TILING BRANCHING MULTIPLICITY SPACES WITH GL₂ PATTERN BLOCKS

SANGJIB KIM

(Received 17 October 2011; accepted 17 October 2012; first published online 10 April 2013)

Communicated by J. Du

Abstract

We study branching multiplicity spaces of complex classical groups in terms of GL₂ representations. In particular, we show how combinatorics of GL₂ representations are intertwined to make branching rules under the restriction of GLₙ to GLₙ₋₂. We also discuss analogous results for the symplectic and orthogonal groups.

2010 Mathematics subject classification: primary 20G05; secondary 05E10.
Keywords and phrases: classical groups, representations, branching rules.

1. Introduction

1.1. Branching rules describe a way of decomposing an irreducible representation of a whole group into irreducible representations of a subgroup. With applications in physics, branching rules for classical groups have been extensively studied. See, for example, [6, 7, 9, 11].

In this paper, we study combinatorial aspects of branching rules for complex classical groups, under the restriction of GLₙ to GLₙ₋₂, Sp₂ₙ to Sp₂ₙ₋₂, and SOₘ to SOₘ₋₂, by investigating the GL₂ module structure of branching multiplicity spaces. Recently, Wallach, Yacobi and the present author studied Sp₂ₙ to Sp₂ₙ₋₂ branching rules in terms of SL₂ representations [5, 10, 12]. Our results for the symplectic group are compatible with those in the above papers once we restrict GL₂ to SL₂.

1.2. A group homomorphism φₜ from the complex torus (ℂ⁺)ᵏ to ℂ⁺ defined by

φₜ(t₁, t₂, ..., tₖ) = t₁α₁ t₂α₂ ··· tₖαₖ

is called a polynomial dominant weight of the complex general linear group GLₖ = GLₖ(ℂ), if it satisfies

α = (α₁, ..., αₖ) ∈ ℤᵏ and α₁ ≥ ··· ≥ αₖ ≥ 0.

This work was supported by the UQ NSRSF.
© 2013 Australian Mathematical Publishing Association Inc. 1446-7887/2013 $16.00
We shall identify the polynomial dominant weight $\phi_\alpha$ with the exponent $\alpha$. We can also identify $\phi_\alpha$ with Young diagram having $\alpha_i$ boxes in the $i$th row for all $i$. The sum $\alpha_1 + \cdots + \alpha_k$ will be denoted by $|\alpha|$.

Then, by theory of highest weight, polynomial dominant weights uniquely label complex irreducible polynomial representations of the general linear group, and we will let $V_\alpha^n$ denote the irreducible representation of $GL_k$ labeled by Young diagram $\alpha$, or equivalently, highest weight $\alpha$. See, for example, [3, Section 9].

1.3. The irreducible representation $V_\lambda^n$ of $GL_n$ labeled by Young diagram $\lambda$ is completely reducible as a $GL_n$ representation. By Schur’s lemma (for example, [1, Section 1.2]), for a pair of polynomial dominant weights $\lambda$ and $\mu$ of $GL_n$ and $GL_{n-2}$ respectively, the branching multiplicity of $V_\mu^{n-2}$ in $V_\lambda^n$ is equal to the dimension of the space

$$V_\lambda^n|_\mu = \text{Hom}_{GL_{n-2}}(V_\mu^{n-2}, V_\lambda^n)$$

(1.1)

of $GL_{n-2}$ homomorphisms, and then, as a $GL_{n-2}$ representation, $V_\lambda^n$ decomposes into isotypic components as

$$V_\lambda^n = \bigoplus_{\mu} V_\mu^{n-2} \otimes \text{Hom}_{GL_{n-2}}(V_\mu^{n-2}, V_\lambda^n)$$

(1.2)

where the summation runs over the highest weights $\mu$ of $V_\mu^{n-2}$ appearing in $V_\lambda^n$. In this sense, we call the space (1.1) a $GL_n$ to $GL_{n-2}$ branching multiplicity space.

