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CONCERNING σ-CONNECTEDNESS OF BAIRE SPACES

A. GARCÍA-MÁYNEZ

1. Introduction. A well known theorem of Sierpiński states that every compact connected Hausdorff space is σ-connected. Hence, if X is locally compact and Hausdorff and X is locally connected at x, then x has a σ-connected neighborhood. However, local connectedness at x is not a necessary condition for x to have a σ-connected neighborhood, because the whole space may be σ-connected without being locally connected at x. One of the purposes of the present paper is then to investigate which points of a given locally compact Hausdorff space have σ-connected neighborhoods. We find also sufficient conditions for a connected, hereditarily Baire space to be σ-connected and prove the impossibility of expressing a connected, Čech-complete, rim compact space as a countable infinite union of mutually disjoint compact sets. Finally, we introduce the concept of D-connected space and relate it to σ-connectedness. We give a condition on a connected, locally compact Hausdorff space to be D-connected and give an example of a closed σ-connected subset of $\mathbf{R}^3$ which is not D-connected.

2. Definitions and preliminary results. Let X be an arbitrary topological space. A sequence $C_1, C_2, \ldots$ of subsets of X is a *σ'-partition* (resp., a *σ-partition*) of X if the C_i's are mutually disjoint, their union is X and at least two of them are non-empty (resp., all of them are non-empty). A σ'-partition is *closed* (resp., *compact*) if all of its elements are closed (resp., compact). X is *σ-connected* (resp., *σ-insular*) if it has no closed σ'-partition (resp., it has no closed σ-partition). Clearly, if X is connected, these two last concepts coincide. We have now two easy but important results:

2.1. *A space X is σ-insular if and only if it is a finite union of σ-connected subspaces.*

Proof. We prove only the necessity, since the sufficiency is obvious. Proceeding by contradiction, assume X is σ-insular but not expressible as a finite union of σ-connected subsets. X has then an infinite number of components (otherwise, some component would have a closed σ-partition and including the remaining components, we would obtain a

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closed σ-partition of X). Hence there exist disjoint, closed and non-empty sets A_1, B_1 such that $X = A_1 \cup B_1$. Clearly, we may assume that B_1 has infinitely many components. By the same argument, there exist disjoint, closed and non-empty sets A_2, B_2 such that $B_1 = A_2 \cup B_2$ and with B_2 having an infinite number of components. Continuing this process, we may find a sequence of disjoint, non-empty closed sets A_1, $A_2, \ldots$ and a decreasing sequence of closed sets $X = B_0 \supset B_1 \supset B_2 \supset \ldots$, each having infinitely many components, and such that $A_i \cap B_i = \emptyset$ and $B_{i-1} = A_i \cup B_i$ for each $i = 1, 2, \ldots$. Let $B^* = B_1 \cap B_2 \cap \ldots$. Then $B^*, A_1, A_2, \ldots$ is a closed σ'-partition of X and, with the possible exception of B^*, all of them are non-empty. Hence, X has a closed σ-partition, contradicting the fact that X is σ-insular.

2.2. *If D is a σ-insular subset of the connected space X and each point of $X - D$ has a σ-connected neighborhood, then X is σ-connected.*

Proof. Assume, on the contrary, that X has a closed σ-partition K_1, $K_2, \ldots$. With no loss of generality, we may assume that $D \subset K_1$. By hypothesis, each point $x \in X - D$ has a σ-connected neighborhood H_x. Since $K_2 \subset X - D$, we must have $H_x \subset K_2$ for each $x \in K_2$ (otherwise H_x would have a closed σ-partition). This implies that K_2 is open. Since K_2 is also closed, this contradicts the connectedness of X.

A space X is said to be *hereditarily Baire* if every closed subset of X is a Baire space. It is a well known fact that every G_δ subset of a countably compact regular space is hereditarily Baire. In particular, Čech-complete spaces (i.e., Tychonoff spaces which are G_δ in their Stone-Čech compactification) are hereditarily Baire.

