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SECOND-ORDER TIME DISCRETIZATION WITH FINITE-ELEMENT METHOD FOR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS WITH A WEAKLY SINGULAR KERNEL

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Abstract

We propose the second-order time discretization scheme with the finite-element approximation for the partial integro-differential equations with a weakly singular kernel. The space discretization is based on the finite element method and the time discretization is based on the Crank-Nicolson scheme with a graded mesh. We show the stability of the scheme and obtain the second-order convergence result for the fully discretized scheme.

1. Introduction

We consider the time discretization method for the following partial integro-differential equation with a weakly singular kernel:

$$\begin{aligned} u_t - \mathscr{A}u(t) &= \int_0^t K(t-s)\mathscr{B}u(s)\,ds + f(x,t), \quad x \in \Omega, \quad \text{for} \quad t > 0, \\ u &= 0, \quad \text{on} \quad \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), \quad \text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

where $\mathscr{A}$ is a linear positive self-adjoint elliptic operator, $\mathscr{B}$ is a general partial differential operator of second order with smooth and time-independent coefficients and K is a weakly singular kernel satisfying

$$|K^i(t)| \le C_K t^{-i-\alpha} \quad \text{with} \quad 0 \le \alpha < 1, \quad \text{for} \quad t > 0, \quad i = 0, 1.$$

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Furthermore, throughout this paper, Ω is a sufficiently smooth domain in $\mathbb{R}^d$, $d \geq 1$ and we assume that f is sufficiently smooth. Problems of this nature arise in several areas, such as the theory of linear viscoelasticity and heat conduction in material with memory; see, for example, [8].

The numerical method considered in this paper is obtained by discretizing in space by the finite-element method, followed by a finite difference and quadrature scheme for the time discretization. For the numerical solutions we assume that we are given a family $\{S_h\}$ of finite-dimensional subspaces of $H_0^1 = H_0^1(\Omega)$ such that

$$\inf_{\chi \in S_h} \{\|v - \chi\| + h\|v - \chi\|_1\} \leq Ch^2\|v\|_2, \qquad \forall v \in H^2 \cap H_0^1, \tag{1.2}$$

where $\|\cdot\|$ is the norm in $L_2 = L_2(\Omega)$ and $\|\cdot\|_s$ is that in $H^s = H^s(\Omega)$.

As a starting point for the discretization of (1.1), we define the semi-discrete solution of (1.1) as the function $u_h : (0, T] \to S_h$ such that

$$\begin{aligned} (u_{h,t}, \chi) + A(u_h, \chi) &= \int_0^t K(t-s)B(u_h(s), \chi)\,ds + (f(t), \chi), \ \forall \chi \in S_h, \ t > 0, \\ u_h(0) &= u_{0h}, \end{aligned} \tag{1.3}$$

where $(\cdot, \cdot)$ is the inner product in L_2, $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ are the bilinear forms on H_0^1 associated with the differential operators $\mathscr{A}$ and $\mathscr{B}$ and where u_{0h} is an appropriate approximation in S_h of initial data in (1.1). In [2], we can find that for each $T > 0$, the error estimate of (1.3) is

$$\|u_h(t) - u(t)\| \leq C_T h^2 \left\{ \|u_0\|_2 + \int_0^t \|u_t\|\,ds \right\} \qquad \text{for} \quad t \leq T.$$

The time-discretization of (1.1) is very interesting because of the nature of "memory effect". The time discretization methods are derived essentially by replacing the derivatives in (1.3) by a difference quotient and using a quadrature rule for the integral terms. The difficulties involved in such time discretization are that all the values of $u(t)$ have to be retained, causing great demands for data storage. To overcome this difficulty, higher-order quadrature formulae or quadrature based on the use of sparser sets of time levels were proposed in literature such as [6, 7] and [9] for partial integro-differential equations with smooth kernels. In the case of weakly singular kernels, the regularity of the solution with respect to time is limited, which makes higher-order quadrature formulae useless, as well as quadratures based on the use of a sparser set of time levels. In fact, for sufficiently smooth data u_0 and f, there exists a unique solution of (1.1) satisfying the following regularities (see [2]):

$$u \in C([0, T]; H^2 \cap H_0^1), \quad u_t \in C([0, T]; L_2) \cap L_1(0, T; H^2 \cap H_0^1),$$
$$u_{tt} \in L_1(0, T; L_2).$$

