 2$.

Case 3. $a, b \notin U(\mathbb{Z}_n)$. Then let $a = up_1^{t_1} \times \cdots \times p_r^{t_r}$ and $b = vp_1^{l_1} \times \cdots \times p_r^{l_j}$, where $u, v \notin p_i\mathbb{Z}$, for every $i, 1 \le i \le s, 1 \le r \le j \le s$, t_i is a positive integer, for every i, $1 \le i \le r$ and l_i is a positive integer, for every $i, 1 \le i \le j$. If $r = j < s$, then a and b are adjacent to $p_1 \times \cdots \times p_s$. If $r = j = s$, then a and b are adjacent to 1. Now, suppose that $r < j$. If $j = s$, then a is adjacent to b. If $r < j < s$, then a and b are adjacent to $p_1 \times \cdots \times p_s$. Therefore, in this case, $d(a,b) \le 2$. This completes the proof. ■

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

Remark 2.7 If $|N(R)| \geq 3$, then let $x, y \in N(R)^*$. It is easy to see that 1 is adjacent to x and y. We know that $1 + x$ is a unit element, and so $1 + x$ is adjacent to x and y. Now, $1 — x — 1+x — y — 1$ is a 4-cycle in $\Gamma_w(R)$. This yields that $\mathrm{gr}(\Gamma_w(R)) \in \{3, 4\}$.

In the following theorem we study the case where $\Gamma_w(R)$ is a forest.

Theorem 2.8 *Let R be a commutative ring. If $\Gamma_w(R)$ is a forest, then the following hold:*

(i) $|N(R)| \leq 2$.
(ii) *If $|N(R)| = 1$, then $\Gamma_w(R)$ is totally disconnected.*
(iii) *If $|N(R)| = 2$, then $\Gamma_w(R)$ is a union of a star and some isolated vertices.*

Proof If $|N(R)| \geq 3$, then by Remark 2.7, $\mathrm{gr}(\Gamma_w(R)) \in \{3, 4\}$. Therefore, $|N(R)| \leq 2$. If $|N(R)| = 1$, then R is a reduced ring and by Remark 2.1, $\Gamma_w(R)$ is totally disconnected. Now, we can assume that $|N(R)| = 2$. Then R has exactly one non-zero nilpotent element, say x. Hence, $x^2 = 0$. We note that every element of $U(R)$ is adjacent to x. If x is adjacent to all vertices, then $\Gamma_w(R)$ is a star. If every vertex that is not adjacent to x, is an isolated vertex, then we are done. Therefore, we can assume that there exists $y \in V(\Gamma_w(R))$ such that $d(y) = 1$ and x is not adjacent to y. Since x is not adjacent to y and $xy \in N(R)$, we conclude that $xy = 0$. Assume that y is adjacent to a. Then $ya = x$ and $y(a + x) = x$. Therefore, y is adjacent to $a + x$, a contradiction. This implies that if $d(y) = 1$, then x is adjacent to y. Since $\Gamma_w(R)$ is a forest, we find that $\Gamma_w(R)$ is a union of a star (with center x) and some isolated vertices. ∎

Remark 2.9 There are some rings R such that $|N(R)| \leq 2$ and $\mathrm{gr}(\Gamma_w(R)) = \infty$. For instance, let $R_1 = \mathbb{Z}_3$ and $R_2 = \mathbb{Z}_4$. Then $|N(R_1)| = 1$, $|N(R_2)| = 2$ and $\mathrm{gr}(\Gamma_w(R_i)) = \infty$, for $i = 1, 2$.

Theorem 2.10 *If R is an Artinian ring, then the following hold:*

(i) *If $\Gamma_w(R)$ is totally disconnected, then $R = \prod_{i=1}^n \mathrm{F}_i$, where F_i is a field, for $1 \leq i \leq n$.*
(ii) *$\Gamma_w(R)$ is a forest if and only if R is isomorphic to one of the rings $\mathbb{Z}_4$, $\mathbb{Z}_2(x)/(x^2)$ and $\prod_{i=1}^n \mathrm{F}_i$, where F_i is a field, for $i = 1, \dots, n$.*
(iii) *$\Gamma_w(R)$ is not a disjoint union of cycles or a unicyclic graph.*

Proof Since R is an Artinian ring, by [6, Theorem 8.7], $R \cong \prod_{i=1}^n R_i$, where $(R_i, \mathfrak{m}_i)$ is a local ring. Moreover, by [6, p. 87], every $\mathfrak{m}_i$ is nilpotent. Let e_i be the $1 \times n$ vector whose the i-th component is 1 and other components are 0.