1.4. After a brief review on the representations of $GL_2$ in Section 2, we describe the $GL_2$ module structure of $GL_n$ to $GL_{n-2}$ branching multiplicity spaces in Section 3. We develop a combinatorial procedure of tiling branching multiplicity spaces with $GL_2$ pattern blocks in Section 4. This procedure will show, in particular, how combinatorics of $GL_2$ representations can be intertwined to make branching rules under the restriction of $GL_n$ to $GL_{n-2}$. We will discuss analogous results for the branching of $Sp_{2n}$ to $Sp_{2n-2}$ and $SO_m$ to $SO_{m-2}$ in Section 5.

2. Irreducible representations of $GL_2$

In this section, we review algebraic and combinatorial models for $GL_2$ representations.

2.1. For a polynomial dominant weight $(x, z) \in \mathbb{Z}^2$ of $GL_2$, the irreducible representation with highest weight $(x, z)$ can be realized as

$$V_2^{(x, z)} = \mathbb{C} \otimes \text{Sym}^{x-z}(\mathbb{C}^2)$$

where $g \in GL_2$ acts on the spaces $\mathbb{C}$ and $\mathbb{C}^2$ via scaling by the factor of $\det(g)^2$ and matrix multiplication, respectively. Here, $\text{Sym}^d(\mathbb{C}^2)$ denotes the $d$th symmetric power of the space $\mathbb{C}^2$, and $\det(g)$ denotes the determinant of the matrix $g \in GL_2$. See, for example, [1, Section 15.5].
2.2. The irreducible representations of \( \text{GL}_k \) can be described in terms of Gelfand–Tsetlin patterns [2]. For \( \text{GL}_2 \), Gelfand–Tsetlin patterns for \( V_2^{(x,z)} \) are triangular arrays of the form

\[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix}
\]

with \( y \in \mathbb{Z} \) and \( x \geq y \geq z \), which can label weight basis vectors \( v \in V_2^{(x,z)} \),

\[
\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot v = (t_1^x t_2^{x+z-y}) v,
\]

for all diagonal matrices \( \text{diag}(t_1, t_2) \) of \( \text{GL}_2 \). See, for example, [3, Section 8.1] or [8]. Then the character of the \( \text{GL}_2 \) representation \( V_2^{(x,z)} \) is

\[
\text{ch}_{(x,z)}(t_1, t_2) = \sum_y t_1^y t_2^{x+z-y} \tag{2.1}
\]

where the summation runs over all integers \( y \) such that \( x \geq y \geq z \), or equivalently, over all Gelfand–Tsetlin patterns with top row \( (x, z) \).

2.3. We remark that if we restrict \( \text{GL}_2 \) to its subgroup \( \text{SL}_2 \), then \( V_2^{(x,z)} \) is isomorphic to \( \text{Sym}^{x-z}(\mathbb{C}^2) \). By taking \( t_1 = t \) and \( t_2 = r^{-1} \) in (2.1), its character can be given as

\[
\text{ch}_{(d)}(t) = t^{-d} + t^{-d+2} + \cdots + t^{d-2} + t^d
\]

where \( d = x - z \). See, for example, [1, Section 11.1] or [3, Section 2.3].

3. Branching multiplicity spaces

In this section, we study the \( \text{GL}_2 \) module structure of \( \text{GL}_n \) to \( \text{GL}_{n-2} \) branching multiplicity spaces.

3.1. Let us recall branching rules for \( \text{GL}_k \) down to \( \text{GL}_{k-1} \), under the embedding of \( \text{GL}_{k-1} \) in the upper left corner of \( \text{GL}_k \). For polynomial dominant weights \( \alpha \) and \( \beta \) of \( \text{GL}_k \) and \( \text{GL}_{k-1} \), respectively, we write \( \beta \sqsubseteq \alpha \) and say that \( \beta \) interlaces \( \alpha \), if

\[
\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{k-1} \geq \beta_{k-1} \geq \alpha_k.
\]

Lemma 3.1 ([3, Section 8.1], [8]). Let \( \alpha \) and \( \beta \) be polynomial dominant weights of \( \text{GL}_k \) and \( \text{GL}_{k-1} \), respectively.