Formally the solution of (1.1) satisfies

$$u_{tt} = \mathscr{A} u_t + K(t)\mathscr{B}u(0) + \int_0^t K(t-s)\mathscr{B}u_s\,ds + f_t$$

or

$$|u_{tt}| \le C_K t^{-\alpha} |\mathscr{B}u(0)| + \text{ more regular terms with respect to } t.$$

In advance, we assume that the solution of (1.1) satisfies

$$\|u_{ttt}\| \le R_0 t^{-\alpha} \text{ and } \|u_{tt}\|_2 \le R_0 t^{-\alpha} \text{ for some } R_0 > 0. \tag{1.4}$$

Furthermore, it is an easy consequence of (1.4) that

$$\|u_{tt}\| \le C t^{-1-\alpha} \quad \text{for some} \quad C > 0. \tag{1.5}$$

In this paper, we consider the graded mesh for the discretization of (1.3) (see also [1] and [3]). Given $M \in \mathbb{N}$, let $\Pi_M := \{t_0, \dots, t_M\}$, $(0 = t_0 < t_1 < \cdots < t_M = T)$, denote a partition of the interval $[0, T]$. With a given partition Π_M of $[0, T]$ we associate the quantities

$$\bar{k} := \max_{n \le M} k_n,$$

where $k_n := t_n - t_{n-1}$ $(n = 1, \dots, M)$. If the mesh points $\{t_n\}_{n=0}^{M}$ are given by

$$t_n := \left(\frac{n}{M}\right)^r T \qquad (n = 0, \dots, M), \tag{1.6}$$

then Π_M is called a *graded mesh*; in the present context, the so-called *grading exponent* $r \in \mathbb{R}$ will always satisfy $r \ge 1$. Let $U^n \in S_h$ be the approximation of the exact solution of (1.3) at time t_n. The time discretization considered here is based on the backward-difference quotient $\bar{\partial}_t U_n = (U^n - U^{n-1})/k_n$. The integral term then has to be evaluated by numerical quadrature from the values of the U^js, but since the integrand is singular, we use product integration. We approximate ϕ in $J_n(\phi) = \int_0^{t_n} K(t_n - s)\phi(s)\,ds$ by piecewise functions

$$\tilde{\phi}(s) = \begin{cases} \phi^0 & s \in (0, t_1], \\ \phi^{j+1/2} & s \in (t_j, t_{j+1}], \quad 1 \le j \le n-2, \\ \frac{t_{n-1/2}-s}{k_{n-1/2}}\phi^{n-3/2} + \frac{s-t_{n-3/2}}{k_{n-1/2}}\phi^{n-1/2} & s \in (t_{n-1}, t_n], \quad n \ge 2, \end{cases} \tag{1.7}$$

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where we denote that $t_{j-1/2} := (t_j + t_{j-1})/2$, $\phi^{j-1/2} := (\phi^j + \phi^{j-1})/2$ and $k_{j-1/2} := (k_j + k_{j-1})/2$. Thus we write the quadrature for $J_n(\phi)$ as

$$\begin{aligned} q^n(\phi) &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} K(t_n - s)\tilde{\phi}(s)\, ds \\ &= \tau_{n0}\phi^0 + \sum_{j=1}^{n-2} \tau_{nj}\phi^{j+1/2} + \tau_{nn-1}\phi^{n-3/2} + \tau_{nn}\phi^{n-1/2} \quad \text{for} \quad n \geq 2 \end{aligned}$$

and

$$q^1(\phi) = \tau_{10}\phi^0 = \int_0^{t_1} K(t_1 - s)\, ds\phi^0,$$

where

$$\tau_{nj} = \begin{cases} \displaystyle\int_{t_j}^{t_{j+1}} K(t_n - s)\, ds & \text{if} \quad j \leq n-2, \\ \displaystyle\int_{t_{n-1}}^{t_n} K(t_n - s)\frac{t_{n-1/2} - s}{k_{n-1/2}}\, ds & \text{if} \quad j = n-1, \\ \displaystyle\int_{t_{n-1}}^{t_n} K(t_n - s)\frac{s - t_{n-3/2}}{k_{n-1/2}}\, ds & \text{if} \quad j = n. \end{cases}$$