(i) Since $\sum_{i=1}^n e_i$ is adjacent to every non-zero element of $\prod_{i=1}^n \mathfrak{m}_i$, $\prod_{i=1}^n \mathfrak{m}_i = 0$. This implies that every R_i is a field, and so $R = \prod_{i=1}^n F_i$. This completes the proof.

(ii) Assume that $\Gamma_w(R)$ is a forest. There are two following cases:

Case 1. $n \geq 2$. If $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$, then we have $e_1 + e_2 — a_1e_1 — e_1 + a_2e_2 — a_1e_1 + e_2 — a_2e_2 — e_1 + e_2$, for $a_i \in \mathfrak{m}_i^*$ and $i = 1, 2$, a contradiction. Therefore, we can assume that R_2 is a field. Similarly, we can assume that $R_2, \dots, R_n$ are fields. If $|\mathfrak{m}_1| \geq 3$, then let $\{a, b\} \subseteq \mathfrak{m}_1$. Since by [6, p. 10], $1 + a$ is a unit element of R_1, we conclude that $(1 + a)e_1 — ae_1 — e_1 + e_2 — be_1 — (1 + a)e_1$ is a 4-cycle, a contradiction. Therefore, $|\mathfrak{m}_1| \leq 2$. If $|\mathfrak{m}_1| = 2$, then by [3, Theorem 3], $R_1 \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$. Therefore $R \cong \mathbb{Z}_4 \times \prod_{i=2}^n \mathrm{F}_i$ or

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

$R \cong \mathbb{Z}_2(x)/(x^2) \times \prod_{i=2}^n \mathrm{F}_i$. If $R \cong \mathbb{Z}_4 \times \prod_{i=2}^n \mathrm{F}_i$, then $e_1 — 2e_1 — 3e_1 — 2e_1 + e_2 — e_1$ is a 4-cycle, a contradiction. If $R \cong \mathbb{Z}_2(x)/(x^2) \times \prod_{i=2}^n \mathrm{F}_i$, then $e_1 — xe_1 — (1+x)e_1 — xe_1 + e_2 — e_1$ is a 4-cycle, a contradiction. Thus, $|\mathfrak{m}_1| = 1$ and so R_1 is a field. Hence, $R \cong \prod_{i=1}^n \mathbb{F}_i$.

Case 2. $n = 1$. Since every element of $U(R)$ is adjacent to each element of $N(R)^*$, we conclude that $|N(R)^*| = 0, 1$. By [3, p. 87], $\mathfrak{m} = N(R)$. Therefore, $\mathfrak{m} = 0$ or $|\mathfrak{m}| = 2$. If $\mathfrak{m} = 0$, then R is a field. If $|\mathfrak{m}| = 2$, then by [3, Theorem 3], $R \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$.

Conversely, if $R \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$, then $\Gamma_w(R) = K_{1,2}$. If $R \cong \prod_{i=1}^n \mathrm{F}_i$, then by Remark 2.1, $\Gamma_w(R)$ is totally disconnected. This completes the proof.

(iii) By contradiction, assume that $\Gamma_w(R)$ is a disjoint union of cycles or a unicyclic graph. There are two cases.

Case 1. $n \geq 2$. If R_1 is a field, then e_1 is an isolated vertex, a contradiction. Therefore, R_1 is not a field. Similarly, R_i is not a field, for $i = 1, \dots, n$. Since $\mathfrak{m}_1 \neq 0$ and by [6, p. 10], $|U(R_1)| \geq 2$. Let $\{1, u\} \subseteq U(R_1)$, $0 \neq a \in \mathfrak{m}_1$ and $0 \neq b \in \mathfrak{m}_2$. Then $e_1 — ae_1 + e_2 — ae_1 + be_2 — e_1$ and $ue_1 — ae_1 + e_2 — ae_1 + be_2 — ue_1$ are two cycles, a contradiction.