1. The multiplicity of a \( \text{GL}_{k-1} \) irreducible representation \( V_{k-1}^{\beta} \) in \( V_k^{\alpha} \), as a \( \text{GL}_{k-1} \) representation, is at most one. It is precisely one when \( \beta \) interlaces \( \alpha \).

2. As a \( \text{GL}_{k-1} \times \text{GL}_1 \) representation, \( V_k^{\alpha} \) decomposes as

\[
V_k^{\alpha} = \bigoplus_{\beta \sqsubseteq \alpha} V_{k-1}^{\beta} \otimes V_1^{(|\alpha|-|\beta|)}
\]

where the summation runs over all \( \beta \) interlacing \( \alpha \).
Next, let us consider polynomial dominant weights \( \lambda \) and \( \mu \) of \( \text{GL}_n \) and \( \text{GL}_{n-2} \), respectively. We say that \( \mu \) \emph{doubly interlaces} \( \lambda \), if there exists a polynomial dominant weight \( \kappa \) of \( \text{GL}_{n-1} \) such that \( \mu \) interlaces \( \kappa \) and \( \kappa \) interlaces \( \lambda \), that is, \( \mu \subseteq \kappa \subseteq \lambda \). By applying the above lemma twice, it is straightforward to see the following proposition.

**Proposition 3.2.**

1. The irreducible representation \( V^\mu_{n-2} \) appears in \( V^\lambda_n \) as a \( \text{GL}_{n-2} \) representation if and only if \( \mu \) doubly interlaces \( \lambda \).
2. The multiplicity of \( V^\mu_{n-2} \) in \( V^\lambda_n \) is equal to the number of all possible \( \kappa \) satisfying \( \mu \subseteq \kappa \subseteq \lambda \).
3. As a \( \text{GL}_{n-2} \times \text{GL}_1 \times \text{GL}_1 \) representation, \( V^\lambda_n \) decomposes as
   \[
   V^\lambda_n = \bigoplus_{\mu \subseteq \kappa} \bigoplus_{\kappa \subseteq \lambda} V^\mu_{n-2} \otimes (V^{|\lambda|-|\mu|}_1 \otimes V^{|\lambda|-|\kappa|}_1)
   \]
   where the summation runs over all \( \mu \) doubly interlacing \( \lambda \) and \( \kappa \) satisfying \( \mu \subseteq \kappa \subseteq \lambda \).

By comparing (1.2) and Proposition 3.2, we can describe the branching multiplicity space \( V^\lambda_{|\mu|} = \text{Hom}_{\text{GL}_{n-2}}(V^\mu_{n-2}, V^\lambda_n) \) in terms of integral sequences \( \kappa \) such that \( \mu \subseteq \kappa \subseteq \lambda \), or arrays of the form

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\
\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_{n-1} \\
\mu_1 & \mu_2 & \cdots & \mu_{n-2}
\end{bmatrix}
\]

where the entries are weakly decreasing along the diagonals from left to right, which we will call \emph{interlacing patterns}.

**3.2.** Our next task is to show that every \( \text{GL}_n \) to \( \text{GL}_{n-2} \) branching multiplicity space can be factored into \( \text{GL}_2 \) representations. For polynomial dominant weights \( \lambda \) and \( \mu \) of \( \text{GL}_n \) and \( \text{GL}_{n-2} \) respectively, let \( IP(\lambda, \mu) \) be the set of interlacing patterns whose top and bottom rows are \( \lambda \) and \( \mu \) respectively. Also, for a sequence \( \sigma \) of weakly decreasing nonnegative integers

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2n-3} \geq \sigma_{2n-2},
\]

let \( GT(\sigma) \) be the set of all \((n-1)\)-tuples of Gelfand–Tsetlin patterns for \( \text{GL}_2 \) whose top rows are \((\sigma_{2i-1}, \sigma_{2i})\) for \( 1 \leq i \leq n-1 \).

