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STOCHASTIC MAXIMUM PRINCIPLE FOR OPTIMAL CONTROL PROBLEM OF FORWARD AND BACKWARD SYSTEM

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Abstract

The maximum principle for optimal control problems of stochastic systems consisting of forward and backward state variables is proved, under the assumption that the diffusion coefficient does not contain the control variable, but the control domain need not be convex.

1. Introduction

The stochastic optimal control problem is important in control theory. A lot of work has been done on the forward stochastic system. See, for example, Ahmed [2], Bensoussan [5], Cadenillas and Karatzas [7], Elliott [8], H. J. Kushner [10], Peng [12]. Recently, another kind of stochastic system, the forward and backward stochastic system, has been developed and studied for its applications in the financial market. In [13], Peng studied the optimal control problem of such a system. The maximum principle he obtained is in local form.

In this paper, we discuss a simplified problem of one in [13], in which the diffusion coefficient does not contain control. We use the "spike variation" method to derive the maximum principle in global form. Thus the control domain is not necessarily convex. For the case when there are initial state constraints and final state constraints, we also obtain a global result by using Ekeland's variational principle. Since some financial models are in the form of forward and backward stochastic systems, our results may have applications in the financial market.

Since the existence problem of optimal control is a different issue, we do not incorporate it in this paper. Some results in this field can be seen in Ahmed [1], [2].

This paper is organized as follows. In Section 2, we state the problem and our main assumptions. In Section 3, we study the variational equations and variational inequality. In Section 4, we obtain the maximum principle in global form. In the last

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section, we show how to obtain the maximum principle in case initial and final state contraints are imposed.

2. Statement of the problem

Let $(\Omega, \mathscr{F}, P)$ be a probability space with filtration $\mathscr{F}^t$ and $W(\cdot)$ a R^d-valued standard Wiener process. We assume $\mathscr{F}^t = \sigma\{W(s) : 0 \le s \le t\}$. We consider the following forward and backward stochastic control system:

$$\begin{cases} \quad dx = f(x, v, t)dt + \sigma(x, t)dW_t, \\ x(0) = x_0, \\ \quad dy = g(x, y, z, v, t)dt + z dW_t, \\ y(T) = h(x(T)), \end{cases} \tag{1}$$

where

$$\begin{aligned} &f : R^n \times R^k \times [0, T] \to R^n, \\ &\sigma : R^n \times [0, T] \to \mathscr{L}(R^d; R^n), \\ &g : R^n \times R^m \times \mathscr{L}(R^d; R^m) \times R^k \times [0, T] \to R^m, \\ &h : R^n \to R^m. \end{aligned}$$

Let U be a non-empty subset of R^k. We set

$$\mathscr{U}_{ad} = \{v(\cdot) \in \mathscr{L}^2_{\mathscr{F}}(0, T; R^k) : v(t) \in U, \ a.e., a.s.\}.$$

Our optimal control problem is to minimize the cost function

$$J(v(\cdot)) = E\gamma(y(0)),$$

over $\mathscr{U}_{ad}$, where $\gamma : R^m \to R^1$.

We assume:

(H_1) f, g, σ, h, γ are continuously differentiable with respect to (x, y, z);

(H_2) the derivatives of f, g and σ with respect to x, y, z are bounded,

$$|f_x| \le C, \qquad \text{for} \qquad f_x = f_x, \sigma_x, g_x, g_y, g_z,$$

and

$$|h_x| \le C(1 + |x|), \qquad |\gamma_y| \le C(1 + |y|).$$

3. Variational equations and variational inequality

The purpose of this section is to introduce the usual first order variational equations and to derive variational inequality. Let $(u(\cdot), x(\cdot), y(\cdot), z(\cdot))$ be an optimal solution of the problem. We introduce the following spike variational control:

$$u^\epsilon(t) = \begin{cases} v, & \tau \le t \le \tau + \epsilon, \\ u(t), & \text{otherwise}, \end{cases}$$

where $\epsilon > 0$ is sufficiently small, v is an arbitrary $\mathscr{F}^\tau$-measurable random variable with values in U, $0 \le t < T$, and $\sup_{\omega\in\Omega} |v(\omega)| < \infty$. Let $(x^\epsilon(\cdot), y^\epsilon(\cdot), z^\epsilon(\cdot))$ be the trajectory of system (1) corresponding to control $u^\epsilon(\cdot)$.