Case 2. $n = 1$. Then $(R, \mathfrak{m})$ is an Artinian local ring and by [6, p. 87], $\mathfrak{m} = N(R)$. We note that every element of $U(R)$ is adjacent to each element of $\mathfrak{m}^*$. This implies that $|\mathfrak{m}^*| \leq 1$. Since otherwise, if $a, b \in \mathfrak{m}^*$, then $1 — a — 1 + a — b — 1$ and $1 — a — 1 + b — b — 1$ are two cycles of $\Gamma_w(R)$, a contradiction. Now, since $|\mathfrak{m}^*| \leq 1$ and by [3, Theorem 3], we conclude that $R \cong \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$. It is easy to see that $\Gamma_w(\mathbb{Z}_2)$ is an isolated vertex and $\Gamma_w(\mathbb{Z}_4) = \Gamma_w(\mathbb{Z}_2(x)/(x^2)) = K_{1,2}$, a contradiction. ∎

Example 2.11 If $\Gamma_w(R)$ is a forest, then by Theorem 2.8, $\Gamma_w(R)$ is totally disconnected or it is a union of a star and some isolated vertices. By the previous theorem, we find that if R is an Artinian ring and $\Gamma_w(R)$ is a forest with at least one edge, then $\Gamma_w(R)$ is a star. We note that there are some rings R for which $\Gamma_w(R)$ has at least one edge, but it is a disconnected graph. For instance, let $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. It is easy to see that $(0,1)$ is an isolated vertex of $\Gamma_w(R)$ and $\Gamma_w(R)$ is not totally disconnected.

Theorem 2.12 *Let R be a commutative ring and $R = \prod_{i=1}^n R_i$. Then the following hold:*

(i) $\omega(\Gamma_w(R)) \geq \prod_{i=1}^n \omega(\Gamma_w(R_i))$.
(ii) *Let $\chi(\Gamma_w(R)) = \chi$ and $\chi(\Gamma_w(R_i)) = \chi_i$ for every i, $1 \leq i \leq n$. If χ_i is finite for every i, $1 \leq i \leq n$, then $\chi \leq \sum_{J \in P} \prod_{i \in J} \chi_i$, where P is the set of all subsets of* $\{1, \dots, n\}$

Proof (i) Let C_i be a clique in $\Gamma_w(R_i)$, for $1 \leq i \leq n$. It is easy to see that $C = \{(a_1, \dots, a_n) \mid a_i \in C_i, 1 \leq i \leq n\}$ is a clique in $\Gamma_w(R)$. This completes the proof.

(ii) First assume that $n = 2$ and $(x, y) \in R$. If $x, y \neq 0$, then we define $f((x, y)) = (\chi_1(x), \chi_2(y))$. If $x = 0$ and $y \neq 0$, then let $f((x, y)) = (0, \chi_2(y))$. Otherwise, since $(x, y) \neq 0$, we conclude that $x \neq 0$ and $y = 0$. In this case, suppose that $f((x, y)) = (\chi_1(x), 0)$. Obviously, f is a proper vertex coloring for R^*. Hence, $\chi \leq \chi_1 + \chi_2 + \chi_1\chi_2$. Now, assume that $n \geq 3$. By induction, one can easily prove that $\chi \leq \sum_{J \in P} \prod_{i \in J} \chi_i$, where P is the set of all subsets of $\{1, \dots, n\}$. ∎

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

In the rest of this section we study the weakly nilpotent graph of a commutative local ring.

Theorem 2.13 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{m} \neq 0$. Then the following hold:*

(i) *If $\mathfrak{m}^2 = 0$, then $\omega(\Gamma_w(R)) = 2$.*
(ii) *If R is a finite ring, then $\chi(\Gamma_w(R)) \leq |\mathfrak{m}|$.*

Proof (i) Let $0 \neq x \in \mathfrak{m}$. Clearly, $\{1, x\}$ is a clique for $\Gamma_w(R)$. Therefore, $\omega(\Gamma_w(R)) \geq 2$. If C is a clique for $\Gamma_w(R)$ with maximum cardinal, then C has at most one unit element. Hence, $|C \cap U(R)| \leq 1$. On the other hand since $\mathfrak{m}^2 = 0$, $|C \cap \mathfrak{m}| \leq 1$. Thus, $|C| \leq 2$. This completes the proof.