The following proposition appears as an exercise in [**1**]. For the sake of completeness, we have included a proof:

2.3. *Every connected, locally connected and hereditarily Baire space is σ-connected.*

Proof. Assume, on the contrary, that X has a closed σ-partition K_1, $K_2, \ldots$. Let Q be the union of the sets $\operatorname{int} K_n$ ($n = 1, 2, \ldots$). Since X is a Baire space, Q is dense in X. Therefore:

$$X - Q = \operatorname{Fr} Q = \bigcup_{n=1}^{\infty} \operatorname{Fr} K_n.$$

Since $\operatorname{Fr} Q$ is also a Baire space, there exists an open set V in X and an index i such that $\emptyset \neq V \cap \operatorname{Fr} Q \subset \operatorname{Fr} K_i$. If R is any component of V intersecting $\operatorname{Fr} K_i$, necessarily $R \cap K_j = \emptyset$ for every $j \neq i$ (because R intersects $X - K_j$ but not $\operatorname{Fr} K_j$). Therefore, $R \subset K_i$. The local connectedness of X implies that R is open. Hence, $R \subset \operatorname{int} K_i$, a contradiction.

Combining 2.2 and 2.3, we obtain:

2.4. *Let X be a connected, hereditarily Baire space and let*

$$D_X = \{x \in X \mid X \text{ is not locally connected at } x\}.$$

If there exists a σ-insular set L containing D_X, then X is σ-connected.

Proof. We have only to observe that L^- is also σ-insular and that every component of $X - D_X^-$ is a σ-connected region.

Example 3 in [**4**] is a connected, locally connected subset of the plane which admits a decomposition into a countably infinite collection of closed segments and which is locally compact at each point of an open dense set (the union of interiors of the segments). Hence 2.3 is false if we omit the word "hereditarily" in its statement. Examples of connected, locally compact Hausdorff spaces which are not σ-connected are plentiful. See, for instance, example 4.3 in [**2**].

We will have the opportunity to use the next theorem in the example at the end of this paper.

2.5. *Let $C_1, C_2, \ldots$ be a sequence of σ-connected sets in a connected, hereditarily Baire space X and assume $C_1 \cup C_2 \cup \ldots$ is dense in X. Let $C = \liminf C_n$. If $C \cap C_n \neq \emptyset$ for every n and $C \subset \limsup (C \cap C_n)$, then X is σ-connected.*

Proof. Proceeding by contradiction, let $H_1, H_2, \ldots$ be a closed σ-partition of X. If some H_i contains infinitely many C_j's, then $C \subset H_i$. But since $C \cap C_n \neq \emptyset$ for every n, we would have $X = H_i$, a contradiction. Assume then that every H_i contains, at most, finitely many C_j's. Consequently, $C \cap H_1, C \cap H_2, \ldots$ is a closed σ'-partition of C and, since C is a Baire space, there exists a j such that $\operatorname{int}_C(C \cap H_j)$ is non-empty. Let V be an open set in X such that $\emptyset \neq C \cap V \subset H_j$. Since

$$C \subset \limsup (C \cap C_n),$$

$V \cap C \cap C_i$ is non-empty for infinitely many indices. But each $V \cap C \cap C_i$ lies in H_j. Hence, H_j contains infinitely many C_i's, a contradiction.

We give next a sufficient condition for a space to have a compact σ-partition:

2.6. *Every locally compact, 0-dimensional, Lindelöf, non-compact space X has a compact σ-partition.*

Proof. For each $x \in X$, let V_x be a compact open neighborhood of x. Since X is a Lindelöf space, there exists a countable subfamily $\{V_1, V_2,$

$\ldots\}$ of $\{V_x | x \in X\}$ covering X. Let

$$K_1 = V_1, K_2 = V_2 - V_1, \ldots, K_n = V_n - (V_1 \cup \ldots \cup V_{n-1}).$$

Clearly, the sets $K_1, K_2, \ldots$ are compact, mutually disjoint and cover X. Since X is non-compact, for infinitely many indices i, K_i is non-empty. Hence, X has a compact σ-partition.

