1. Introduction. By a von Neumann algebra \( M \) we mean a weakly closed, self-adjoint algebra of operators on a Hilbert space \( \mathcal{H} \) which contains \( I \), the identity operator. A factor is a von Neumann algebra whose centre consists of scalar multiples of \( I \).

In all that follows \( \phi: M \rightarrow N \) will be a one to one, *-linear map from the von Neumann factor \( M \) onto the von Neumann algebra \( N \) such that both \( \phi \) and \( \phi^{-1} \) preserve commutativity. Our main result states that if \( M \) is not of type \( I_2 \) then \( \phi = \phi_0 + \lambda \theta \) where \( \theta \) is an isomorphism or an anti-isomorphism, \( c \) is a non-zero scalar, and \( \lambda \) is a *-linear map from \( M \) into \( Z_N \), the centre of \( N \).

Our interest in this problem was aroused by several recent results. In [1], Choi, Jafarian, and Radjavi proved that if \( S \) is the real linear space of \( n \times n \) matrices over any algebraically closed field, \( n \geq 3 \), and \( \psi \) a linear operator on \( S \) which preserves commuting pairs of matrices, then either \( \psi(S) \) is commutative or there exists a unitary matrix \( U \) such that

\[
\psi(A) = cU^*AU + f(A)I \quad \text{or} \quad \psi(A) = cUA^*U + f(A)I
\]

for all \( A \) in \( S \). They proved an analogous result for the collection of all bounded self-adjoint operators on an infinite dimensional Hilbert space when \( \psi \) is one to one. Subsequently, Omladic [7] proved that if \( \psi:L(X) \rightarrow L(X) \) is a bijective linear operator preserving commuting pairs of operators where \( X \) is a non-trivial Banach space, then

\[
\psi(A) = cUAU^{-1} + f(A)I \quad \text{or} \quad \psi(A) = UA^*U^{-1} + f(A)I
\]

where \( U \) is a bounded invertible operator on \( X \) and \( A' \) is the adjoint of \( A \).

We viewed this problem as one involving mappings between the Lie algebras \( M \) and \( N \) which preserve the zero brackets. Our technique is to show, as in [6] where bracket preserving maps were studied, that on projections \( P \) in \( M \),

\[
\phi(P) = \theta(P) + \lambda(P)I \quad \text{or} \quad \phi(P) = -\theta(P) + \lambda(P)I
\]
where $\theta$ is a projection orthoisomorphism. This representation is harder to achieve than in [6], but once having it the techniques of [6] are applied together with results concerning the linear span of projections in a factor to give the result. A key tool used in [6] is a theorem of Dye [3] relating projection orthoisomorphisms to $C^*$-isomorphisms.

The techniques of this paper give the result as long as the dimension of the underlying Hilbert space is $>4$. However, since the Choi, Jafarian, Radjavi theorem implies our theorem for all type $I_n$ factors, $n > 2$, and since we would have to invoke their theorem for $n = 3, 4$, we shall assume that $M$ is not a finite factor of type $I$. We use [2] as a general reference for the theory of von Neumann algebras.

2. The decomposition $\phi = \theta + \lambda$.

**Lemma 1.** $N$ is a factor.

**Proof.** Let $Z_M$, $Z_N$ be the centers of $M$, $N$ respectively. Since $\phi(Z_M) = Z_N$ and $Z_M$ is 1-dimensional, $Z_N$ is 1-dimensional.

**Lemma 2.** We can assume, by dividing by an appropriate constant, that $\phi(I) = I$.

**Proof.** Since $Z_N = CI$ and since $\phi$ is one to one, $\phi(I) = \beta I$ for $\beta \neq 0$. Replace $\phi$ by $(1/\beta)\phi$.

**Definition.** A von Neumann subalgebra $M_0 \subseteq M$ is normal in $M$ if

$$M_0 = (M_0' \cap M)' \cap M$$

where, for any subset $S \subseteq \mathcal{B}(H)$,

$$S' = \{ Y \in \mathcal{B}(H) \mid XY = YX \forall X \in S \}.$$  

**Lemma 3.** If $M_0$ is a normal subalgebra of $M$, then $N_0 = \phi(M_0)$ is a normal subalgebra of $N$ with the same linear dimension.