**Theorem 3.3.** Let \( \lambda \) and \( \mu \) be polynomial dominant weights of \( \text{GL}_n \) and \( \text{GL}_{n-2} \), and \( \sigma = \sigma(\lambda, \mu) \) be the sequence \((x_1, z_1, \ldots, x_{n-1}, z_{n-1})\) obtained by rearranging the sequence

\[
(\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_{n-2})
\]
in weakly decreasing order, that is, \( x_1 \geq z_1 \geq \cdots \geq x_{n-1} \geq z_{n-1} \). Then, the map from \( IP(\lambda, \mu) \) to \( GT(\sigma) \) sending

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} & \lambda_n \\
\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_{n-1} \\
\mu_1 & \mu_2 & \cdots & \mu_{n-2}
\end{bmatrix}
\]

to

\[
\begin{bmatrix}
x_1 & z_1 \\
\kappa_1 \\
x_2 & z_2 \\
\kappa_2 \\
\vdots \\
x_{n-1} & z_{n-1} \\
\kappa_{n-1}
\end{bmatrix}
\]

is a bijection.

We will prove the theorem in the context of pattern-tiling in Proposition 4.3. Our proof will show in particular how combinatorics of \( GL_2 \) representations are intertwined to make branching rules under the restriction of \( GL_n \) to \( GL_{n-2} \). We also note that a direct proof can be given by using the observation that if \( \mu \) doubly interlaces \( \lambda \), then \( x_1 = \lambda_1, z_{n-1} = \lambda_n \), and

\[
z_j = \max(\lambda_{j+1}, \mu_j) \quad \text{and} \quad x_{j+1} = \min(\lambda_{j+1}, \mu_j)
\]

for \( 1 \leq j \leq n - 2 \).

As an immediate consequence of Theorem 3.3, since there are exactly \( x - z + 1 \) possible Gelfand–Tsetlin patterns with top row \((x, z)\), we have the following corollary.

**Corollary 3.4.** For \( \mu \) doubly interlacing \( \lambda \), the multiplicity of \( V_{\mu}^{\mu} \) in \( V_{\lambda}^{\lambda} \), or equivalently the dimension of the branching multiplicity space \( V_{\lambda}^{\lambda}|_{\mu} \), is

\[
\prod_{j=1}^{n-1}(x_j - z_j + 1)
\]

where the \( x_j \) and \( z_j \) are defined from the rearrangement \((x_1, z_1, \ldots, x_{n-1}, z_{n-1})\) of the sequence \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2})\) in weakly decreasing order.

We note that this formula can be derived from [12, Proposition 3.2]. See the remark after Theorem 3.5.

**3.3.** In the setting of Proposition 3.2, consider the diagonal block \( GL_2 \) complement to \( GL_{n-2} \) in \( GL_n \):

\[
\begin{bmatrix}
g_1 & 0 \\
0 & g_2
\end{bmatrix} \in GL_n
\]

where \( g_1 \in GL_{n-2} \) and \( g_2 \in GL_2 \). This \( GL_2 \) commutes with \( GL_{n-2} \) acting on \( V_{\mu}^{\mu} \) in (1.2), and therefore, the \( GL_n \) to \( GL_{n-2} \) branching multiplicity space carries the structure of a \( GL_2 \) module.
Theorem 3.5. For μ doubly interlacing λ, the \( GL_n \) to \( GL_{n-2} \) branching multiplicity space \( V^{|\mu|}_n \nu \) is, as a \( GL_2 \) representation, isomorphic to the tensor product of \( GL_2 \) irreducible representations

\[
\text{Hom}_{GL_{n-2}}(V^{|\mu|}_{n-2}, V^{|\nu|}_n) \cong \mathbb{C} \otimes V^{|x_1|}_{2} \otimes V^{|x_2|}_{2} \otimes \cdots \otimes V^{|x_{n-1}|}_{2} \otimes V^{|z_{n-1}|}_{2}
\]

where \( \mathbb{C} \) is the one-dimensional representation given by \( \det(g)^{-|\mu|} \) for \( g \in GL_2 \); and \( x_j \) and \( z_j \) are defined from the rearrangement \( (x_1, z_1, \ldots, x_{n-1}, z_{n-1}) \) of the sequence \( (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2}) \) in weakly decreasing order.