We introduce the following variational equations:

$$\begin{cases} dx_1 = \left[f_x x_1 + f(u^\epsilon) - f(u)\right]dt + \sigma_x x_1 dW_t, \\ x_1(0) = 0, \\ dy_1 = \left[g_x x_1 + g_y y_1 + g_z z_1 + g(u^\epsilon) - g(u)\right]dt + z_1 dW_t, \\ y_1(T) = h_x(x(T))x_1(T). \end{cases} \tag{2}$$

For convenience, we use the following notation in this paper.

$$\begin{aligned} f_x &\triangleq f_x(x(t), u(t), t), & g_x &\triangleq g_x(x(t), y(t), z(t), u(t), t), \\ f(u^\epsilon) &\triangleq f(x(t), u^\epsilon(t), t), & f(t) &\triangleq f(x(t), u(t), t), \quad \text{etc.} \end{aligned}$$

The variational inequality can be obtained from the fact $J(u^\epsilon(\cdot)) - J(u(\cdot)) \ge 0$. The following lemmas are needed to establish the inequality.

LEMMA 1. Suppose (H_1) and (H_2) hold. For the first order variations x_1, y_1, z_1, we have the following estimations:

$$\sup_{0\le t\le T} E|x_1(t)|^2 \le C\epsilon^2, \tag{3}$$

$$\sup_{0\le t\le T} E|x_1(t)|^4 \le C\epsilon^4, \tag{4}$$

$$\sup_{0\le t\le T} E|y_1(t)|^2 \le C\epsilon^2, \tag{5}$$

$$\sup_{0\le t\le T} E|y_1(t)|^4 \le C\epsilon^4, \tag{6}$$

$$E\int_0^T (z_1(s))^2\,ds \le C\epsilon^2, \tag{7}$$

$$E\left(\int_0^T (z_1(s))^2\,ds\right)^2 \le C\epsilon^4. \tag{8}$$

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PROOF. We first prove (3) and (4). The first equation of (2) yields

$$
\begin{aligned}
E|x_1(t)|^2 &= E\left(\int_0^t \left[f_x x_1 + f(u^\epsilon) - f(u)\right] ds + \int_0^t \sigma_x x_1 \, dW_s\right)^2 \\
&\le 3\left[E\left(\int_0^t f_x x_1 \, ds\right)^2 + E\left(\int_0^t \left[f(u^\epsilon) - f(u)\right] ds\right)^2 + E\int_0^t (\sigma_x x_1)^2 \, ds\right] \\
&\le 6C^2 T E \int_0^t {x_1}^2 \, ds + 3E\left(\int_0^t \left(f(u^\epsilon) - f(u)\right) ds\right)^2 .
\end{aligned}
$$

Applying Gronwall's inequality,

$$
E|x_1(t)|^2 \le C\epsilon^2, \qquad \text{for} \quad t \in [0, T] \quad \text{uniformly}.
$$

Similarly (4) holds.

We next estimate y_1 and z_1. Squaring both sides of

$$
\begin{aligned}
&-y_1(t) - \int_t^T z_1(s) \, dW_s \\
&= -h_x(x(T))x_1(T) + \int_t^T \left(g_x x_1 + g_y y_1 + g_z z_1 + g(u^\epsilon) - g(u)\right) ds,
\end{aligned}
$$

and using the fact that

$$
E y_1(t) \int_t^T z_1(s) \, dW_s = 0,
$$

we get

$$
\begin{aligned}
&E|y_1(t)|^2 + E\int_t^T (z_1(s))^2 \, ds \\
&\quad = E\left(-h_x(x(T))x_1(T) + \int_t^T \left(g_x x_1 + g_y y_1 + g_z z_1 + g(u^\epsilon) - g(u)\right) ds\right)^2 \\
&\quad \le 5C^2 E x_1^2(T) + 5C^2 T E \int_t^T x_1^2(s) \, ds + 5C^2 T E \int_t^T y_1^2(s) \, ds \\
&\qquad + 5C^2(T - t) E \int_t^T z_1^2(s) \, ds + 5E\left(\int_t^T \left(g(u^\epsilon) - g(u)\right) ds\right)^2 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
E|y_1(t)|^2 + \frac{1}{2} E\int_t^T z_1^2(s) \, ds &\le 5C^2 E x_1^2(T) + 5C^2 T E \int_0^T x_1^2(s) \, ds \\
&\quad + 5C^2 T E \int_t^T y_1^2(s) \, ds + 5E\left(\int_t^T \left(g(u^\epsilon) - g(u)\right) ds\right)^2 ,
\end{aligned}
$$

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with $\delta = \dfrac{1}{10c^2}$, $t \in [T-\delta, T]$.