**Proof.** If $S$ is any subset of $M$, $\phi(S' \cap M') = \phi(S)' \cap N$. Since $M_0$ is normal, $M_0 = (M_0' \cap M)' \cap M$ so that

$$\phi(M_0) = (\phi(M_0)' \cap \phi(M))' \cap \phi(M) = (\phi(M_0)' \cap N)' \cap N.$$  

Since $M_0$ is a self-adjoint collection, so is $\phi(M_0)$ which implies that $(\phi(M_0)' \cap N)' \cap N$ is a von Neumann algebra. Hence $N_0 = \phi(M_0)$ is a von Neumann algebra and is normal in $N$.

**Lemma 4.** If $P$ is a non-central projection in $M$, then $\phi(P) = \alpha Q + \lambda I$ where $Q$ is a non-central projection in $N$ and $\alpha \neq 0$.

**Proof.** By [5, Theorems 1 and 4], a finite-dimensional subalgebra of a factor is normal. Let $M_0 = \text{lin.sp.}\{I, P\}$. $M_0$ is a 2-dimensional subalgebra of $M$ and is thus normal in $M$. By Lemma 3, $\phi(M_0) = N_0$ is a 2-dimensional von Neumann subalgebra of $N$, say
$\phi(M_0) = \text{lin.sp.}\{I, Q\}$

where $Q$ is a non-central projection. We have $\phi(P) \in \phi(M_0)$ so $\phi(P) = \alpha Q + \lambda I$. If $\alpha = 0$ then $P$ would be central by the commutativity preserving property of $\phi$.

**Lemma 5.** If $P$ is a non-central projection and

$$\phi(P) = \alpha Q + \lambda I = \alpha' Q' + \lambda' I$$

with $\alpha, \alpha' \neq 0$, $Q$ and $Q'$ non-central projections in $N$, then either (i) $Q = Q'$ and $\alpha = \alpha'$, or (ii) $Q = I - Q'$ and $\alpha = -\alpha'$.

**Proof.** For an operator $A \in B(H)$, let $\sigma(A)$ be its spectrum. We have

$$\{\alpha + \lambda, \lambda\} = \sigma(\alpha Q + \lambda I) = \sigma(\alpha' Q' + \lambda' I) = \{\alpha' + \lambda', \lambda'\}.$$ 

If $\alpha + \lambda = \alpha' + \lambda'$ then $Q = Q'$. If $Q = Q'$ then clearly $\lambda = \lambda'$ so that $\alpha = \alpha'$. If $\alpha + \lambda = \lambda'$ and $\alpha' + \lambda' = \lambda$ then $\alpha = -\alpha'$ and $\lambda \neq \lambda'$ since $\alpha \neq 0$. We would then have

$$Q + Q' = \left(\frac{\lambda - \lambda'}{\alpha}\right)I.$$ 

This forces

$$\frac{\lambda - \lambda'}{\alpha} = 1.$$ 

If $Q = I - Q'$ it is easy to see that $\alpha = -\alpha'$.

**Lemma 6.** Let $P_1, P_2$ be non-central orthogonal projections in $M$ with $P_1 + P_2 \neq I$ There exist orthogonal non-central projections $Q_1, Q_2$ in $N$ and non-zero scalars $\alpha_1, \alpha_2$, such that

$$\phi(P_i) = \alpha_i Q_i + \lambda_i I \quad i = 1, 2.$$ 

**Proof.** Let $M_0 = \text{lin.sp.}\{I, P_1, P_2\}$. $M_0$ is a 3-dimensional abelian subalgebra of $M$ so that $N_0 = \phi(M_0)$ is a 3-dimensional abelian subalgebra of $N$. We claim that

$$N_0 = \text{lin.sp.}\{I, Q_1, Q_2\}$$

where $\phi(P_i) = \alpha_i Q_i + \lambda_i I$ as in Lemma 4. Clearly $Q_1, Q_2 \in N_0$ since $I \in N_0$, $\phi(P_i) \in N_0$, and $\alpha_i \neq 0$. Suppose

$$\alpha I + \beta Q_1 + \gamma Q_2 = 0.$$ 

Since

$$\phi(I) = I \quad \text{and} \quad Q_i = \phi\left(\frac{1}{\alpha_i}P_i - \lambda_i I\right). \quad i = 1, 2.$$ 