Our fully discretized scheme based on the Crank-Nicolson scheme is now defined by

$$\begin{aligned} (\bar{\partial}_t U^n, \chi) + A(U^{n-1/2}, \chi) &= q^{n-1/2}(B(U, \chi)) + (f^{n-1/2}, \chi), \qquad n \geq 2, \\ (\bar{\partial}_t U^1, \chi) + A(U^1, \chi) &= q^1(B(U, \chi)) + (f^1, \chi), \end{aligned} \tag{1.8}$$

where

$$q^{n-1/2}(B(U, \chi)) = \frac{1}{2}\left\{q^n(B(U, \chi)) + q^{n-1}(B(U, \chi))\right\}.$$

The purpose of this paper is to show stability and to obtain error estimates for the scheme (1.8).

2. Stability and convergence

We show stability and obtain the error estimates for the fully discretized scheme (1.8). The following three lemmas from Kim and Choi [5] are required for our analysis.

LEMMA 1. *If the grading exponent $r > 1$, then we have the following estimates. There is a positive constant C dependent on T and r such that*

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(i) $|\tau_{nj}| \le C(n-j)^{-\alpha} k_{j+1}^{1-\alpha}$,
(ii) $\sum_{n=2}^{M} \sum_{j=1}^{n} |\tau_{nj} - \tau_{n-1j-1}| \le C$.

LEMMA 2. *For each $\epsilon > 0$, there is a constant C_ϵ such that*

$$\left| \sum_{l=1}^{n} k_l \sum_{j=0}^{l-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} f_j f_l \right| \le \epsilon \sum_{l=1}^{n} k_l f_l^2 + C_\epsilon \sum_{l=0}^{n-1} (n-l)^{-\alpha} k_{l+1}^{1-\alpha} \sum_{j=0}^{l} k_{j+1} f_j^2.$$

Also we need the following discrete version of the Gronwall lemma.

LEMMA 3. *Let $\{w_n\}$ be a sequence of nonnegative real numbers satisfying*

$$w_n \le \beta_n + \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} w_j,$$

where $\beta_n \ge 0$. Then there is a positive constant C such that

$$w_n \le \beta_n + C \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \beta_j, \quad n \ge 0.$$

We recall that the bilinear form $A(\cdot,\cdot)$ is coercive and bounded if there are positive constants c_0 and c_1 satisfying

$$c_0 \|u\|_1^2 \le A(u,u) \le c_1 \|u\|_1^2 \qquad \forall u \in H^2 \cap H_0^1. \tag{2.1}$$

The following theorem is our conditional stability result for the fully discretized scheme (1.8).

THEOREM 1. *Suppose that $\bar{k} := \max_{n \le M} k_n$ is so small that*

$$\lambda := \int_0^{\bar{k}} |K(s)|\, ds < \frac{c_0}{2\|B\|} \quad \textit{for all} \quad n \ge 2.$$

Then scheme (1.8) *is stable, that is, there is a positive constant $C_T := C(T, r, \gamma)$ such that*

$$\|U^n\| \le C_T \left(\|U^0\| + \sum_{j=1}^{n} k_j \|f^j\| \right) \quad \textit{for} \quad n \ge 1.$$

PROOF. Letting $\tilde{U}^n = U^{n-1/2}$ for $n \geq 2$, $\tilde{U}^0 = U^0$ and $\tilde{U}^1 = U^1$, (i) of Lemma 1 implies that

$$|q^1(B(U,\phi))| \leq Ck_1^{1-\alpha}\|U^0\|_1\|\phi\|_1, \tag{2.2}$$

$$|q^n(B(U,\phi))| \leq C\left(\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}^{1-\alpha}\|\tilde{U}_j\|_1\|\phi\|_1\right) + \lambda\|B\|\|\tilde{U}^n\|_1\|\phi\|_1, \quad n \geq 2.$$

Taking $\phi = \tilde{U}^n$ for $n \geq 1$ in (1.8), we have

$$\begin{aligned} &\frac{1}{2}\left(\|U^n\|^2 - \|U^{n-1}\|^2\right) + k_n(c_0 - \lambda\|B\|)\|\tilde{U}^n\|_1^2 \\ &\quad \leq Ck_n\|\tilde{U}^n\|_1\left\{\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}^{1-\alpha}\|\tilde{U}^j\|_1\right\} + k_n\|f^{n-1/2}\|\|\tilde{U}^n\| \quad \text{for} \quad n \geq 2. \end{aligned} \tag{2.3}$$