Applying Gronwall's inequality,

$$E|y_1(t)|^2 \le C\epsilon^2, \qquad t \in [T-\delta, T],$$

$$E\int_t^T z_1^2(s)\,ds \le C\epsilon^2, \quad t \in [T-\delta, T].$$

Similarly we have

$$\begin{aligned} -y_1(t) &- \int_t^{T-\delta} z_1(s)\,dW_s \\ &= -Y_1(T-\delta) + \int_t^{T-\delta} \big(g_x x_1 + g_y y_1 + g_z z_1 + g(u^\epsilon) - g(u)\big)\,ds. \end{aligned}$$

So

$$\begin{aligned} E|y_1(t)|^2 + E\int_t^{T-\delta} z_1^2(s)\,ds \le 5E|y_1(T-\delta)|^2 &+ 5C^2TE\int_t^{T-\delta} x_1^2(s)\,ds \\ &+ 5C^2TE\int_t^{T-\delta} y_1^2(s)\,ds \\ &+ 5C^2(T-\delta-t)E\int_t^{T-\delta} z_1^2(s)\,ds \\ &+ 5E\left(\int_t^{T-\delta}\big(g(u^\epsilon)-g(u)\big)\,ds\right)^2. \end{aligned}$$

Thus

$$E|y_1(t)|^2 \le C\epsilon^2, \qquad t \in [T-2\delta, T],$$

$$E\int_t^{T-\delta} z_1^2(s)\,ds \le C\epsilon^2, \quad t \in [T-2\delta, T].$$

After a finite number of iterations, (5) and (7) are obtained. (6) and (8) can be proved by using a similar method and the inequality

$$E\left(\int_t^T z_1(s)\,dW_s\right)^4 \ge \beta E\left(\int_t^T z_1^2(s)\,ds\right)^2, \quad \beta > 0.$$

LEMMA 2. Suppose (H_1) and (H_2) hold. Then we have the following estimations:

$$\sup_{0\le t\le T} E|x^\epsilon(t) - x(t) - x_1(t)|^2 \le C_\epsilon \epsilon^2, \quad C_\epsilon \to 0, \tag{9}$$

$$\sup_{0\le t\le T} E|y^\epsilon(t) - y(t) - y_1(t)|^2 \le C_\epsilon \epsilon^2, \quad C_\epsilon \to 0, \tag{10}$$

$$E\int_0^T |z^\epsilon(t) - z(t) - z_1(t)|^2\, ds \le C_\epsilon \epsilon^2 \quad C_\epsilon \to 0. \tag{11}$$

PROOF. To prove (9), we observe that

$$\begin{aligned}
&\int_0^t f(x+x_1, u^\epsilon)\, ds + \int_0^t \sigma(x+x_1)\, dW_s \\
&\quad = \int_0^t \left[f(x, u^\epsilon) + \int_0^1 f_x(x+\lambda x_1, u^\epsilon) d\lambda x_1 \right] ds \\
&\qquad\qquad + \int_0^t \left[\sigma(x) + \int_0^1 \sigma_x(x+\lambda x_1) d\lambda x_1 \right] dW_s \\
&\quad = \int_0^t f(x,u)\, ds + \int_0^t \sigma(x)\, dW_s + \int_0^t [f_x x_1 + f(u^\epsilon) - f(u)]\, ds \\
&\qquad\qquad + \int_0^t \sigma_x x_1\, dW_s + \int_0^t A^\epsilon\, ds + \int_0^t B^\epsilon\, dW_s \\
&\quad = x(t) - x_0 + x_1(t) + \int_0^t A^\epsilon\, ds + \int_0^t B^\epsilon\, dW_s
\end{aligned}$$

in which

$$\begin{aligned}
A^\epsilon &= \int_0^1 [f_x(x+\lambda x_1, u^\epsilon) - f_x(x,u)]\, d\lambda x_1, \\
B^\epsilon &= \int_0^1 [\sigma_x(x+\lambda x_1) - \sigma_x(x)]\, d\lambda x_1.
\end{aligned}$$