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we have
\[ 0 = aI + \beta Q_1 + \gamma Q_2 = \phi\left( (\alpha - \beta\lambda_1 - \gamma\lambda_2)I + \frac{\beta}{\alpha_1} P_1 + \frac{\gamma}{\alpha_2} P_2 \right). \]

This implies
\[ \frac{\beta}{\alpha_1} P_1 + \frac{\gamma}{\alpha_2} P_2 \in Z_M \]

since \( \phi \) is one to one. Since \( P_1 P_2 = 0 \) and the \( P_i \) are non-central we have \( \beta = \gamma = 0 \). This forces \( \alpha = 0 \). Thus \( \{I, Q_1, Q_2\} \) is a linearly independent subset of the three-dimensional algebra \( N_0 \).

Case (1). \( Q_1 Q_2 = 0 \), and we need do no more.
If \( Q_1 P_2 \neq 0 \) then, since \( Q_1 Q_2 \in N_0 \) we have
\[ (\ast) \quad Q_1 Q_2 = aI + \beta Q_1 + \gamma Q_2 \]

where not all of \( \alpha, \beta, \gamma \) are zero. Multiplying \( (\ast) \) by \( Q_1 Q_2 \) we get \( \alpha + \beta + \gamma = 1 \). Multiplying by \( Q_1 \) we see that
\[ (1 - \gamma)Q_1Q_2 = (1 - \gamma)Q_1. \]

Case (2). \( 1 - \gamma \neq 0 \). Then \( Q_1 = Q_1 Q_2 \) or \( Q_1 \subseteq Q_2 \). If \( Q_1 = Q_2 \) then \( \{I, Q_1, Q_2\} \) would span a two-dimensional subalgebra so we must have \( Q_1 \cong Q_2 \). In this case we replace \( Q_2 \) by \( I - Q_2 \) and note that
\[ \alpha_2 Q_2 + \lambda_2 I = \alpha_2 (I - Q_2) + (\lambda_2 - \alpha_2) I. \]

If \( \gamma = 1 \) then \( (\ast) \) becomes \( Q_1 Q_2 = aI + \beta Q_1 + Q_2 \) so that
\[ (1 - \beta)Q_1Q_2 = (1 + \alpha)Q_2. \]

Case (3). \( \beta \neq 1 \). Then \( 1 - \beta = 1 + \alpha \) and \( Q_1 Q_2 = Q_2 \). As in (2), \( Q_1 \neq Q_2 \), and we replace \( Q_1 \) by \( I - Q_1 \).

Case (4). \( \beta = 1 \). Then \( \alpha = -1 \) and \( Q_1 Q_2 = -I + Q_1 + Q_2 \). That is, \( I - Q_1 \perp I - Q_2 \) so we replace both \( Q_1 \) and \( Q_2 \) by \( I - Q_1 \) and \( I - Q_2 \) respectively.

**Lemma 7.** If \( P_1, P_2, Q_1, Q_2 \) and \( \alpha_1, \alpha_2 \) are as in Lemma 6 then \( \alpha_1 = \alpha_2 \).

**Proof.** Let \( M_0 = \text{lin.sp.}\{I, P_1 + P_2\} \). Then \( M_0 \) is a 2-dimensional subalgebra of \( M \), so that \( \phi(M_0) = N_0 \) is a two-dimensional subalgebra of \( N \), say \( N_0 = \text{lin.sp.}\{I, Q\} \). Thus
\[ \phi(P_1 + P_2) = \alpha_1 Q_1 + \alpha_2 Q_2 + (\lambda_1 + \lambda_2) I = \alpha Q + \lambda I \]

where the \( \alpha_i \) and \( \lambda_i, i = 1, 2 \) are as in Lemma 6. Since \( \alpha \neq 0 \) and \( Q \) not central, the spectrum of \( \alpha Q + \lambda I \) consists of two points. Thus if
\[ A = \alpha_1 Q_1 + \alpha_2 Q_2 + (\lambda_1 + \lambda_2) I, \]

\( \sigma(A) \) consists of two points. Since \( Q_1 \perp Q_2 \) and \( Q_1 + Q_2 \neq I \) we have
\[ \sigma(A) = \{ \alpha_1 + \lambda_1 + \lambda_2, \alpha_2 + \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 \} \]

and so two of these points coincide. Now \( \alpha_1, \alpha_2 \neq 0 \) so we must have
\[ \alpha_1 + \lambda_1 + \lambda_2 = \alpha_2 + \lambda_1 + \lambda_2 \]
and so \( \alpha_1 = \alpha_2 \).