(ii) Suppose that $\mathfrak{m}^* = \{x_1, \dots, x_{|\mathfrak{m}|-1}\}$. We define $f(x_i) = i$, for every i, $1 \leq i \leq |\mathfrak{m}| - 1$ and $f(u) = |\mathfrak{m}|$, for every $u \in U(R)$. Clearly, f is a proper vertex coloring for $V(\Gamma_w(R))$. Therefore, $\chi(\Gamma_w(R)) \leq |\mathfrak{m}|$. ∎

Remark 2.14 There are some rings R and S such that $\Gamma_w(R) = \Gamma_w(S)$ and R is not isomorphic to S. For instance, $\Gamma_w(\mathbb{Z}_4) = \Gamma_w(\mathbb{Z}_2[x]/(x^2))$.

The next result gives some properties of $\Gamma_w(S^{-1}R)$, where $S = R \setminus Z(R)$.

Theorem 2.15 *Let R be a commutative ring and $S = R \setminus Z(R)$. Then the following hold:*

(i) *If $x/s_1 — x/s_2$ is an edge of $\Gamma_w(S^{-1}R)$, then $x \in N(R)^*$.*
(ii) *If $x_1/s_1 — x_2/s_2$ is an edge of $\Gamma_w(S^{-1}R)$, for some $x_1 \neq x_2$, then $x_1 — x_2$ is an edge of $\Gamma_w(R)$.*
(iii) *If x is an isolated vertex of $\Gamma_w(R)$, then x/s is an isolated vertex of $\Gamma_w(S^{-1}R)$, for every $s \in S$.*
(iv) *If $x_1 — x_2$ is an edge of $\Gamma_w(R)$, then $x_1/s_1 — x_2/s_2$ is an edge of $\Gamma_w(S^{-1}R)$.*
(v) *$\Gamma_w(R)$ is a subgraph of $\Gamma_w(S^{-1}R)$. Moreover, if $\Gamma_w(S^{-1}R)$ is a connected graph, then $\Gamma_w(R)$ is a connected graph and $\operatorname{diam}(\Gamma_w(R)) \leq \operatorname{diam}(\Gamma_w(S^{-1}R))$.*

Proof (i) Since x/s_1 is adjacent to x/s_2, $(x^2/s_1s_2)^n = 0$, for some positive integer n. Hence, $tx^{2n} = 0$, for some $t \in S$. Since $t \notin Z(R)$, $x \in N(R)^*$.

(ii) Since x_1/s_1 is adjacent to x_2/s_2, $t(x_1x_2)^n = 0$, for some positive integer n and some $t \in S$. We note that $t \notin Z(R)$. Therefore, $x_1x_2 \in N(R)^*$, and so x_1 is adjacent to x_2.

(iii) By contradiction assume that x/s is adjacent to x_1/s_1. If $x_1 = x$, then by (i), $x \in N(R)^*$, and so x is adjacent to 1, a contradiction. If $x_1 \neq x$, then by (ii), x is adjacent to x_1, a contradiction.

(iv) Since x_1 is adjacent to x_2, $(x_1x_2)^n = 0$, for some positive integer n. Obviously, x_1/s_1 is adjacent to x_2/s_2.

(v) Suppose that $\Gamma_w(S^{-1}R)$ is a connected graph and $x, y \in R^*$. Then there exists a path between $x/1$ and $y/1$. Let $x_1/s_1 — x_2/s_2 — \cdots — x_n/s_n$ be the shortest path between $x/1$ and $y/1$, where $x_1 = x$, $s_1 = 1$, $x_n = y$ and $s_n = 1$. If $x_i \neq x_j$, for every $i \neq j$, then by (ii), we have $x_1 — x_2 — \cdots — x_n$, and so $d(x, y) \leq d(x/1, y/1)$. We show that $x_i \neq x_j$, for $i \neq j$. By contradiction, suppose that $x_i = x_j$. If $1 \leq i < j \leq n - 1$, then by