It follows easily from Lemma 1 that

$$\sup_{0\le t\le T} E\left\{ \left(\int_0^t A^\epsilon\, ds \right)^2 + \left(\int_0^t B^\epsilon\, dW_s \right)^2 \right\} = o(\epsilon^2). \tag{12}$$

Since

$$x^\epsilon(t) - x_0 = \int_0^t f(x^\epsilon, u^\epsilon)\, ds + \int_0^t \sigma(x^\epsilon)\, dW_s,$$

we get

$$\begin{aligned}
x^\epsilon(t) - x(t) - x_1(t) = \int_0^t C^\epsilon(s)(x^\epsilon - x - x_1)\, ds + \int_0^t D^\epsilon(s)(x^\epsilon - x - x_1)\, dW_s \\
+ \int_0^t A^\epsilon ds + \int_0^t B^\epsilon\, dW_s,
\end{aligned}$$

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with

$$C^\epsilon(s) = \int_0^1 f_x(x + x_1 + \lambda(x^\epsilon - x - x_1), u^\epsilon)\, d\lambda,$$
$$D^\epsilon(s) = \int_0^1 \sigma_x(x + x_1 + \lambda(x^\epsilon - x - x_1), u^\epsilon)\, d\lambda.$$

Using Gronwall's inequality, (9) follows from the above relation and (12).

We next prove (10) and (11). It can be easily checked that

$$\int_t^T g(x + x_1, y + y_1, z + z_1, u^\epsilon)\, ds + \int_t^T (z(s) + z_1(s))\, dW_s$$
$$= h(x(T)) + h_x(x(T))x_1(T) - y(t) - y_1(t) + \int_t^T G^\epsilon\, ds,$$

where

$$G^\epsilon = \int_0^1 \big(g_x(x + \lambda x_1, y + \lambda y_1, z + \lambda z_1, u^\epsilon) - g_x\big)\, d\lambda x_1$$
$$+ \int_0^1 \big(g_y(x + \lambda x_1, y + \lambda y_1, z + \lambda z_1, u^\epsilon) - g_y\big)\, d\lambda y_1$$
$$+ \int_0^1 \big(g_z(x + \lambda x_1, y + \lambda y_1, z + \lambda z_1, u^\epsilon) - g_z\big)\, d\lambda z_1.$$

So we have

$$-\big(y^\epsilon(t) - y(t) - y_1(t)\big) = -\big(h(x^\epsilon(T)) - h(x(T))\big) + h_x(x(T))x_1(T)$$
$$+ \int_t^T [g(x^\epsilon, y^\epsilon, z^\epsilon, u^\epsilon) - g(x + x_1, y + y_1, z + z_1, u^\epsilon)]ds$$
$$+ \int_t^T \big(z^\epsilon(s) - z(s) - z_1(s)\big)dW_s + \int_t^T G^\epsilon\, ds.$$

Thus it follows that

$$E|y^\epsilon(t) - y(t) - y_1(t)|^2 + E\int_t^T |z^\epsilon(s) - z(s) - z_1(s)|^2\, ds$$
$$= E\Big\{ - \big(h(x^\epsilon(T)) - h(x(T) + x_1(T))\big)$$
$$- \int_0^1 \big[h_x(x(T) + \lambda x_1(T)) - h_x(x(T))\big]\, d\lambda x_1(T) + \int_t^T G^\epsilon\, ds$$
$$+ \int_t^T \big[g(x^\epsilon, y^\epsilon, z^\epsilon, u^\epsilon) - g(x + x_1, y + y_1, z + z_1, u^\epsilon)\big]\, ds\Big\}^2.$$

From Lemma 1 and (9), we see that

$$\sup_{0\le t\le T} E\left(\int_t^T G^\epsilon\, ds\right)^2 = \mathrm{o}(\epsilon^2),$$
$$E\big[h(x^\epsilon(T)) - h(x(T) + x_1(T))\big]^2 = \mathrm{o}(\epsilon^2).$$

We get (10) and (11) by applying the iterative method used in Lemma 1 to the above relation.