**Lemma 8.** If \( P_1, P_2 \) are non-central, orthogonal, equivalent projections in \( M \) with \( P_1 + P_2 \neq I \) there exist non-central, orthogonal, equivalent projections \( Q_1, Q_2 \) in \( N \) and \( \alpha \neq 0 \) such that \( \phi(P_1) = \alpha Q_1 + \lambda I \).

**Proof.** Let the \( Q_i \) and \( \alpha \) be chosen as in Lemma 7, let \( V \) be a partial isometry in \( M \) such that \( V^*V = P_1, VV^* = P_2 \), and let \( \mathcal{A} \) be the non-commutative 5-dimensional von Neumann subalgebra of \( M \) generated by \( \{ I, P_1, P_2, V, V^* \} \). Then \( \mathcal{B} = \phi(\mathcal{A}) \) is a 5-dimensional von Neumann subalgebra of \( N \) generated by
\[ \{ I, \alpha Q_1 + \lambda_1 I, \alpha Q_2 + \lambda_2 I, X, X^* \} \]
where \( X = \phi(V) \). We have that
\[ Z_{\mathcal{B}} = \text{lin.sp.} \{ I, Q_1 + Q_2 \} \]
since
\[ Z_{\mathcal{A}} = \text{lin.sp.} \{ I, P_1 + P_2 \} . \]
Since \( \mathcal{B} \) is a non-commutative 5-dimensional von Neumann algebra,
\[ \mathcal{B} = M_1 \oplus M_2 \cong \mathbb{C} \oplus M_2(\mathbb{C}) \]
where \( M_2(\mathbb{C}) \) is the algebra of \( 2 \times 2 \) matrices over \( \mathbb{C} \). Let \( I_1 \) and \( I_2 \) be the central projections of \( \mathcal{B} \) which are the identities of \( M_1 \) and \( M_2 \) respectively. We have \( I_1 + I_2 = I \). Now \( Q_1 + Q_2 \) is a non-zero central projection in \( \mathcal{B} \) so either \( Q_1 + Q_2 = I_1 \) or \( Q_1 + Q_2 = I_2 \). But \( I_1 \) is not the sum of non-zero orthogonal projections so we have \( Q_1 + Q_2 = I_2 \). This implies that \( Q_1 \) and \( Q_2 \) are in \( M_2 \) and so are equivalent since they are non-central.

**Lemma 9.** Let \( M \) be a factor of type \( \Gamma_{\infty} \), II, or III and let \( P \in M \) be a non-central projection. There exists \( \alpha \in \mathbb{C}, \alpha \neq 0 \), independent of \( P \) and a non-central projection \( Q \in N \) such that \( \phi(P) = \alpha Q + \lambda I \).

**Proof.** Let \( P = P_1 \) and let \( P_2 \neq P_1 \) be any other non-central projection in \( M \). One of \( P_1 \vee P_2, (I - P_1) \vee P_2, P_1 \vee (I - P_2) \) or \( (I - P_1) \vee (I - P_2) \) has codimension \( \geq 2 \). Suppose it is \( P_1 \vee P_2 \), the other cases being similar. Thus \( I - (P_1 \vee P_2) \) is the sum of two orthogonal projections. (In the type II and III cases we need only that \( I - (P_2 \vee P_2) \neq 0 \) and then could “halve” \( I - (P_1 \vee P_2) \) to get equivalent orthogonal projections. In the type I case the codimension \( \geq 2 \) as long as the dimension of \( \mathcal{A} \geq 5 \).) Let \( P_3 \) be one of them. Then \( P_1 \perp P_3 \) and \( P_1 + P_3 \neq I \). Applying Lemma 7 to \( P_1 \) and \( P_3 \) we get
\[ \phi(P_1) = \alpha Q_1 + \lambda_1 I, \quad \phi(P_3) = \alpha Q_3 + \lambda_3 I. \]