Summing (2.3) from $n = 1$ to N and applying Lemma 2 with a suitable ϵ, we obtain

$$\begin{aligned} \|U^N\|^2 + \sum_{n=1}^{N}k_n\|\tilde{U}^n\|_1^2 \leq \|U^0\|^2 &+ C\left(k_1\|f^1\|\|U^1\| + \sum_{n=2}^{N}k_n\|f^{n-1/2}\|\|\tilde{U}^n\|\right) \\ &+ C\sum_{n=1}^{N-1}(N-n)^{-\alpha}k_{n+1}^{1-\alpha}\sum_{j=0}^{n-1}k_{j+1}\|\tilde{U}^j\|_1^2. \end{aligned}$$

It follows from Lemma 3 that

$$\|U^N\|^2 + \sum_{n=1}^{N}k_n\|\tilde{U}^n\|_1^2 \leq C_T\left\{\|U^0\| + k_1\|f^1\| + \sum_{n=2}^{N}k_n\|f^{n-1/2}\|\right\}\max_{n\leq N}\|U^n\|.$$

Hence we have

$$\|U^N\| \leq \max_{n\leq N}\|U^n\| \leq C_T\left(\|U^0\| + k_1\|f^1\| + \sum_{n=2}^{N}k_n\|f^{n-1/2}\|\right) \quad \text{for } 1 \leq N \leq M.$$

Next we derive the error estimate for the fully discretized scheme (1.8). For the analysis, we introduce the "discrete Ritz-Volterra projection" V_h defined for an appropriately smooth function u by

$$A((V_hu - u)(t_n), \chi) = q^n(B(V_hu - u, \chi)), \qquad \forall\chi \in S_h \quad \text{for} \quad n \geq 0. \tag{2.4}$$

We have the following two lemmas which state the error estimate for the discrete Ritz-Volterra projection (2.4).

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LEMMA 4. *Assume that* $\bar{k} := \max_{n\le M} k_n$ *is so small that*

$$\lambda := \int_0^{\bar{k}} |K(s)|\,ds < \frac{c_0}{\|B\|} \quad \textit{for all} \quad n \ge 2$$

and that $u(t) \in H^2 \cap H_0^1$ *and* $u(t) \in C^2(\Omega)$. *Then we have for* V_h *that*

$$\begin{aligned}\|(V_h u^n - u^n)\| &\le C_T h^2 \sup_{j\le n} \|u^j\|_2 \\ &\le C_T h^2 \left(\|u_0\|_2 + \int_0^{t_n} \|u_t(s)\|_2\,ds \right).\end{aligned}$$

PROOF. Let $\rho = V_h u - u$ and let R_h be the Ritz projection defined by

$$A(R_h u - u, \chi) = 0, \qquad \forall \chi \in S_h.$$

It is a well-known estimate for R_h that

$$\|(R_h u - u)(t)\| + h\|(R_h u - u)(t)\|_1 \le C h^2 \|u(t)\|_2.$$

Letting $\theta^n = V_h u^n - R_h u^n$ under the definition of V_h, we have that with $c_0 > 0$,

$$\begin{aligned} c_0\|\theta^n\|_1^2 &\le A(\theta^n, \theta^n) = A(\rho^n, \theta^n) = q^n(B(\rho, \theta^n)) \\ &\le C_T \|\theta^n\|_1 \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1} \|\rho^j\|_1 + \lambda \|B\| \|\rho^n\|_1 \|\theta^n\|_1, \end{aligned}$$

or

$$(c_0 - \lambda\|B\|)\|\rho^n\|_1 \le C\|R_h u^n - u^n\|_1 + C_T \left(\sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\rho^j\|_1 \right).$$

Thus Lemma 3 implies that

$$\|\rho^n\|_1 \le C_T \sup_{j\le n} \|R_h u^j - u^j\|_1 \le C_T h \sup_{j \le n} \|u^j\|_2.$$

In order to obtain the L_2-estimate for ρ^n, we use a duality argument defined by

$$\|\rho^n\| = \sup_{\|\phi\|=1} (\rho^n, \phi).$$

For each such ϕ, we let Ψ be the solution of

$$\mathscr{A}\Psi = \emptyset \quad \text{in} \quad \Omega, \qquad \Psi = 0 \quad \text{on} \quad \partial\Omega.$$