LEMMA 3. Under the assumptions (H_1) and (H_2), the following variational inequality holds:

$$E\gamma_y(y(0))y_1(0) \ge \mathrm{o}(\epsilon).$$

PROOF. From Lemma 2, we have the estimation

$$E\big[\gamma(y^\epsilon(0)) - \gamma(y(0) + y_1(0))\big] = \mathrm{o}(\epsilon).$$

Therefore

$$\begin{aligned} 0 &\le E\big[\gamma(y(0) + y_1(0)) - \gamma(y(0))\big] + \mathrm{o}(\epsilon) \\ &= E\gamma_y(y(0))y_1(0) + \mathrm{o}(\epsilon). \end{aligned}$$

4. The maximum principle in global form

We introduce the adjoint equations and the Hamilton function for our problem. From the variational inequality obtained in Lemma 3, the maximum principle can be proved by applying Itô's formula.

The adjoint equations are

$$\begin{cases} -dp = (f_x^* p + g_x^* q + \sigma_x^* k)dt - k\,dW_t, \\ p(T) = -h_x^*(x(T))q(T), \\ -dq = g_y^* q\,dt + g_z^* q\,dW_t, \\ q(0) = -\gamma_y(y(0)), \end{cases} \tag{13}$$

and the Hamiltonian function is

$$H(x, y, z, v, p, q, k, t) \triangleq \langle p, f(x, v, t)\rangle + \langle q, g(x, y, z, v, t)\rangle + \langle k, \sigma(x)\rangle,$$

where

$$\begin{aligned} H:\quad & R^n \times R^m \times \mathscr{L}(R^d; R^m) \times R^k \times R^n \times R^m \times \mathscr{L}(R^d; R^n) \\ & \times [0, T] \to R^n. \end{aligned}$$

Relations (13) can be rewritten as

$$\begin{cases} -dp = H_x dt - k dW_t, \\ p(T) = -h_x^*(x(T))q(T), \\ -dq = H_y dt + H_z dW_t, \\ q(0) = -\gamma_y(y(0)). \end{cases} \tag{14}$$

From (14) and Lemma 3, we have

THEOREM 1. Suppose (H_1) and (H_2) hold. Let $(u(\cdot), x(\cdot), y(\cdot), z(\cdot))$ be an optimal control and its corresponding trajectory of (1), $(p(\cdot), q(\cdot), k(\cdot))$ be the corresponding solution of (14). Then the maximum principle holds, that is

$$\begin{aligned} &H(x(t), y(t), z(t), v, p(t), q(t), k(t), t) \\ &\quad \geq H(x(t), y(t), z(t), u(t), p(t), q(t), k(t), t), \ \forall v \in U, \ a.e., a.s. \end{aligned} \tag{15}$$

PROOF. By applying Itô's formula to $\langle p, x_1 \rangle$ and $\langle q, y_1 \rangle$, it follows from (2) and (14) that

$$\begin{aligned} o(\epsilon) &\leq E\gamma_y(y(0))y_1(0) \\ &= E\int_0^T \big[H(x(t), y(t), z(t), u^\epsilon(t), p(t), q(t), k(t)) \\ &\qquad -H(x(t), y(t), z(t), u(t), p(t), q(t), k(t)\big] dt. \end{aligned}$$

From the above inequality, (15) can be easily derived.

5. Problem with state constraints

In this section, we discuss briefly the case when there are initial state constraints and final state constraints on the state variables:

$$EG_1(x(T)) = 0,$$
$$EG_0(y(0)) = 0,$$

where

$$\begin{aligned} G_1 &: R^n \to R^{n_1}, \qquad n_1 < n, \\ G_0 &: R^m \to R^{m_1}, \qquad m_1 < m. \end{aligned}$$

We assume

(H_3) G_0, G_1 are continuously differentiable and G_{0x}, G_{1x} are bounded;

(H_4) the control domain U is closed.

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We apply Ekeland's variational principle to solve this optimal control problem. We first define the metric in $\mathscr{U}_{ad}$. For $u(\cdot), v(\cdot) \in \mathscr{U}_{ad}$, let

$$d(u(\cdot), v(\cdot)) = E\text{mes}\{t \in [0, T] : u(t) \neq v(t)\}.$$

With this metric, $(\mathscr{U}_{ad}, d(\cdot, \cdot))$ is a complete space.