Applying Lemma 7 to \( P_2 \) and \( P_3 \) we get
\[ \phi(P_2) = \alpha' Q_2' + \lambda_2' I, \quad \phi(P_3) = \alpha' Q_3' + \lambda_3' I. \]

Applying Lemma 5 to the two representations of \( \phi(P_3) \) we get \( \alpha' = \pm \alpha \). If \( \alpha' = -\alpha \), write
\[ \phi(P_2) = \alpha(I - Q_2') + (\lambda_2' - \alpha) I. \]

We now replace \( \phi \) by \((1/\alpha)\phi\).

**Lemma 10.** Let \( M \) be a factor of type I\(_{\infty} \), II, or III, and \( P \) a non-central projection. Then \( \phi(P) \) can be expressed uniquely in one of two ways

(i) \( \phi(P) = \theta(P) + \lambda(P) I, \) or

(ii) \( \phi(P) = -\theta'(P) + \lambda'(P) I \)

where \( \theta(P), \theta'(P) \) are non-central projections in \( N \), and \( \lambda(P), \lambda'(P) \) are scalars.

**Proof.** With the above normalization
\[ \phi(P) = Q + \lambda I = -(I - Q) + (1 + \lambda) I \]
so we let \( \theta(P) = Q, \lambda(P) = \lambda, \theta'(P) = I - Q, \lambda'(P) = 1 + \lambda \). If
\[ Q + \lambda I = Q' + \lambda' I \]
where \( Q \) commutes with \( Q' \) then
\[ (\lambda - \lambda')^2 I = (Q' - Q)^2 = Q' + Q - 2QQ'. \]
This happens if and only if \( Q = Q' \).

3. The C*-isomorphism theorem.

**Lemma 11.** \( \theta(I - P) = I - \theta(P), \theta'(I - P) = I - \theta'(P) \).

**Proof.** See [6, Lemma 4].

**Lemma 12.** If \( P \) and \( Q \) are orthogonal projections in \( M \) then either
\[ \theta(P) \perp \theta(Q) \quad \text{or} \quad I - \theta(P) \perp I - \theta(Q). \]

**Proof.** This follows from Lemma 5 and Lemma 9.

**Definition.** If \( M \) is a von Neumann algebra let \( M_p \) be the collection of projections in \( M \). A projection orthoisomorphism between von Neumann algebras \( M \) and \( N \) is a map \( \theta: M_p \to N_p \) which is one to one, onto, and such that if \( P, Q \in M_p \) with \( PQ = 0 \) then \( \theta(P)\theta(Q) = 0 \).
Lemma 13. If $\mathcal{A}$ is an abelian von Neumann subalgebra of $M$ of dimension $\geq 3$ then either $\theta$ or $\varphi$ is an orthoisomorphism on $\mathcal{A}_p$, and these possibilities are mutually exclusive. If $\theta$ is an orthoisomorphism then both $\theta$ and $\lambda$ are additive on mutually orthogonal projections in $\mathcal{A}_p$. A similar statement holds for $\varphi'$ and $\lambda'$.

Proof. See [6, Lemma 6].

Lemma 14. Let $P_1, \ldots, P_n$, $n \geq 3$ be mutually orthogonal equivalent projections in $M$. If the $\theta(P_i)$ are orthogonal then they are equivalent in $N$. If the $\theta'(P_i)$ are mutually orthogonal then they are equivalent in $N$.

Proof. Applying Lemma 8 we have that if $\theta(P_1) \perp \theta(P_2)$ then $\theta(P_1) \sim \theta(P_2)$ in $N$, etc.

Theorem 1. Let $\phi : M \to N$ be a commutativity preserving map of the infinite factor $M$ onto the von Neumann algebra $N$. Then $N$ is an infinite factor and if $P \in M_p$, $\phi(P) = \theta(P) + \lambda(P)$ where $\theta$ is an orthoisomorphism, or

$$
\phi(P) = -\theta'(P) + \lambda'(P)
$$

where $\theta'$ is an orthoisomorphism. If $M$ is a finite factor, so is $N$ and a similar conclusion holds for $\phi$.