Proof. By taking \( GL_1 \times GL_1 \) in Proposition 3.2 as a maximal torus of \( GL_2 \), we can consider the following formula as the \( GL_2 \) character of the branching multiplicity space

\[
\text{ch}(V^{|\mu|}_n) = \sum_{\kappa} t_1^{\kappa - |\mu|} t_2^{n - 1 - |\kappa|}
\]

where \( (t_1, t_2) \in GL_1 \times GL_1 \) and the summation runs over all \( \kappa \) such that \( \mu \subseteq \kappa \subseteq \lambda \). Then

\[
(t_1 t_2)^{|\mu|} \cdot \text{ch}(V^{|\mu|}_n) = \sum_{\kappa} t_1^{\kappa - |\mu|} t_2^{n - 1 - |\kappa|} = \sum_{\kappa} t_1^{(\kappa_1 + \cdots + \kappa_{n-1})} t_2^{(x_1 + z_1 + \cdots + x_{n-1} + z_{n-1}) - (\kappa_1 + \cdots + \kappa_{n-1})} = \prod_{j=1}^{n-1} \sum_{\kappa_j} t_1^{\kappa_j} t_2^{x_j + z_j - \kappa_j}
\]

and, by Theorem 3.3, \( x_j \geq \kappa_j \geq z_j \) for each \( j \). This shows that \( \text{ch}(V^{|\mu|}_n) \) is the product of \( (t_1 t_2)^{-|\mu|} \), the character of the one-dimensional representation twisted by \( \det(g)^{-|\mu|} \), and the characters of the \( V^{|x_j|}_2 \). This finishes our proof. \( \square \)

The following \( SL_2 \) module structure of the branching multiplicity space was studied by Yacobi in his thesis (see [12, Proposition 3.2]):

\[
\text{Hom}_{SL_{n-2}}(V^{|\mu|}_{n-2}, V^{|\nu|}_n) \cong \text{Sym}^{x_1 - z_1}(\mathbb{C}^2) \otimes \cdots \otimes \text{Sym}^{x_{n-1} - z_{n-1}}(\mathbb{C}^2).
\]

Our theorem can be understood as a result obtained by lifting \( SL_2 \) to \( GL_2 \).

4. Tiling branching multiplicity spaces

In this section we develop a combinatorial procedure of tiling branching multiplicity spaces with Gelfand–Tsetlin patterns for \( GL_2 \), thereby proving Theorem 3.3.

4.1. First, in order to consider some directed paths in a graph, we place vertices on the coordinate plane as

\[
P_n = \{(a, b) : b = 0, 1 \leq a \leq n\} \cup \{(a, b) : b = -1, 2 \leq a \leq n - 1\}.
\]
For example, $P_7$ is

Then we consider directed paths from $u = (1, 0)$ to $v = (n, 0)$ in $2n - 3$ steps visiting each point in $P_n$ exactly once, when we are only allowed to move right($\rightarrow$) or up($\uparrow$) or down($\downarrow$) or up-right($\nearrow$) or down-right($\searrow$) at each step.