Let $(u(\cdot), x(\cdot), y(\cdot), z(\cdot))$ be an optimal solution of the problem. For $v(\cdot) \in \mathscr{U}_{ad}$, we define

$$J_\rho(v(\cdot)) = \big\{E|G_1(x(T; v))|^2 + E|G_0(y(0; v))|^2 \\ +\big[E\gamma(y(0; v)) - E\gamma(y(0)) + \rho\big]^2\big\}^{\frac{1}{2}}.$$

It can be checked that $J_\rho(v(\cdot)) : \mathscr{U}_{ad} \to R^1$ is continuous, and

$$J_\rho(v(\cdot)) \geq 0, \quad \forall v(\cdot) \in \mathscr{U}_{ad},$$
$$J_\rho(u(\cdot)) = \rho.$$

Obviously, we have

$$J_\rho(u(\cdot)) \leq \inf_{v(\cdot)\in\mathscr{U}_{ad}} J_\rho(v(\cdot)) + \rho.$$

From Ekeland's variational principle, there exists $v_\rho(\cdot) \in \mathscr{U}_{ad}$ such that

$$\begin{aligned} &(1)\ J_\rho(v_\rho(\cdot)) \leq J_\rho(u(\cdot)) = \rho, \\ &(2)\ d(v_\rho(\cdot), u(\cdot)) \leq \sqrt{\rho}, \\ &(3)\ J_\rho(w(\cdot)) \geq J_\rho(v_\rho(\cdot)) - \sqrt{\rho}\, d(w(\cdot), v_\rho(\cdot)), \quad \text{for} \quad w(\cdot) \in \mathscr{U}_{ad}. \end{aligned} \tag{16}$$

Making "spike variation"

$$v_\rho^\epsilon(t) = \begin{cases} v, & \tau \leq t \leq \tau + \epsilon, \\ v_\rho(t), & \text{otherwise}, \end{cases}$$

for v_ρ as in Section 3, it follows from (16) that

$$J_\rho(v_\rho^\epsilon(\cdot)) - J_\rho(v_\rho(\cdot)) + \sqrt{\rho}\, d(v_\rho^\epsilon(\cdot); v_\rho(\cdot)) \geq 0. \tag{17}$$

Let (x_ρ, y_ρ, z_ρ) and $(x_\rho^\epsilon, y_\rho^\epsilon, z_\rho^\epsilon(\cdot))$ be the corresponding trajectories to $v_\rho\ (\cdot)$ and $v_\rho^\epsilon(\cdot)$ respectively. The variational equations is the same as the one in Section 3, with $(x(\cdot), y(\cdot), z(\cdot)) = (x_\rho(\cdot), y_\rho(\cdot), z_\rho(\cdot))$, $u(\cdot) = v_\rho(\cdot)$. Similarly to the approach in Lemma 2, it can be shown that

$$\sup_{0\leq t\leq T} E|x_\rho^\epsilon(t) - x_\rho(t) - x_{\rho 1}(t)|^2 \leq C_\epsilon \epsilon^2, \qquad C_\epsilon \to 0,$$

$$\sup_{0\le t\le T} E|y^\epsilon_\rho(t) - y_\rho(t) - y_{\rho 1}(t)|^2 \le C_\epsilon \epsilon^2, \qquad C_\epsilon \to 0,$$

$$E\int_0^T |z^\epsilon_\rho(t) - z_\rho(t) - z_{\rho 1}(t)|^2 \le C_\epsilon \epsilon^2, \qquad C_\epsilon \to 0.$$

Thus

$$\begin{aligned} J^2_\rho(v^\epsilon_\rho(\cdot)) - J^2_\rho(v_\rho(\cdot)) = {} & 2\langle EG_1(x_\rho(T)), EG_{1x}(x_\rho(T))x_{\rho 1}(T)\rangle \\ & +2\langle EG_0(y_\rho(0)), EG_{0x}(y_\rho(0))y_{\rho 1}(0)\rangle \\ & +2\langle E(\gamma(y_\rho(0)) - \gamma(y(0)) + \rho), E\gamma_y(y_\rho(0))y_{\rho 1}(0)\rangle + o(\epsilon). \end{aligned} \tag{18}$$

Let

$$\begin{aligned} h^\epsilon_{\rho 1} &= \frac{2EG_1(x_\rho(T))}{J_\rho(v^\epsilon_\rho(\cdot)) + J_\rho(v_\rho(\cdot))}, \\ h^\epsilon_{\rho 2} &= \frac{2EG_0(y_\rho(0))}{J_\rho(v^\epsilon_\rho(\cdot)) + J_\rho(v_\rho(\cdot))}, \\ h^\epsilon_{\rho 0} &= \frac{2E[\gamma(y_\rho(0)) - \gamma(y(0)) + \rho]}{J_\rho(v^\epsilon_\rho(\cdot)) + J_\rho(v_\rho(\cdot))}. \end{aligned}$$