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Then, for $\chi \in S_h$, we have from (2.2) that

$$\begin{aligned}(\rho^n, \phi) &= A(\rho^n, \Psi) = A(\rho^n, \Psi - \chi) + A(\rho^n, \chi)\\ &= A(\rho^n, \Psi - \chi) + q^n(B(\rho, \chi - \Psi)) + q^n((\rho, \mathscr{B}^*\Psi))\\ &\le C\|\rho^n\|_1\|\Psi - \chi\|_1 + C\sup_{j\le n}\|\rho^j\|_1\|\chi - \Psi\|_1\\ &\quad + C_T\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}^{1-\alpha}\|\rho^j\| + \lambda\|\mathscr{B}^*\Psi\|\|\rho^n\|.\end{aligned}$$

Take $\chi = R_h\Psi$ and note that $c_0\|\mathscr{A}^{-1}\| \le 1$ is an easy consequence of (2.1) and thus $1 - \lambda\|\mathscr{B}^*\Psi\| \ge \frac{1}{2}$. Then we obtain that

$$\|\rho^n\| \le Ch\sup_{j\le n}\|\rho^j\|_1 + C_T\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}^{1-\alpha}\|\rho^j\|.$$

Thus Lemma 3 implies $\|\rho^n\| \le Ch\sup_{j\le n}\|\rho^j\|_1$, which completes the proof.

LEMMA 5. *Under the assumptions of Lemma 4, we have for $\rho = V_hu - u$,*

$$\sum_{j=1}^{n} k_j\|\bar{\partial}_t\rho^j\| \le Ch^2\left\{\|u_0\|_2 + \int_0^{t_n}\|u_t(s)\|_2\,ds\right\}.$$

PROOF. For the sake of convenience, we denote $\delta^n = k_n\bar{\partial}_t\rho^n$, $\delta^0 = \rho^0$, $\omega_{nj} = \tau_{nj} - \tau_{n-1j-1}$, $W_n = \sum_{j=1}^{n}|\omega_{nj}|$ for $n \ge 2$, and $W_1 = W_0 = 0$. Then we obtain directly from (2.4) that for all $\chi \in S_h$,

$$\begin{aligned}A(\delta^n, \chi) &= k_n\bar{\partial}_t q^n(B(\rho, \chi))\\ &= q^n(B(\delta, \chi)) + \sum_{j=1}^{n-2}\omega_{nj}B(\rho^{j-1/2}, \chi)\\ &\quad + \omega_{nn-1}B(\rho^{n-5/2}, \chi) + \omega_{nn}B(\rho^{n-3/2}, \chi) \qquad (2.5)\\ &\le C_T\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+j}^{1-\alpha}|B(\delta^j, \chi)| + \lambda|B(\delta^n, \chi)| + C\max_{j\le n}\|\rho^j\|_1\|\chi\|_1 W_n.\end{aligned}$$

Taking $\theta^n = \bar{\partial}_t(V_hu^n - R_hu^n)$, we have with $c_0 > 0$,

$$\begin{aligned}c_0k_n\|\theta^n\|_1^2 &= A(\delta^n, \theta^n)\\ &\le C_T\|\theta^n\|_1\left\{\sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}^{1-\alpha}\|\delta^j\|_1 + \max_{j\le n}\|\rho^j\|_1 W_n\right\} + \lambda\|B\|\|\delta^n\|_1.\end{aligned}$$

Hence we get

$$\|\delta^n\|_1 \le C_T \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\delta^j\|_1 + C \max_{j\le n} \|\rho^j\|_1 W_n + C k_n \|\bar{\partial}_t (R_h u^n - u^n)\|_1.$$

It follows from Lemma 3 that

$$k_n \|\bar{\partial}_t \rho^n\|_1 \le C_T \max_{j\le n} \|\rho^j\|_1 \left(\sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} W_j + W_n \right) + C_T h \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \int_{t_j}^{t_{j+1}} \|u_s(s)\|_2 \, ds. \tag{2.6}$$

We can easily verify that for $\beta_j \ge 0$,

$$\sum_{n=1}^{N} \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \beta_j = \sum_{j=0}^{N-1} \beta_j \sum_{n=j+1}^{N} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \le C \sum_{j=0}^{N-1} \beta_j. \tag{2.7}$$