A space X is *D-connected of order* $\leqq 0$ if X is compact and connected (i.e., X is a continuum). Inductively, if α is a positive ordinal, we say X is *D-connected of order* $\leqq \alpha$ if for each pair of points $a, b \in X$, there exists a finite sequence of ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$ less than α and a finite sequence $C_1, C_2, \ldots, C_n$ of subspaces of X such that $a \in C_1^-$, $b \in C_n^-$, $C_i^- \cap C_{i+1}^- \neq \emptyset$ for each $i < n$ and C_i is D-connected of order $\leqq \alpha_i$ for each $i \leqq n$. X is *D-connected of order* α if X is D-connected of order $\leqq \alpha$ and for each ordinal $\beta < \alpha$, it is false that X is D-connected of order $\leqq \beta$. In general, X is *D-connected* if there exists an ordinal number α such that X is D-connected of order α. D-connected spaces of order $\leqq 1$ receive the name of *semicontinua*. Observe that if $\{C_j | j \in J\}$ is a family of D-connected sub-spaces of X of order $\leqq \alpha$ with non-empty intersection, then their union is also D-connected of order $\leqq \alpha$. Using Sierpiński's theorem, it is not difficult to prove:

2.7. *Every Hausdorff D-connected space is σ-connected.*

The example at the end of this paper will prove that the converse of 2.7 is false, even for closed subsets of Euclidean spaces.

In [**2**] it is proved that for every space, X, we can find an ordinal number α such that every D-connected subspace of X has order $\leqq \alpha$. The least ordinal number for which this happens will be called the *D-order* of X. For each $p \in X$, the *D-component* of p is the union of all D-connected subspaces of X containing p. Clearly, every D-component of X is closed and D-connected and any two different D-components are disjoint. The *constituant* of p is the union of all continua in X containing p.

3. Main results. We start this section with a modified version of a standard theorem on separation of Hausdorff spaces.

3.1. *Let N be a closed subset of a locally compact Hausdorff space X. If E is a component of N with compact boundary and U is any neighborhood of E, then there exists an open V in X such that $E \subset V \subset V^- \subset U$, $N \cap \operatorname{Fr} V = \emptyset$ and such that $V^- - \operatorname{int} E$ is compact.*

Proof. Let S be an open set with compact closure such that $\operatorname{Fr} E \subset S \subset S^- \subset U$. Let $K = (N \cap S)^-$ and $K^* = K - \operatorname{int} E$. Since K^* is compact and no component of K^* intersects both sets $\operatorname{Fr} E$ and $K^* \cap \operatorname{Fr} S$ (a connected set in K^* intersecting $\operatorname{Fr} E$ has to lie in E because K^* lies

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in N and E is a component of N), we have a separation $K^* = A \cup B$, where $\operatorname{Fr} E \subset A$ and $K^* \cap \operatorname{Fr} S \subset B$. Since X is regular and Hausdorff and A is a compact subset of the open set $S - B$, there exists an open set W such that

$$A \subset W \subset W^- \subset S - B.$$

Let $V = W \cup \operatorname{int} E$. Then

$$E \subset V \subset V^- \subset W^- \cup E \subset U, \operatorname{Fr} V \subset \operatorname{Fr} W \subset S \quad \text{and}$$
$$K^* \cap \operatorname{Fr} W = \emptyset.$$

Since

$$N \cap \operatorname{Fr} V \subset (N - E) \cap \operatorname{Fr} W \cap S \subset K^* \cap \operatorname{Fr} W = \emptyset,$$

we have also

$$N \cap \operatorname{Fr} V = \emptyset.$$

Finally, $V^- - \operatorname{int} E$ is compact because it is closed and it is contained in the compact set S^-.