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

(i) we have that x_i/s_i is adjacent to x_{j+1}/s_{j+1}, a contradiction. If $1 < i < j = n$, then x_{i-1}/s_{i-1} is adjacent to x_n/s_n, a contradiction. Now, suppose that $i = 1$ and $j = n$; then we conclude that $x = y$, a contradiction. Thus, $\mathrm{diam}(\Gamma_w(R)) \le \mathrm{diam}(\Gamma_w(S^{-1}R))$. ■

Example 2.16 By (v) of the previous theorem, if $\mathrm{gr}(\Gamma_w(R)) < \infty$, then $\mathrm{gr}(\Gamma_w(S^{-1}R)) < \infty$. But there are some rings R, for which $\mathrm{gr}(\Gamma_w(S^{-1}R)) < \infty$ and $\mathrm{gr}(\Gamma_w(R)) = \infty$. For instance, $\mathrm{gr}(\Gamma_w(\mathbb{Z}_4)) = \infty$ but $\mathrm{gr}(\Gamma_w(S^{-1}\mathbb{Z}_4)) = 4$, where $S = \{1, 3\}$. We note that $1/1 — 2/1 — 3/1 — 2/3 — 1/1$ is a 4-cycle of $\Gamma_w(S^{-1}\mathbb{Z}_4)$.

3 The Complement of the Weakly Nilpotent Graph of a Commutative Ring

As we mentioned in the introduction, the complement of the weakly nilpotent graph of R, $\overline{\Gamma_w(R)}$, is a graph with the vertex set R^* and two distinct vertices x and y in R^*, are adjacent if and only if $xy \notin N(R)^*$. Clearly, if R is a reduced ring, then $N(R)^* = \varnothing$ and so $\overline{\Gamma_w(R)}$ is a complete graph. In this section, we determine $\mathrm{diam}(\overline{\Gamma_w(R)})$, where R is an Artinian ring. We study the clique number and the girth of $\overline{\Gamma_w(R)}$. Among other results, we characterize all commutative rings R for which $\overline{\Gamma_w(R)}$ is a cycle. We start with the following theorem.

Theorem 3.1 *If R is an Artinian ring. Then the following holds:*

(i) *If R is a local ring, then* $\overline{\Gamma_w(R)}$ *is connected if and only if R is a field.*
(ii) *If R is a non-local ring, then* $\mathrm{diam}(\overline{\Gamma_w(R)}) \le 4$.

Proof (i) One side is clear. For the other side, assume that $(R, \mathfrak{m})$ is a local ring and $\overline{\Gamma_w(R)})$ is connected. If $\mathfrak{m} \ne 0$, then there is not any path between a and u, for every $a \in \mathfrak{m}^*$ and $b \in U(R)$. This yields that $\overline{\Gamma_w(R)})$ is disconnected, a contradiction.

(ii) By [6, Theorem 8.7], we know that $R \cong \prod_{i=1}^n R_i$, where $n \ge 1$ and $(R_i, \mathfrak{m}_i)$ is a local ring, for every i, $1 \le i \le n$. Let e_i be the $1 \times n$ vector whose i-th component is 1 and other components are 0. Let $a = \sum_{i=1}^n a_i e_i$, $b = \sum_{i=1}^n b_i e_i \in V(\overline{\Gamma_w(R)})$. There are three cases.

Case 1. $a, b \in U(R)$. Then a is adjacent to b.

Case 2. $a \in U(R)$ and $b \notin U(R)$. If $b_i \in \mathfrak{m}_i$, for every i, $1 \le i \le n$, then $b_j \ne 0$, for some j, $1 \le j \le n$. Suppose that r is the least positive integer such that $b_j^r = 0$. Hence, $a — b_j^{r-1} a_j e_j — b$ is a path. If $b_j \in U(R)$, for some j, $1 \le j \le n$, then a is adjacent to b.