**Example 4.1.** These are two paths for $P_6$ out of 16 possible ones.

Each directed path can be presented by a sequence of allowed steps. For example, the two paths for $P_6$ in Example 4.1 can be presented as, respectively,

$$[\ \downarrow \ \uparrow \ \rightarrow \ \downarrow \ \rightarrow \ \uparrow \ \downarrow \ \uparrow \ \rightarrow \ ],$$
$$[\ \rightarrow \ \downarrow \ \nearrow \ \rightarrow \ \downarrow \ \rightarrow \ \uparrow \ \rightarrow \ ].$$  

At each step of a path, it is clear whether we are on the line $y = 0$ or the line $y = -1$; and if we are on $y = 0$ then the next step should be down($\downarrow$), and if we are on $y = -1$ then the next step should be up($\uparrow$). Therefore, in presenting directed paths for $P_n$ from $(1, 0)$ to $(n, 0)$, we may omit up($\uparrow$) and down($\downarrow$) arrows. Then, by denoting moving right($\rightarrow$) on the line $y = 0$ and on the line $y = -1$ by harpoon-up($\leftrightarrow$) and harpoon-down($\rightarrow$), respectively, we can present every path uniquely with the following four arrows:

$\downarrow$, $\rightarrow$, $\rightarrow$, $\nearrow$.

**4.2.** From this observation, we define *pattern blocks* attached to arrows and a *tiling* given by a directed path.
4.2. For each $i$ with $1 \leq i \leq n - 1$, the $i$th pattern block corresponding to the down-right, harpoon-up, harpoon-down and up-right arrows is

\[
\begin{array}{c|c|c|c|c}
\downarrow & \rightarrow & \rightarrow & \uparrow \\
x_i & x_i & z_i & y_i & y_i \\
y_i & z_i & y_i & x_i & \varepsilon_i
\end{array}
\]

(2) For each directed path from $(1, 0)$ to $(n, 0)$ of $P_n$, its tiling is the concatenation of pattern blocks defined by the sequence of arrows presenting the path such that:

(a) $y_i$ is at coordinate $(i + 0.5, -0.5)$;
(b) $x_i$ and $z_i$ above $y_i$ are at coordinates $(i, 0)$ and $(i + 1, 0)$, respectively;
(c) $x_i$ and $z_i$ below $y_i$ are at coordinates $(i, -1)$ and $(i + 1, -1)$, respectively for $1 \leq i \leq n - 1$.

With this definition, the two paths given in Example 4.1 can be given as

\[
\begin{bmatrix}
\downarrow & \rightarrow & \rightarrow & \downarrow & \rightarrow \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{bmatrix}
\]

and the corresponding tilings are

\[
\begin{bmatrix}
x_1 & x_2 & z_2 & x_4 & x_5 & z_5 \\
y_1 & y_2 & y_3 & y_4 & y_5 \\
z_1 & x_3 & z_3 & z_4 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
x_1 & z_1 & z_2 & z_3 & x_5 & z_5 \\
y_1 & y_2 & y_3 & y_4 & y_5 \\
x_2 & x_3 & x_4 & z_4 \\
\end{bmatrix}
\]

respectively.

4.3. For each tiling, we identify two subsequences of $(x_1, z_1, \ldots, x_{n-1}, z_{n-1})$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the subsequence on the line $y = 0$; and $\mu = (\mu_1, \ldots, \mu_{n-2})$ be the subsequence on the line $y = -1$. In the above example, $\lambda$ and $\mu$ are, respectively,

\[
\lambda = (x_1, x_2, z_2, x_4, x_5, z_5) \quad \text{and} \quad \mu = (z_1, x_3, z_3, z_4); \\
\lambda = (x_1, z_1, z_2, z_3, x_5, z_5) \quad \text{and} \quad \mu = (x_2, x_3, x_4, z_4).
\]

We note that, with the order $x_1 \geq z_1 \geq x_2 \geq z_2 \geq \cdots$, the entries of the sequences $\lambda$ and $\mu$ satisfy the identities (3.1).
The following proposition shows that the tiling procedure given in Definition 4.2 provides the correspondence stated in Theorem 3.3.