3.2. COROLLARY. *Let H be a proper subset of a connected, locally compact Hausdorff space X. If E is a component of H with compact boundary, then $E^- \cap \operatorname{Fr} H \neq \emptyset$.*

Proof. Assume, on the contrary, that $E^- \cap \operatorname{Fr} H = \emptyset$. Then $E = E^-$ and E is a component of $\operatorname{int} H$. Let T be an open set in X such that $E \subset T \subset T^- \subset \operatorname{int} H$. By 3.1, there exists an open set V such that $E \subset V \subset V^- \subset T$ and such that $T^- \cap \operatorname{Fr} V = \emptyset$. Therefore, $\operatorname{Fr} V$ is empty and V is a proper open and closed subset of X, contradicting the connectedness of X.

Lemma 3.3 below will be used in the proof of Theorem 3.5:

3.3. LEMMA. *Let $K_1, K_2, \ldots$ be a closed σ'-partition of a locally compact Hausdorff space X. If for n sufficiently large, K_n is compact, then X is not connected.*

Proof. Assume, on the contrary, that X is connected. Let $X^* = X \cup \{p\}$ be the one-point compactification of X and suppose K_n is compact for every $n > s$. If $B = \{p\} \cup K_1 \cup K_2 \cup \ldots \cup K_s$, then B, $K_{s+1}, K_{s+2}, \ldots$ is a closed σ'-partition of X^*, contradicting Sierpiński's theorem.

Combining 3.1 and 3.3, we obtain a much better result:

3.4. *Let $K_1, K_2, \ldots$ be a closed σ-partition of a locally compact, connected Hausdorff space X. Then no K_i has a component with compact boundary.*

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Proof. Proceeding by contradiction, assume, for instance, that K_1 has a component E with compact boundary. Using 3.1, we can find an open set V such that $E \subset V \neq X$, with $K_1 \cap \mathrm{Fr}\, V = \emptyset$ and such that $V^- - \mathrm{int}\, E$ is compact. Let E^* be the component of V^- containing E. According to 3.2, we have $E^* \cap \mathrm{Fr}\, V \neq \emptyset$. Therefore, $E^* \cap K_1, E^* \cap K_2, \ldots$ is a closed σ'-partition of E^* and, with the possible exception of $E^* \cap K_1$, all of its elements are compact. This contradicts Lemma 3.3.

Before going on, we need a definition:

Let $L \subset X$. X is said to be a *semicontinuum with respect to* L if each $p \in X - L$ belongs to a continuum $H_p \subset X$ intersecting L.

We prove now our first important result:

3.5. *Let $\mathscr{G}$ be a family of non-empty regions (i.e., open connected sets) in a connected, locally compact Hausdorff space X. Let L be a component of $X - \bigcup \mathscr{G}$ with compact boundary. Then there exist $G_1, \ldots, G_n \in \mathscr{G}$ and semicontinua $G_1^*, \ldots, G_n^*$, L^* with respect to $G_1, \ldots, G_n$, L such that $S = L^* \cup \bigcup_{i=1}^n G_i^*$ is a region in X.*

Proof. According to 3.1, there exists an open set $U \neq X$ such that $L \subset U$, $\mathrm{Fr}\, U \subset \bigcup \mathscr{G}$ and such that $U^- - \mathrm{int}\, L$ is compact. Since $\mathrm{Fr}\, U$ is compact, there exist $G_1, \ldots, G_m \in \mathscr{G}$ such that

$$\mathrm{Fr}\, U \subset \bigcup_{i=1}^m G_i = R \quad \text{and}$$

$$G_i \cap \mathrm{Fr}\, U \neq \emptyset \quad \text{for each } i = 1, \ldots, m.$$