Proof. If $M$ is infinite choose mutually orthogonal equivalent projections $P_i$, $i = 1, 2, 3, 4$ such that

$$
\sum_{i=1}^{4} P_i = I
$$

and assume the $\theta(P_i)$ are orthogonal. Then the $\theta(P_i)$ are equivalent. Since $P_1 \sim P_3 \sim P_1 + P_2$ we have, from Lemma 8 and the additivity of $\theta$, that $\theta(P_1) \sim \theta(P_1) + \theta(P_2)$ so that $N$ is infinite. Now

$$
I = \phi(I) = \sum_{i=1}^{4} \phi(P_i) = \sum_{i=1}^{4} \theta(P_i) + \left( \sum_{i=1}^{4} \lambda(P_i) \right) I
$$

which implies

$$
\sum_{i=1}^{4} \theta(P_i) = I \quad \text{and} \quad \sum_{i=1}^{4} \lambda(P_i) = 0
$$

since the $\theta(P_i)$ are orthogonal. Thus $\theta(I) = I$. In the $\theta'$ case, $\theta'(I) = -I$. The proof in the infinite case now follows [6, Theorem 2].

If $M$ is finite, and hence of type $\text{II}_1$ since we are ruling out the type $\text{II}_n$ case, then so is $N$ since the above reasoning could be applied to $\phi^{-1}$ if $N$ were infinite. $N$ cannot be of type $\text{II}_n$ since $\phi^{-1}$ preserves linear dimension.
Hence $N$ is also of type $\text{II}_1$. The proof for $M$ and $N$ being $\text{II}_1$-factors now follows [6, Theorem 3].

**Theorem 2.** Let $\phi: M \to N$ be a commutativity preserving map from the factor $M$ onto the von Neumann algebra $N$. Then $\phi = c\bar{\theta} + \lambda$ where $c \in \mathbb{C}$, $c \neq 0$, $\bar{\theta}$ is an isomorphism or an anti-isomorphism of $M$ onto $N$, and $\lambda$ is a $\ast$-linear map from $M$ into $Z_N = \text{Cl}$.

**Proof.** On projections

$$\phi(P) = \theta(P) + \lambda(P)I \quad \text{or} \quad \phi(P) = -\theta'(P) + \lambda'(P)I$$

as in Theorem 1. Taking the case where $\theta$ is an orthoisomorphism there is, by a theorem of Dye [3, Theorem 1], a $C^*$-isomorphism $\bar{\theta}$ of $M$ on $N$ which agrees with $\theta$ on $M_P$. By [8, Theorem 6] every self-adjoint operator in a properly infinite von Neumann algebra is a real linear combination of eight projections, and it was proved in [4] that every operator in a $\text{II}_1$-factor is a linear combination of projections. Thus for any factor $M$, if $A \in M$ then

$$A = \sum_{i=1}^{n} \alpha_i P_i.$$

We have

$$\phi(A) = \phi\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i \left(\theta(P_i) + \lambda(P_i)I\right)$$

$$= \sum_{i=1}^{n} \alpha_i \bar{\theta}(P_i) + \left(\sum_{i=1}^{n} \alpha_i \lambda(P_i)\right)I$$

$$= \bar{\theta}(A) + (\sum \alpha_i \lambda(P_i))I.$$

That is, $\phi(A) - \bar{\theta}(A) \in Z_N = \text{Cl}$ for each $A \in M$. Setting $\phi(A) - \bar{\theta}(A) = \lambda(A)$ we see that $\lambda(A)$ is a $\ast$-linear map from $M$ into $Z_N$, and

$$\phi(A) = \bar{\theta}(A) + \lambda(A).$$

A similar argument applies in the $\theta'$ case to give

$$\phi(A) = -\bar{\theta}(A) + \lambda(A).$$

We recall that $\phi$ was normalized in Lemma 2 and after Lemma 9 so what we have really proved is

$$\frac{1}{c^\ast} \phi = \pm \bar{\theta} + \lambda$$

where $\bar{\theta}$ is a $C^*$-isomorphism. Since a $C^*$-isomorphism on a factor is either an isomorphism or an anti-isomorphism we have the result.
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