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A NOTE ON DERIVATIONS OF LIE ALGEBRAS

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Abstract

In this note, we will prove that a finite-dimensional Lie algebra L over a field of characteristic zero, admitting an abelian algebra of derivations $D \leq \mathrm{Der}(L)$, with the property

$$L^n \subseteq \sum_{d \in D} d(L),$$

for some $n > 1$, is necessarily solvable. As a result, we show that if L has a derivation $d : L \to L$ such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable.

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In [3], Ladisch proved that a finite group G, admitting an element a with the property $G' = [a, G]$, is solvable. Here, $[a, G]$ is the set of all commutators $[a, x]$, with $x \in G$. Using this result, one can prove that a finite group is solvable if it has a fixed point free automorphism. In this note, we prove a similar result for Lie algebras in a more general framework; we show that a finite-dimensional Lie algebra L of characteristic zero is solvable if it has an abelian subalgebra A with the property $L^n \subseteq [A, L]$, for some $n > 1$, where $[A, L]$ denotes the linear subspace generated by all commutators of the form $[a, x]$, with $a \in A$ and $x \in L$. Next, we use this result to prove that a finite-dimensional Lie algebra L over a field of characteristic zero, admitting an abelian algebra of derivations $D \leq \mathrm{Der}(L)$ with the property

$$L^n \subseteq \sum_{d \in D} d(L),$$

for some $n > 1$, is necessarily solvable. As a special case, we conclude that if the Lie algebra L admits a derivation $d : L \to L$, such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable. Note that a similar result was obtained by Jacobson in [2]: a finite-dimensional Lie algebra over an algebraically closed field of characteristic zero, admitting an invertible derivation, is nilpotent.

Our main result (Corollary 2 below) is also true for connected compact Lie groups, and so it may be also true for finite groups. Therefore, we ask the following question.

Let G be a finite group admitting an abelian subgroup A with the property $\gamma_n(G) \subseteq \{[a, x] : a \in A, x \in G\}$, for some $n > 1$. Is it true that G is solvable?

In this note, L is a finite-dimensional Lie algebra over a field K of characteristic zero. We denote by L^n and $L^{(n)}$ the nth terms of the lower central series and derived series of L, respectively. Also we denote by Der(L) the algebra of derivations of L.

Theorem 1. *Let L be a finite-dimensional Lie algebra over K and suppose that U is an ideal. Suppose that there is an abelian subalgebra A such that $U^n \subseteq [A, U]$, for some $n > 1$. Then U is solvable.*

Proof. First, we assume that K is algebraically closed, so we can apply the solvability criterion of Cartan. Let $S = U^n$ and define a symmetric bilinear form on L by

$$\kappa_S(x, y) = \mathrm{Tr}(ad_S x \circ ad_S y),$$

where ad_S denotes the restriction of ad on S. Note that, since S is an ideal of L, κ_S is associative, that is,

$$\kappa_S([x, y], z) = \kappa_S(x, [y, z]).$$

We show that $\kappa_S(S, S') = 0$. However, since $S' \subseteq U^n \subseteq [A, U]$, we prove that $\kappa_S(S, [A, U]) = 0$. By the associativity of κ_S, this is equivalent to $\kappa_S([S, U], A) = 0$. But we have $[S, U] \subseteq U^n \subseteq [A, U]$, therefore it is enough to prove that $\kappa_S([A, U], A) = 0$. By the associativity again, this is just $\kappa_S([A, A], U) = 0$, which is true, because A is abelian. Hence S is solvable by Cartan's criterion and, since $U^{(n-2)} \subseteq U^{n-1} = S$, U is solvable.

We now suppose that K is not necessarily algebraically closed. Let $\overline{K}$ be its algebraic closure and $\overline{L} = \overline{K} \otimes_K L$. Now $\overline{L}$ is a finite-dimensional Lie algebra over $\overline{K}$ in which $\overline{K} \otimes_K U$ is an ideal and $\overline{K} \otimes_K A$ is an abelian subalgebra. Further, we have

$$(\overline{K} \otimes_K U)^n = \overline{K} \otimes_K U^n \subseteq [\overline{K} \otimes_K A, \overline{K} \otimes_K U].$$

So $\overline{K} \otimes_K U$ is solvable, that is, there is a number m such that $(\overline{K} \otimes_K U)^{(m)} = 0$. On the other hand,

$$(\overline{K} \otimes_K U)^{(m)} = \overline{K} \otimes_K U^{(m)},$$

therefore $U^{(m)} = 0$. □

As a result, if we assume that $U = L$, we obtain the following corollary.

Corollary 2. *Suppose that there exist an abelian subalgebra $A \leq L$ and an integer $n > 1$, such that $L^n \subseteq [A, L]$. Then L is solvable.*

As another result, we have the following corollary.

Corollary 3. *Suppose that L is semisimple and A is an abelian subalgebra. Then $[A, L] \subsetneq L$.*

Using the Lie functor, we can restate Corollary 2 for connected compact Lie groups.

Corollary 4. *Suppose that a connected compact Lie group G has an abelian Lie subgroup A, such that $\gamma_n(G) \subseteq \{[a, x] : a \in A, x \in G\}$, for some $n > 1$. Then G is solvable.*

Finally, we can apply Theorem 1 to derivations of Lie algebras to obtain a sufficient condition for solvability.

COROLLARY 5. *Suppose that there exist an abelian subalgebra* $D \leq \mathrm{Der}(L)$ *and an integer* $n > 1$ *such that*

$$L^n \subseteq \sum_{d \in D} d(L).$$

Then L is solvable.

PROOF. Let $\hat{L} = D \ltimes L$, the natural semidirect product. Then L is an ideal and D is an abelian subalgebra in $\hat{L}$. Note that in the semidirect product,

$$[D, L] = \sum_{d \in D} d(L),$$

hence the assumption is just $L^n \subseteq [D, L]$. So, L is solvable by Theorem 1. □

As a special case, if the Lie algebra L admits a derivation $d : L \to L$ such that $L^n \subseteq d(L)$, for some $n > 1$, then L is solvable. Note that our results are not true for Lie algebras of positive characteristics, since there are simple Lie algebras over fields of nonzero characteristics, that admit invertible derivations; see, for example, [1].

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