On the other hand, each component of U^- has compact boundary, so each component of U^- intersects $\mathrm{Fr}\, U$ by 3.2. We prove the component L^* of U^- containing L is a semicontinuum with respect to L. Take $x \in L^* - L$. Let T be the component of $U^- - \mathrm{int}\, L$ containing x. Clearly, T is compact. Also, $T \cap \mathrm{Fr}\, L \neq \emptyset$. Assume, on the contrary, $T \cap \mathrm{Fr}\, L = \emptyset$. Hence T is a compact component of $L^* - L$ and this latter set is a proper open subset of L^*. By 3.2,

$$T \cap \mathrm{Fr}_{L^*}(L^* - L) \neq \emptyset.$$

Therefore, $T \cap L \neq \emptyset$, a contradiction. Observe now that each component of U^- different from L^* is a continuum intersecting $\mathrm{Fr}\, U$. Consequently, the union G_i^* of G_i and all components of U^- intersecting $G_i \cap \mathrm{Fr}\, U$ is a semicontinuum with respect to G_i and

$$U^- \cup R = L^* \cup \bigcup_{i=1}^m G_i^*.$$

Since $\mathrm{Fr}\, U \subset R$, $U^- \cup R$ is open and has at most m components, the component S of $U^- \cup R$ containing L^* is also open. Ordering the G_i's in

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such a way that for some $n \leqq m$, $G_i \subset S$ if and only if $i \leqq n$, we obtain the theorem.

3.5.1. COROLLARY. *If, besides the conditions in* 3.5, *L and each element of* $\mathscr{G}$ *is σ-connected (resp., D-connected), then L lies in a σ-connected (resp., D-connected) region.*

3.5.2. COROLLARY. *Let* $D_X = \{x \in X | X$ *is not locally connected* at $x\}$, *where X is as in* 3.5. *If L is a component of* D_X^- *with compact boundary, then there exists a region S in X which is the union of a semicontinuum with respect to L and a finite collection of semicontinua. Hence, if L is σ or D-connected, so is S.*

3.5.3. COROLLARY. *Let X and* D_X *be as in* 3.5.2. *If all the components of* D_X^- *are σ-connected and, with at most a finite number of exceptions, all of them have compact boundary, then X is σ-connected.*

We find next two sufficient conditions for a connected Čech-complete space to be a semicontinuum.

3.6. *Let X be a connected Čech-complete space. Then X is a semicontinuum in each of the following cases:*

a) *X is locally connected.*

b) *X is locally compact and each component of* D_X^- *is compact.*

Proof. a) Fix $a, b \in X$ and let $G_1 \supset G_2 \supset \ldots$ be a decreasing sequence of open sets in $Z = \beta X$ such that $X = G_1 \cap G_2 \cap \ldots$. For each $x \in X$, let $R(x, 1)$ be an open set in Z such that

$$x \in R(x, 1) \subset \mathrm{Cl}_Z R(x, 1) \subset G_1.$$

Let $S(x, 1)$ be the component of $X \cap R(x, 1)$ containing x. The family $\mathscr{S}_1 = \{S(x, 1) | x \in X\}$ is then a covering of X with regions in X. Let $S(x_1, 1), S(x_2, 1), \ldots, S(x_k, 1)$ be a simple chain from a to b and define M_1 as the component of $\mathrm{Cl}_Z R_1$ containing S_1, where

$$R_1 = \bigcup_{j=1}^{k} R(x_j, 1) \quad \text{and} \quad S_1 = \bigcup_{j=1}^{k} S(x_j, 1).$$

M_1 is then a continuum in Z such that $\{a, b\} \subset M_1 \subset G_1$. For each $x \in S_1$, let $R(x, 2)$ be an open set in Z such that

$$x \in R(x, 2) \subset \mathrm{Cl}_Z R(x, 2) \subset G_2 \cap R_1.$$

Let $S(x, 2)$ be the component of $X \cap R(x, 2)$ containing x. $\mathscr{S}_2 = \{S(x, 2) | x \in S_1\}$ is then a covering of S_1 with regions in X. Let $S(y_1, 2), S(y_2, 2), \ldots, S(y_p, 2)$ be a simple chain from a to b and define M_2 as the component of $\mathrm{Cl}_Z R_2$ containing S_2, where

$$R_2 = \bigcup_{j=1}^{p} R(y_j, 2) \quad \text{and} \quad S_2 = \bigcup_{j=1}^{p} S(y_j, 2).$$