Case 3. $a, b \notin U(R)$. Let

$$I = \{\, i \mid 1 \le i \le n, a_i \in U(R_i) \,\} \quad \text{and} \quad J = \{\, i \mid 1 \le i \le n, b_i \in U(R_i) \,\}.$$

If $I \cap J \ne \varnothing$, then let $t \in I \cap J$. It is easy to see that a and b are adjacent to e_t. Now, assume that $I \cap J = \varnothing$. We have $a — \sum_{i \in I} e_i — \sum_{i=1}^n e_i — \sum_{i \in J} e_i — b$. This completes the proof. ■

Lemma 3.2 *Let R be a commutative ring; then the following hold:*

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

(i) $\omega(\overline{\Gamma_w(R)}) \geq |U(R)|$.
(ii) $\omega(\overline{\Gamma_w(R)}) = |U(R)|$ *if and only R is a local ring with maximal ideal* $N(R)$

Proof (i) If $|U(R)| = 1$, then it is clear that $\omega(\overline{\Gamma_w(R)}) \geq |U(R)|$. Now, let $u, v \in U(R)$; then it is easy to see that u is adjacent to v in $\overline{\Gamma_w(R)}$, and so $\omega(\overline{\Gamma_w(R)}) \geq |U(R)|$.

(ii) First suppose that R is a local ring with maximal ideal $N(R)$. If C is a clique of $\overline{\Gamma_w(R)}$ and $|C| = \omega(\overline{\Gamma_w(R)})$, then $C \subseteq U(R)$ or $C \subseteq N(R)$. Since otherwise, u is adjacent to x, for some $u \in U(R)$ and $x \in N(R)$, which is impossible. On the other hand by [6, p. 10], $1 + N(R) \subseteq U(R)$. This implies that $|N(R)| \leq |U(R)|$, and so $\omega(\overline{\Gamma_w(R)}) = |U(R)|$. Conversely, suppose that $\omega(\overline{\Gamma_w(R)}) = |U(R)|$. Since $U(R)$ is a clique of $\overline{\Gamma_w(R)}$ and $\omega(\overline{\Gamma_w(R)}) = |U(R)|$, u is not adjacent to x, for every $u \in U(R)$ and $x \in R \setminus U(R)$. This yields that x is a nilpotent element. Therefore $R \setminus U(R) = N(R)$, and so $N(R)$ is a maximal ideal. This shows that R is a local ring with maximal ideal $N(R)$. ∎

Remark 3.3 Let R be a commutative ring. If $\mathrm{gr}(\overline{\Gamma_w(R)}) = \infty$, then $|U(R)| \leq 2$.

In the next theorem, we determine the girth of $\overline{\Gamma_w(R)}$.

Theorem 3.4 *If R is a commutative ring, then* $\mathrm{gr}(\overline{\Gamma_w(R)}) \in \{3, \infty\}$.

Proof Assume that $x_1 — x_2 — \cdots — x_n — x_1$ is a cycle and $n \geq 4$. If $|U(R)| \geq 3$, then by Lemma 3.2, $\mathrm{gr}(\overline{\Gamma_w(R)}) = 3$. Therefore, we can assume that $|U(R)| \leq 2$. Now, consider the following two cases.

Case 1. $|U(R)| = 2$. Let $1, u \in U(R)$. If $0 \neq x \in R \setminus (N(R) \cup \{1, u\})$, then $1 — u — x — 1$ is a 3-cycle of $\overline{\Gamma_w(R)}$ and so $\mathrm{gr}(\overline{\Gamma_w(R)}) = 3$. If $R \setminus N(R) = \{1, u\}$, then $N(R)$ is a maximal ideal of R. This implies that R is a local ring with maximal ideal $N(R)$. Since by [6, p. 10], $1 + N(R) \subseteq U(R)$, $|N(R)| \leq 2$. If $|N(R)| = 1$, then $R \cong \mathbb{Z}_2, \mathbb{Z}_3$ and $\overline{\Gamma_w(R)}) = K_1$ and $\overline{K_2}$, respectively, a contradiction. If $|N(R)| = 2$, then by [3, Theorem 2], $R \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$ and $\overline{\Gamma_w(R)} = \overline{K_3}$, a contradiction.

Case 2. $|U(R)| = 1$. Then by [6, p. 10], $1 + N(R) \subseteq U(R)$, and so R is a reduced ring. This implies that 1 is adjacent to x_1 and x_2. Now, $x_1 — x_2 — 1 — x_1$ is a 3-cycle. This completes the proof. ∎

In the next theorem, we characterize all commutative rings R for which $\overline{\Gamma_w(R)}$ is a cycle.

