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A new construction of the Margulis measure for Anosov flows†

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Abstract. The Margulis measure for Anosov flows arises from a Hausdorff measure for a natural distance on unstable leaves. This generalizes work of Ursula Hamenstädt for the case of geodesic flows.

Introduction. Let M be a compact Riemannian manifold and $\varphi^t: M \to M$ a C^2 Anosov flow. On $W^u(z)$ we define a distance and a spherical measure σ which expand uniformly under the flow. σ is equivalent to the conditional *Margulis measure* [**H, M, S**] and for a Lyapunov metric σ equals the conditional Margulis measure on every leaf $W^u(z)$.

DEFINITION 1. [**A**]. A flow $\varphi^t: M \to M$ is called an *Anosov flow* if $\dot{\varphi} \neq 0$ and the tangent bundle is a Whitney sum $TM = E^u \oplus E^s \oplus E^o$, where E^o is generated by $\dot{\varphi}$ and there are $a > 0$, $b \geq 1$ so that

$$\|D\varphi^t u\| \leq b \cdot \|u\| \cdot e^{at} \quad \text{for } t \leq 0,\, u \in E^u$$

and (∗)

$$\|D\varphi^t v\| \leq b \cdot \|v\| \cdot e^{-at} \quad \text{for } t \geq 0,\, v \in E^s.$$

Remark. E^u, E^s, $E^{ou} := E^u \oplus E^o$ and $E^{os} := E^s \oplus E^o$ are tangent to foliations W^u, W^s, W^{ou} and W^{os} respectively, which are continuous in the C^1-topology. Every unstable leaf $W^u(z)$ has a distance d_z (and thus notions of openness and compactness) induced by Riemannian lengths of curves in $W^u(z)$.

DEFINITION 2. [**M**]. $S \subset W^u(z)$ and $S' \subset W^u(z')$ are called *ε-equivalent* if there is a continuous $\phi: S \times [0, 1] \to M$ so that $\phi(\cdot, 0) = \text{id}$, $\psi := \phi(\cdot, 1): S \to S'$ is a homeomorphism, $\phi(x, [0, 1])$ is contained in $W^{os}(x)$ and is a curve of length less than ε for all $x \in S$.

Remark. [**A**]. After possibly changing a there exists a Riemannian metric on M, equivalent to the given metric and called a *Lyapunov adapted metric*, such that (∗) holds with $b = 1$.

DEFINITION 3. Fix $R \in \mathbb{R}$. For x, $y \in W^u(z)$ let $\eta(x, y) := \eta_{z,R}(x, y) := \exp(-\sup\{t \in \mathbb{R}: d_{\varphi t_z}(\varphi^t(x), \varphi^t(y)) \leq R\})$.

Remark. $\eta \circ \varphi^t = e^t \cdot \eta$, $\eta_{z',R} = \eta_{z,R}$ for $z' \in W^u(z)$, $\eta \geq 0$, $\eta(x, y) = \eta(y, x)$ and $\eta(x, y) = 0$ iff $x = y$.

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LEMMA 1. *For* $x_1, x_2, y \in W^u(z)$ *and* a, b *as in* $(*)$ *we have* $(\eta(x_1, x_2))^a/b \leq (\eta(x_1, y))^a + (\eta(x_2, y))^a$.

Remark. In particular η^a is a distance if M is equipped with a Lyapunov metric.

Proof. Let $t = -\log \eta(x_1, x_2)$, $r_i = d_{\varphi^t z}(\varphi^t x_i, \varphi^t y)$. If any $r_i > Rb$, $\eta^a(x_i, y) > \eta^a(x_1, x_2)$, so we are done.

If $r_i < Rb$ then $\eta^a(x_i, y) \geq e^{-at} r_i / Rb$ since

$$d_{\varphi^{t+\tau} z}(\varphi^{t+\tau} x_i, \varphi^{t+\tau} y) > R \quad \text{for } \tau > \frac{1}{a} \cdot \log \frac{Rb}{r_i} > 0.$$

(Join $\varphi^{t+\tau} x_i$ and $\varphi^{t+\tau} y$ by a curve $\gamma \subset W^u(\varphi^{t+\tau} z)$ so that $l(\gamma) = d_{\varphi^{t+\tau} z}(\varphi^{t+\tau} x_i, \varphi^{t+\tau} y)$. By $(*)$

$$r_i \leq l(\varphi^{-\tau} \circ \gamma) \leq b \cdot e^{-a\tau} \cdot l(\gamma) = b \cdot e^{-a\tau} \cdot d_{\varphi^{t+\tau} z}(\varphi^{t+\tau} x_i, \varphi^{t+\tau} y).)$$

Since $r_1 + r_2 \geq d_{\varphi^t z}(\varphi^t x_1, \varphi^t x_2) = R$ we obtain $\eta^a(x_1, x_2)/b \leq e^{-at}(r_1 + r_2)/Rb \leq \eta^a(x_1, y) + \eta^a(x_2, y)$. □

LEMMA 2. *Omitting z in the subscript, we have* $\eta_R \leq \eta_r \leq (Rb/r)^{1/a} \cdot \eta_R$ *for* $0 < r \leq R$.