Clearly M_2 is a continuum in Z and $\{a, b\} \subset M_2 \subset M_1 \cap G_2$. Continuing this process indefinitely, we can find a decreasing sequence of continua $M_1 \supset M_2 \supset \ldots$ in Z such that $\{a, b\} \subset M_n \subset M_{n-1} \cap G_n$ for every n. $M = M_1 \cap M_2 \cap \ldots$ is then a continuum in X about a and b.

b) Let $\mathcal{G}$ be the decomposition of X into components of D_X^- and points of $X - D_X^-$ and let $g: X \rightarrow X/\mathcal{G} = Q$ be the canonical identification. The assumptions about X and D_X^- imply that g is a perfect mapping of X onto Q. Hence Q is also connected, Hausdorff and locally compact. Besides, D_Q is totally disconnected. According to 2.2, p. 104 in [**6**], Q is locally connected and hence Q is a semicontinuum. Fix a, b in X and let H be a continuum in Q about $g(a)$ and $g(b)$. Then $K = g^{-1}(H)$ is a continuum in X about a and b and the proof is complete.

From 3.5.2 and 3.6 b), we easily obtain:

3.7. *Let X and D_X be as in* 3.5.2. *Assume there exists an ordinal number α such that every component of D_X^- is D-connected of order $\leqq \alpha$ and has compact boundary. Then X is D-connected of order $\leqq \alpha + 1$.*

Proof. If $\alpha = 0$, 3.7 follows from 3.6 b). Assume then $\alpha > 0$. By 3.5.2, every point of X has an open neighborhood which is D-connected of order $\leqq \alpha + 1$. This clearly implies that X is D-connected of order $\leqq \alpha + 1$.

We investigate now the nature of closed σ-partitions of certain Čech-complete spaces.

3.8. *Let $K_1, K_2, \ldots$ be a closed σ-partition of the connected, rim compact and Čech-complete space X. Then, for infinitely many indices i, K_i is non-compact.*

Proof. Assume, on the contrary, that with at most a finite number of exceptions, K_i is compact. With no loss of generality, we may assume that K_i is compact for all $i \geqq 2$. Let FX be the Freudenthal compactification of X. (Main properties of FX can be found in [**3**]). If $Z = K_1^- \cup (FX - X)$, then Z is a G_δ and F_δ subset of FX (because $FX - Z = \bigcup_{i=2}^\infty K_i$ is G_δ and F_δ in FX). Each compact subset L of Z is clearly contained in a compact set $L^* \subset Z$ which is a G_δ in FX. Therefore it is possible to express Z as a countable union $Z = \bigcup_{i=1}^\infty H_i$ of compact sets, each of which is a G_δ in FX and, with no loss of generality, we may assume that $K_1^- \subset H_1 \subset H_2 \subset \ldots$. Since each space $H_{n+1} - H_n$ is contained in $FX - X$, which is a 0-dimensional space, 2.6 implies that $H_{n+1} - H_n$ is either compact or has a compact σ-partition. Hence Z, as well as $FX - Z$, admits a compact σ-partition. But then also FX admits a compact σ-partition, contradicting Sierpiński's theorem.

The following three examples exhibit limitations to improving 3.8:

3.9. *Example.* A Čech-complete and connected subset of the plane which admits a compact σ-partition.