Theorem 3.5 *Let R be a commutative ring. Then $\overline{\Gamma_w(R)}$ is a cycle if and only if R is a field of order* 4 *or* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof Clearly, if R is a field of order 4 or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\overline{\Gamma_w(R)}$ is a cycle.

Conversely, assume that $\overline{\Gamma_w(R)}$ is a cycle. Then by the previous theorem, $\overline{\Gamma_w(R)}$ is a 3-cycle. This implies that $|R| = 4$. If R is a local ring, then by [3, Theorem 3], R is a field or $R \cong \mathbb{Z}_4, \mathbb{Z}_2(x)/(x^2)$. Obviously, $\mathrm{gr}(\Gamma_w(\mathbb{Z}_4)) = \mathrm{gr}(\Gamma_w(\mathbb{Z}_2(x)/(x^2))) = \infty$. Therefore, R is a field of order 4. Now, suppose that R is a non-local ring. By

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press

Theorem 2.15, $\overline{\Gamma_w(R)}$ is a 3-cycle, and so $|R| = 4$. Hence, by [6, Theorem 8.7], $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as desired. ∎

Acknowledgment The authors would like to thank the referee for her/his valuable and fruitful comments.

References

[1] G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr, and F. Shaveisi, *On the coloring of the annihilating-ideal graph of a commutative ring.* Discrete Math. **312**(2012), no. 17, 2620–2626. http://dx.doi.org/10.1016/j.disc.2011.10.020

[2] S. Akbari, D. Kiani, F. Mohammadi, and S. Moradi, *The total graph and regular graph of a commutative ring.* J. Pure Appl. Algebra **213**(2009), no. 12, 2224–2228. http://dx.doi.org/10.1016/j.jpaa.2009.03.013

[3] S. Akbari and A. Mohammadian, *Zero-divisor graphs of non-commutative rings.* J. Algebra **296**(2006), no. 2, 462–479. http://dx.doi.org/10.1016/j.jalgebra.2005.07.007

[4] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring.* J. Algebra **217**(1999), no. 2, 434–447. http://dx.doi.org/10.1006/jabr.1998.7840

[5] S. E. Atani, A. Yousefian Darani, and E. R. Puczylowski, *On the diameter and girth of ideal-based zero-divisor graphs.* Publ. Math. Debrecen **78**(2011), no. 3–4, 607–612. http://dx.doi.org/10.5486/PMD.2011.4800

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, ON, 1969.

[7] P. Chen, *A kind of graph structure of rings.* Algebra Colloq. **10**(2003), no. 2, 229–238.

[8] A.-H. Li and Q.-S. Li, *A kind of graph structure on von-neumann regular rings.* Int. J. Algebra **4**(2010), no. 5–8, 291–302.

[9] ———, *A kind of graph structure on non-reduced rings.* Algebra Colloq. **17**(2010), no. 1, 173–180. http://dx.doi.org/10.1142/S1005386710000180

[10] M. J. Nikmehr and F. Heydari, *The unit graph of a left artinian ring.* Acta Math. Hungar. **139**(2013), no. 1–2, 134–146. http://dx.doi.org/10.1007/s10474-012-0250-3

[11] M. J. Nikmehr and S. Khojasteh, *On the nilpotent graph of a ring.* Turkish J. Math. **37**(2013), no. 4, 553–559.

[12] M. J. Nikmehr and F. Shaveisi, *The regular digraph of ideals of a commutative ring.* Acta Math. Hungar. **134**(2012), no. 4, 516–528. http://dx.doi.org/10.1007/s10474-011-0139-6

[13] N. J. Rad, H. Jafari, and D. A. Mojdeh, *On domination in zero-divisor graphs.* Canad. Math. Bull. **56**(2013), no. 2, 407–411. http://dx.doi.org/10.4153/CMB-2011-156-1

Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran
e-mail: s_khojasteh@liau.ac.ir

Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
e-mail: nikmehr@kntu.ac.ir

https://doi.org/10.4153/CMB-2016-096-1 Published online by Cambridge University Press