Since

$$J_\rho(v(\cdot)) > 0, \qquad J_\rho(v^\epsilon_\rho(\cdot)) > 0, \qquad J_\rho(v^\epsilon_\rho(\cdot)) \to J_\rho(v_\rho(\cdot)), \qquad \epsilon \to 0,$$

it follows from (17) and (18) that

$$\begin{aligned} &\langle h^\epsilon_{\rho 1}, EG_{1x}(x_\rho(T))x_{\rho 1}(T)\rangle + \langle h^\epsilon_{\rho 2}, EG_{0x}(y_\rho(0))y_{\rho 1}(0)\rangle \\ &\quad + \langle h^\epsilon_{\rho 0}, E\gamma_y(y_\rho(0))y_{\rho 1}(0)\rangle + \epsilon\sqrt{\rho} + o(\epsilon) \ge 0. \end{aligned} \tag{19}$$

Let $(p^\epsilon_\rho, k^\epsilon_\rho, q^\epsilon_\rho)$ be the solution of

$$\begin{cases} -dp^\epsilon_\rho = [f^*_x(x_\rho, v_\rho)p^\epsilon_\rho + g^*_x(x_\rho, y_\rho, z_\rho, v_\rho)q^\epsilon_\rho + \sigma^*_x(x_\rho)k^\epsilon_\rho]dt - k^\epsilon_\rho dW_t, \\ p^\epsilon_\rho(T) = G_{1x}(x_\rho(T))h\rho_1{}^\epsilon - h^*_x(x_\rho(T))q^\epsilon_\rho(T), \\ -dq^\epsilon_\rho = g^*_y(x_\rho, y_\rho, z_\rho, v_\rho)q^\epsilon_\rho dt + g_z(x_\rho, y_\rho, z_\rho, v_\rho)q^\epsilon_\rho dW_t, \\ q^\epsilon_\rho(0) = -(G_{0x}(y_\rho(0))h^\epsilon_{\rho 2} + \gamma_y(y_\rho(0))h^\epsilon_{\rho 0}). \end{cases} \tag{20}$$

Using Itô's formula, (19) can be rewritten as

$$\begin{aligned} E\int_0^T \big[&H(x_\rho, y_\rho, z_\rho, v^\epsilon_\rho, p^\epsilon_\rho, q^\epsilon_\rho, k^\epsilon_\rho) \\ &-H(x_\rho, y_\rho, z_\rho, v_\rho, p^\epsilon_\rho, q^\epsilon_\rho, k^\epsilon_\rho)\big]dt + \epsilon\sqrt{\rho} + o(\epsilon) \ge 0, \end{aligned} \tag{21}$$

where

$$H(x, y, z, v, p, q, k) \triangleq \langle p, f(x, v)\rangle + \langle q, g(x, y, z, v)\rangle + \langle k, \sigma(x)\rangle.$$

Since

$$\lim_{\epsilon\to 0}(|h^\epsilon_{\rho 0}|^2 + ||h^\epsilon_{\rho 1}||^2 + ||h^\epsilon_{\rho 2}||^2) = 1,$$

there exists a convergent subsequence $(h^\epsilon_{\rho 0}, h^\epsilon_{\rho 1}, h^\epsilon_{\rho 2})$ with

$$\begin{aligned} &(h^\epsilon_{\rho 0}, h^\epsilon_{\rho 1}, h^\epsilon_{\rho 2}) \to (h_{\rho 0}, h_{\rho 1}, h_{\rho 2}), \quad \epsilon \to 0, \\ &|h_{\rho 0}|^2 + ||h_{\rho 1}||^2 + ||h_{\rho 2}||^2 = 1. \end{aligned} \tag{22}$$

Let (p_ρ, q_ρ, k_ρ) be the solution of

$$\begin{cases} -dp_\rho = \left[f^*_x(x_\rho, v_\rho)p_\rho + g^*_x(x_\rho, y_\rho, z_\rho, v_\rho)q_\rho + \sigma^*_x(x_\rho)k_\rho\right]dt - k_\rho dW_t, \\ p_\rho(T) = G_{1x}(x_\rho(T))h_{\rho 1} - h^*_x(x_\rho(T))q_\rho(T), \\ -dq_\rho = g^*_y(x_\rho, y_\rho, z_\rho, v_\rho)q_\rho dt + g^*_z(x_\rho, y_\rho, z_\rho, v_\rho)q_\rho dW_t, \\ q_\rho(0) = -(G_{0x}(y_\rho(0))h_{\rho 2} + \gamma_y(y_\rho(0))h_{\rho 0}). \end{cases}$$