Proof. Clearly $\eta_R \leq \eta_r$. $d_{\varphi^t z}(\varphi^t x, \varphi^t y) = r$ for $t = -\log \eta_r(x, y)$, so $\eta_R(x, y) \geq (r/Rb)^{1/a} \eta_r(x, y)$ as in the proof of Lemma 1. □

DEFINITION 4. [H]. For $S \subset W^u(z)$ let

$$\sigma_\varepsilon(S) := \inf \left\{ \sum_{j=1}^{\infty} \varepsilon_j^h : S \subset \bigcup_{j=1}^{\infty} B_{\eta_z}(x_j, \varepsilon_j) \quad \text{with } x_j \in W^u(z) \quad \text{and } \varepsilon_j \leq \varepsilon \right\}$$

and

$$\sigma(S) := \sigma_z(S) := \sup_{\varepsilon > 0} \sigma_\varepsilon(S).$$

Here h is topological entropy and $B_\eta(x_j, \varepsilon_j)$ are ε_j-balls for η around x_j.

Remark. σ is the h/a-dimensional *spherical measure* [F] on $W^u(z)$ arising from the distance η^a. $\sigma_{\varphi^t z} \circ \varphi^t = e^{ht} \cdot \sigma_z$ and σ is Borel regular, i.e. $\sigma(S) = \sup\{\sigma(C): C \subset S$ compact$\}$ (see [F]).

LEMMA 3. [M, S]. *For a* C^2 *Anosov flow* φ^t *with dense leaves* W^u *and* W^s *we can construct the* Margulis measure μ. *Its restriction* μ_z^u *to* $W^u(z)$ *is positive on open and finite on compact sets.* $\mu_{\varphi^t z}^u \circ \varphi^t = e^{ht} \cdot \mu_z^u$ *and for* $\delta > 0$ *there exists* $\varepsilon > 0$ *such that if* $S \subset W^u(z)$ *and* $S' \subset W^u(z')$ *are* ε*-equivalent then* $(1-\delta) \cdot \mu^u(S) < \mu^u(S') < (1+\delta) \cdot \mu^u(S)$. *Furthermore* μ^u *is Borel regular.*

LEMMA 4. *There exist* $0 < \alpha_1 < \alpha_2 < \infty$ *such that* $\alpha_1 \cdot \varepsilon^h < \mu^u(B_\eta(x, \varepsilon)) < \alpha_2 \cdot \varepsilon^h$ *for all* $x \in M$.

Proof. Suppose $\mu^u(B_\eta(x_i, 1)) \to 0$ for some $\{x_i\}_{i=1}^{\infty} \subset M$. By compactness of M assume $x_i \to x$. For i large $S = \bar{B}_\eta(x, \frac{1}{2})$ is ε-equivalent to some $S' \subset B_\eta(x_i, 1)$ and $\mu^u(B_\eta(x_i, 1)) \geq \mu^u(S') \geq \frac{1}{2} \cdot \mu^u(S) > 0$ by Lemma 3, a contradiction. So $0 < \alpha_1 < \mu^u(B_\eta(x, 1))$ and similarly $\mu^u(B_\eta(x, 1)) < \alpha_2 < \infty$. The claim follows, since

$$\mu^u(B_\eta(x, \varepsilon)) = \mu^u(\varphi^{\log \varepsilon}(B_\eta(\varphi^{-\log \varepsilon} x, 1))) = \varepsilon^h \cdot \mu^u(B_\eta(\varphi^{-\log \varepsilon} x, 1)).$$ □

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LEMMA 5.

$$\alpha_2^{-1} \cdot \mu^u \le \sigma \le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \mu^u.$$

Proof. (1) Let $S \subset W^u(z)$. By definition of σ_ε there is a covering

$$S \subset \bigcup_{j=1}^{\infty} B_\eta(x_j, \varepsilon_j), \quad \varepsilon_j \le \varepsilon, x_j \in W^u(z),$$

so that

$$\sigma_\varepsilon(S) + \delta \ge \sum_{j=1}^{\infty} \varepsilon_j^h \ge \alpha_2^{-1} \cdot \sum_{j=1}^{\infty} \mu^u(B_\eta(x_j, \varepsilon_j)) \ge \alpha_2^{-1} \cdot \mu^u(S).$$

Let $\delta \to 0$ and $\varepsilon \to 0$.