Summing (2.6) from $n = 1$ to N and applying the inequality (2.7) and (ii) of Lemma 1, we have

$$\sum_{n=1}^{N} k_n \|\bar{\partial}_t \rho^n\|_1 \le C h \left(\max_{n\le N} \|u^n\|_2 + \int_0^{t_N} \|u_s(s)\|_2 \, ds \right).$$

With the same argument of Lemma 4, we can write with $\chi = R_h \Psi$,

$$\begin{aligned} k_n(\bar{\partial}_t \rho^n, \phi) &= k_n A(\bar{\partial}_t \rho^n, \Psi) = k_n A(\bar{\partial}_t \rho^n, \Psi - \chi) + k_n A(\bar{\partial}_t \rho^n, \chi) \\ &= k_n A(\bar{\partial}_t \rho^n, \Psi - \chi) + k_n \bar{\partial}_t q^n (B(\rho, \chi - \Psi)) + k_n \bar{\partial}_t q^n ((\rho, \mathscr{B}^* \Psi)) \\ &\le G_n + C_T \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\delta^j\| + \lambda \|\mathscr{B}^* \Psi\| \|\rho^n\|, \end{aligned}$$

where

$$G_n := C h \left(k_n \|\bar{\partial}_t \rho^n\|_1 + \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\delta^j\|_1 \right) + C_T \left(h \max_{j\le n} \|\rho^j\|_1 + \max_{j\le n} \|\rho^j\| \right) W_n.$$

Thus we have

$$k_n \|\bar{\partial}_t \rho^n\| \le C h \left(k_n \|\bar{\partial}_t \rho^n\|_1 + \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\delta^j\|_1 \right) + C h^2 \sup_{j\le n} \|u^j\|_2 W_n + C_T \sum_{j=0}^{n-1} (n-j)^{-\alpha} k_{j+1}^{1-\alpha} \|\delta^j\|.$$

Applying Lemma 4 again, summing from $n = 1$ to N and using the inequality (2.7) and Lemma 1, we obtain that

$$\sum_{n=1}^{N} \|\bar{\partial}_t \rho^n\| \le Ch \sum_{j=1}^{N} k_n \|\bar{\partial}_t \rho^n\|_1 + \max_{n\le N} \|u^n\|_2 \sum_{n=1}^{N} W_n$$
$$\le Ch^2 \left(\|u^0\|_2 + \int_0^{t_N} \|u_s(s)\|_2 \, ds \right),$$

which completes the proof.

We also need the following error estimate for our quadrature scheme (1.7).

LEMMA 6. *Suppose that $f \in C^2(0, T]$, $f' \in C[0, T]$ and $|f''(t)| \le Ct^{-\alpha}$. If a grading exponent $r \ge 2/(2-\alpha)$, then there is a constant C_T depending on f and T such that*

$$\left| \sum_{n=1}^{M} k_n \left(\int_0^{t_k} K(t_k - s) f(s)\, ds - q^n(f) \right) \right| \le C_T(f) \bar{k}^2.$$

PROOF. Refer to Kim and Choi [5].

Finally, we obtain the second-order convergence result for the fully discretized scheme (1.8).

THEOREM 2. *Let u and $\{U^n\}$ be the solution of (1.1) and (1.8) respectively. We assume that for sufficiently smooth data u_0 and f, u satisfies $u \in C([0, T]; H^2 \cap H_0^1) \cap C^3((0, T]; L^2(\Omega))$, $u_t \in L_1(0, T; H^2 \cap H_0^1)$ and $u_{tt} \in L_1(0, T; H^2) \cap C^1((0, T]; H^2)$. Furthermore, we assume that $\|u_{tt}\|_2 \le R_0 t^{-\alpha}$ for some $R_0 > 0$. If a grading exponent r is greater than $2/(2-\alpha)$, then there exists a constant C_T independent of h and k such that*

$$\|u^n - U^n\| \le C_T(u) \left(h^2 + \bar{k}^2\right).$$

PROOF. Let $\tilde{u} = V_h u$ for all $t_k \ge 0$ and $e^n = U^n - V_h u^n + V_h u^n - u^n = \theta^n + \rho^n$. Comparing (1.8) with the variational form of (1.1) and introducing (2.4) we have the following identity for $n \ge 2$:

$$(\bar{\partial}_t \theta^n, \phi) + A(\theta^{n-1/2}, \phi) = q^{n-1/2}(B(\theta, \phi)) + I_1^n + I_2^n, \tag{2.8}$$

where we denote I_1^n and I_2^n as follows:

$$I_1^n = (u_t^{n-1/2}, \phi) - (\bar{\partial}_t V_h u^n, \phi) = (u_t^{n-1/2} - \bar{\partial}_t u^n, \phi) - (\bar{\partial}_t \rho^n, \phi),$$
$$I_2^n = q^{n-1/2}(B(u, \phi)) - \frac{1}{2}(J_n(B(u, \phi)) + J_{n-1}(B(u, \phi))).$$

With the same argument as that used in Theorem 1 and taking $\phi = \tilde{\theta}^n$ in (2.8), we have

$$\frac{1}{2}(\|\theta^n\|^2 - \|\theta^{n-1}\|^2) + k_n\|\tilde{\theta}^n\|_1^2 \leq Ck_n \sum_{j=0}^{n-1}(n-j)^{-\alpha}k_{j+1}\|\tilde{\theta}^j\|_1\|\tilde{\theta}^n\|_1 + Ck_n|I_1^n + I_2^n|. \tag{2.9}$$

Summing (2.9) from $n = 1$ to N and applying Lemma 2, we immediately obtain

$$\|\theta^N\|^2 + \sum_{n=1}^{N} k_n\|\tilde{\theta}^n\|_1^2 \leq \|\theta^0\|^2 + C\sum_{n=1}^{N-1}(N-n)^{-\alpha}k_{n+1}^{1-\alpha}\sum_{j=0}^{n} k_{j+1}\|\tilde{\theta}^j\|_1^2 + C\sum_{n=1}^{N} k_n(|I_1^n| + |I_2^n|). \tag{2.10}$$

We now turn to the estimates for I_1 and I_2. Since $u \in C^2((0, t_1]; L_2)$, the Taylor formula with the integral form of the remainder implies that

$$k_1|(u_t^1 - \bar{\partial}_t u^1, \tilde{\theta}^1)| \leq k_1^2\|\theta^1\| \int_0^{t_1} \|u_{tt}\|\, ds \leq C(u)k_1^{2-\alpha}\|\tilde{\theta}^1\|. \tag{2.11}$$

If $u \in C^3((0, T]; L_2)$, then still by the Taylor formula, we get

$$k_n|(u_t^{n-1/2} - \bar{\partial}_t u^n, \tilde{\theta}^n)| \leq k_n^2\|\tilde{\theta}^n\| \int_{t_{n-1}}^{t_n} \|u_{ttt}\|\, ds \leq C_T(u)k_n^3 t_k^{-1-\alpha} \quad \text{for} \quad n \geq 2. \tag{2.12}$$

We can easily verify that

$$k_n \leq r\frac{T}{M}\left(\frac{n}{M}\right)^{r-1} \quad \text{for} \quad n \leq M \quad \text{and} \quad \bar{k} \geq r\left(\frac{1}{2}\right)^{r-1}\frac{T}{M}. \tag{2.13}$$

Denoting $r = 2 + p/(2-\alpha)$ for some $p > 0$, we immediately have from (2.11)–(2.13)

$$\sum_{n=1}^{N} k_n|I_1^n| \leq \max_{n\leq N}\|\theta^j\| \left\{C_T(u)\left(\bar{k}^2 + \bar{k}^2\sum_{n=2}^{N}\frac{1}{M}\left(\frac{n}{M}\right)^{-1+p}\right) + \sum_{n=1}^{N} k_n\|\bar{\partial}_t\rho^n\|\right\}.$$

Also the estimate for I_2^n is directly obtained by Lemma 6:

$$\sum_{n=1}^{N} k_n|I_2^n| \leq C_T(u)\bar{k}^2 \max_{n\leq N}\|\theta^n\|.$$

Thus, from Lemma 3 and Lemma 5, we obtain

$$\|\theta^n\| \leq \max_{n\leq N}\|\theta^n\| \leq Ch^2\left\{\|u_0\|_2 + \int_0^{t_n}\|u_t\|_2\, ds\right\} + C_T(u)\bar{k}^2.$$

Since the estimate for $\|\rho^n\|$ is given in Lemma 5, we complete the proof of the theorem.

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