It can be easily proved that

$$\begin{aligned} p^\epsilon_\rho &\to p_\rho \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^n), \\ q^\epsilon_\rho &\to q_\rho \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^m), \\ k^\epsilon_\rho &\to k_\rho \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, \mathscr{L}(R^d, R^n)). \end{aligned}$$

So from (21) we have

$$\begin{aligned} &H(x_\rho, y_\rho, z_\rho, v, p_\rho, q_\rho, k_\rho) \\ &- H(x_\rho, y_\rho, z_\rho, v_\rho, p_\rho, q_\rho, k_\rho) + \sqrt{\rho} \geq 0. \qquad \forall v \in U, \quad a.e., a.s. \end{aligned} \tag{23}$$

Similarly from (22), there exists a subsequence of $(h_{\rho 0}, h_{\rho 1}, h_{\rho 2})$ which converges to (h_0, h_1, h_2) with

$$|h_0|^2 + ||h_1||^2 + ||h_2||^2 = 1.$$

Since $v_\rho(\cdot) \to u(\cdot) \quad \rho \to 0$, consequently,

$$\begin{aligned} x_\rho(\cdot) &\to x(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^n), \\ y_\rho(\cdot) &\to y(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^m), \\ z_\rho(\cdot) &\to z(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, \mathscr{L}(R^d, R^n)), \\ x_{\rho 1}(\cdot) &\to x_1(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^n), \\ y_{\rho 1}(\cdot) &\to y_1(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, R^m), \\ z_{\rho 1}(\cdot) &\to z_1(\cdot) \quad &&\text{in} \quad \mathscr{L}^2_{\mathscr{F}}(0, T, \mathscr{L}(R^d, R^n)), \end{aligned}$$

where $(x_1(\cdot), y_1(\cdot), z_1(\cdot))$ is the solution of the variational equations whose forms are the same as (2).

We introduce the adjoint equations of the above variational equations as

$$\begin{cases} -dp = \left[f_x^*(x,u)p + g_x^*(x,y,z,u)q + \sigma_x^*(x)k\right]dt - kdW_t, \\ p(T) = G_{1x}(x(T))h_1 - h_x^*(x(T))q(T), \\ -dq = g_y^*(x,y,z,u)qdt + g_z^*(x,y,z,u)qdW_t, \\ q(0) = -(G_{0x}(y(0))h_2 + \gamma_y(y(0))h_0). \end{cases} \tag{24}$$

It can be proved that

$$\begin{aligned} p_\rho(\cdot) &\to p(\cdot) && \text{in} && \mathscr{L}^2_{\mathscr{F}}(0, T, R^n), \\ q_\rho(\cdot) &\to q(\cdot) && \text{in} && \mathscr{L}^2_{\mathscr{F}}(0, T, R^m), \\ k_\rho(\cdot) &\to k(\cdot) && \text{in} && \mathscr{L}^2_{\mathscr{F}}(0, T, \mathscr{L}(R^d, R^n)). \end{aligned}$$

Let $\rho \to 0$ in (23). Then the following inequality holds.

$$\begin{aligned} &H(x(t), y(t), z(t), v, p(t), q(t), k(t)) - \\ &H(x(t), y(t), z(t), u(t), p(t), q(t), k(t)) \geq 0, \quad \forall v \in U, \quad a.e., a.s. \end{aligned} \tag{25}$$

So we have the following theorem.

THEOREM 2. Assume (H_1), (H_2), (H_3) and (H_4) hold. Let $(u(\cdot), x(\cdot), y(\cdot), z(\cdot))$ be an optimal solution of the optimal control problem stated at the beginning of this section, and $(p(\cdot), q(\cdot), k(\cdot))$ be the corresponding solution of the adjoint equations (24). Then the maximum principle (25) holds.

Remark

For the forward stochastic system in which control enters into the diffusion coefficient, the maximum principle in global form can be found in Arkin and Saksonov [4], Bismut [6], Cadenillas and Karatzas [7], and Peng [12]. But for the forward and backward stochastic system, such an optimal control problem is still an open problem.

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