Let L_n be the segment in $\mathbf{R}^2$ with end points $(1/n, 0)$ and $(1/n, 1)$ $(n = 1, 2, \ldots)$. Let C_n be the set of points in the circle $x^2 + y^2 = (1/n)^2$ having at least one coordinate $\leqq 0$. Define: $K_n = L_n \cup C_n$, $K = \bigcup_{n=1}^{\infty} K_n$. Then clearly $K_1, K_2, \ldots$ is a compact σ-partition of the connected set K. Being a G_δ in the plane, K is Čech-complete.

3.10. *Example.* A rim compact, connected, Baire subset of the plane which admits a compact σ-partition.

Simply adjoin the points $(0, r)$ (r rational, $0 < r < 1$) to the space K described in 3.9. The space K^* obtained this way is a Baire space because it is locally compact at each point of an open dense subset. However, K^* is not Čech-complete.

3.11. *Example.* A Čech-complete, rim compact, connected subset of the plane which admits a closed σ-partition with infinitely many compact elements.

Adjoin the points $(0, y)$ (y irrational, $0 < y < 1$) to the space K described in 3.9.

S. Mazurkiewicz describes in [**5**] a closed and connected subset of the plane which satisfies the following properties:

i) K has a closed σ-partition $K_0, K_1, K_2, \ldots$;

ii) $K_0 = \mathscr{C} \times \mathbf{R}^+$, where $\mathscr{C}$ is the Cantor discontinuum and $\mathbf{R}^+$ is the set of non-negative reals.

iii) For $i > 0$, K_i is a constituant of K.

iv) For each $t \in \mathbf{R}^+$ and each integer $i > 0$, the line $y = t$ intersects K_i.

v) $\liminf K_i = \emptyset$.

Using K, we may construct:

3.12. *Example.* A closed σ-connected subset of $\mathbf{R}^3$ of D-order 1 which is not D-connected.

For each integer $n > 0$, let

$$A_n = \{i/3^n | i \neq 0 (\text{mod } 3), i/3^n \in \mathscr{C}\}$$

and let $L_n = A_n \times \mathbf{R}^+$. Let B_n be a 1-dimensional continuum in the plane $y = n$ such that B_n intersects K_n and each component of L_n in exactly one point and such that the intersection of B_n with the plane $z = 0$ lies completely in $K_n \cup L_n$ (for instance, B_n may be a finite union of "hooks" in $y = n$ joining irreducibly a fixed point of K_n to each component of L_n). Then $C_n = K_n \cup B_n \cup L_n$ is a closed semicontinuum in $\mathbf{R}^3$ and $X = (C_1 \cup C_2 \cup \ldots)^-$ is clearly closed, connected and contains

K. Since

$$C = \liminf C_n = \limsup (C \cap C_n) = K_0,$$

2.5 implies that X is σ-connected. Observe now that a continuum in X cannot intersect more than one component of $K_0 - (L_1 \cup L_2 \cup \ldots)$. For assume, on the contrary, there is such a continuum H. Being bounded, H cannot intersect infinitely many B_i's, say $H \cap B_i = \emptyset$ for $i > m$. On the other side, the σ-connectedness of H implies that H intersects uncountably many components of K_0. Let $H \cap K_0 = A \cup B$ be a separation, where B contains all the components of $H \cap K_0$ lying in L_i for each $i = 1, 2, \ldots, m$. Since $H \cap K_i = \emptyset$ for $i > m$ (because otherwise $H \cap [K_0 \cup C_1 \cup \ldots \cup C_m], H \cap K_{m+1}, \ldots$ would be a closed σ'-partition of H), we have

$$H \subset A \cup (B \cup C_1 \cup \ldots \cup C_m)$$

and both sets A and $B \cup C_1 \cup \ldots \cup C_m$ are separated. This contradicts the connectedness of H. The constituants of X are then $C_1, C_2, \ldots$ and all components of K_0 disjoint from $L_1 \cup L_2 \cup \ldots$. Since each constituant is closed, they are precisely the D-components of X.

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Universidad Nacional Autonoma de México,
México