(2) Let $S \subset W^u(z)$ be compact, $\varepsilon > 0$, $S_\varepsilon = \{x \in W^u(z): \exists y \in S: \eta(x, y) < \varepsilon/(2b)^{1/a}\}$ and $\{x_j\}_{j=1}^m \subset S$ a maximal subset so that the $B_\eta(x_j, \varepsilon/(2b)^{1/a})$ are pairwise disjoint. Since $S \subset \bigcup_{j=1}^m B_\eta(x_j, \varepsilon)$ by Lemma 1,

$$\sigma_\varepsilon(S) \le \sum_{j=1}^{m} \varepsilon^h \le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \sum_{j=1}^{m} \mu^u(B_\eta(x_j, \varepsilon/(2b)^{1/a})) \le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \mu_u(S_\varepsilon)$$

by Lemma 4 and disjointness of $B_\eta(x_j, \varepsilon/(2b)^{1/a}) \subset S_\varepsilon$. Letting $\varepsilon \to 0$ gives $\sigma(S) \le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \mu^u(S)$ and Borel-regularity of σ and μ^u yields the second inequality. □

LEMMA 6. *If σ is constructed from a Lyapunov metric then for $\delta > 0$ there is an $\varepsilon > 0$ so that if $S \subset W^u(z)$ and $S' \subset W^u(z')$ are ε-equivalent then $(1-\delta) \cdot \sigma(S) < \sigma(S') < (1+\delta) \cdot \sigma(S)$.*

Proof. We will use that if $\{x\}$, $\{x'\}$ are ε-equivalent and $\{x''\} := W^{ou}(x') \cap W^s(x)$ then $x'' = \varphi^\tau x'$ for some $\tau \in \mathbb{R}$, so there is a $C \in \mathbb{R}$ such that $\varphi^t x$, $\varphi^t x'$ are $C \cdot \varepsilon$-equivalent for $t > 0$. For $\delta_j < \delta < 1$ cover $S \subset \bigcup_{j=1}^\infty B_{\eta_z}(x_j, \delta_j)$ so that $\sum_{j=1}^\infty \delta_j^h \le \sigma(S) + \delta$.

Claim. $S' \subset \bigcup_{j=1}^\infty B_{\eta_{\psi(z)}}(\psi(x_j), \iota(\varepsilon) \cdot \delta_j)$ with $\lim_{\varepsilon \to 0} \iota(\varepsilon) = 1$.

Proof of Claim. If $y \in S \cap B_{\eta_z}(x, \delta)$ then $d_{\varphi^{-\log\delta} z}(\varphi^{-\log\delta} x, \varphi^{-\log\delta} y) < R$.

$$d_{\psi(\varphi^{-\log d} z)}(\psi(\varphi^{-\log\delta} x), \psi(\varphi^{-\log\delta} y)) < R + \theta(C\varepsilon)$$

with $\lim_{\varepsilon \to 0} \theta(\varepsilon) = 0$ by uniform continuity of E^u. By Lemma 2

$$\eta_{\psi(z)}(\psi(x), \psi(y)) < \iota(\varepsilon) \cdot \delta := \left(\frac{R + \theta(C\varepsilon)}{R}\right)^{1/a} \cdot \delta.$$

Therefore $\psi(S \cap B_{\eta z}(x_j, \delta_j)) \subset S' \cap B_{\eta_{\psi(z)}}(\psi(x_j), \iota(\varepsilon)\delta_j)$ and the claim follows. □

Thus $\sigma(S') \le \iota(\varepsilon)^h \cdot \sigma(S)$. The other inequality follows similarly.

THEOREM. *Let M be a compact C^2-manifold and φ^t as in* Lemma 3. *Equip M with a Lyapunov metric. Then after normalization the measures σ of* Definition 4 *agree with the conditionals of the Margulis measure on every leaf.*

Proof. The μ^u are defined up to a global constant (not just up to a constant on each leaf [M]). σ has measurable densities f_z: $W^u(z) \to \mathbb{R}$ with respect to μ^u. Since μ is

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ergodic and $f: M \to \mathbb{R}$, $z \mapsto f_z(z)$ φ^t-invariant ($\mu^u \circ \varphi^t = e^{ht} \cdot \mu^u$ and $\sigma \circ \varphi^t = e^{ht} \cdot \sigma$!) and measurable by Lemmata 3 and 6, $f \equiv$ constant μ—a.e. By Lemmata 3 and 6, after normalizing, $f_z \equiv 1$ μ^u—a.e. on each leaf $W^u(z)$. □

Remarks. (1) If M carries an arbitrary Riemannian metric then after normalization the measure σ agrees with the conditionals of the Margulis measure on μ-almost every leaf. (The above proof goes through: f is measurable since $(x, y, z, R) \mapsto \eta_{z,R}(x, y)$ is lower semicontinuous.)

(2) The above results are also true for the h-dimensional Hausdorff measure [**F**].

(3) It is interesting to compare this construction with the one in [**B**].

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