**Proposition 4.3.**

1. For a given tiling, let us impose the order
   
   \[ x_1 \geq z_1 \geq x_2 \geq z_2 \geq \cdots \geq x_{n-1} \geq z_{n-1}, \]

   on the entries \( x_i \) and \( z_i \) of pattern blocks, and let \( \lambda \) and \( \mu \) be its subsequences placed on the lines \( y = 0 \) and \( y = -1 \), respectively. If \( y_i \) satisfies \( x_i \geq y_i \geq z_i \) for each pattern block, then \( \mu \subseteq (y_1, \ldots, y_{n-1}) \subseteq \lambda \), that is, for all \( r \) and \( s \),
   
   \[ \lambda_r \geq y_r \geq \lambda_{r+1} \quad \text{and} \quad y_s \geq \mu_s \geq y_{s+1}. \]

2. Conversely, let an interlacing pattern \( \mu \subseteq (y_1, \ldots, y_{n-1}) \subseteq \lambda \) be given. If we place its entries \( \lambda_i, \mu_j \) and \( y_k \) on coordinates \((i, 0), (j+1, -1)\) and \((k+0.5, -0.5)\) for all \( i, j \) and \( k \), then we obtain a tiling defined by the directed path connecting the \( \lambda_i \) and \( \mu_j \) in weakly decreasing order. That is, if \((x_1, z_1, \ldots, x_{n-1}, z_{n-1})\) is the rearrangement of the sequence \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_{n-2})\) in weakly decreasing order, then \( x_i, y_i \) and \( z_i \) form a pattern block and satisfy
   
   \[ x_i \geq y_i \geq z_i \]

   for \( 1 \leq i \leq n - 1 \).

**Proof.** It is enough to check out the inequalities for all possible pairs of consecutive pattern blocks in a tiling listed below. Note that these are also all possible partial interlacing patterns with two triples \((x, y, z)\) and \((x', y', z')\).

\[
\begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix} \quad \begin{bmatrix}
  x & x' & z' \\
  y & y' & z
\end{bmatrix} \quad \begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix} \\
\begin{bmatrix}
  x' & z' \\
  y & y' \\
  z & z'
\end{bmatrix} \quad \begin{bmatrix}
  x & x' & z' \\
  y & y' & z
\end{bmatrix} \quad \begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix} \\
\begin{bmatrix}
  x' & z' \\
  y & y' \\
  z & z'
\end{bmatrix} \quad \begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix} \\
\begin{bmatrix}
  x' & z' \\
  y & y' \\
  z & z'
\end{bmatrix} \quad \begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix} \\
\begin{bmatrix}
  x' & z' \\
  y & y' \\
  z & z'
\end{bmatrix} \quad \begin{bmatrix}
  x & x' \\
  y & y' \\
  z & z'
\end{bmatrix}
\]

In the first case, \((\lambda_1, \lambda_2) = (x, x')\) and \((\mu_1, \mu_2) = (z, z')\). With \( x \geq z \geq x' \geq z' \), we have \( x \geq y \geq z \) and \( x' \geq y' \geq z' \) if and only if

\[ x \geq y \geq x' \geq y' \quad \text{and} \quad y \geq z \geq y' \geq z'. \]
In the second case, \((\lambda_1, \lambda_2, \lambda_3) = (x, x', z')\) and \(\mu_1 = z\). With \(x \geq z \geq x' \geq z'\), we have \(x \geq y \geq z\) and \(x' \geq y' \geq z'\) if and only if
\[
x \geq y \geq x' \geq y' \geq z' \quad \text{and} \quad y \geq z \geq y'.
\]
The rest of the cases can be shown similarly. \(\square\)

4.4. We give an example illustrating tiling procedures, and therefore showing the \(GL_2\) module structure of branching multiplicity spaces. Let us consider polynomial dominant weights \((x_i, z_i) \in \{(8, 5), (4, 2), (1, 0)\}\) of \(GL_2\), and Gelfand–Tsetlin patterns
\[
\left(\begin{bmatrix} 8 & 5 & 2 & 0 \\ y_1 & y_2 & y_3\end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 0\end{bmatrix}\right)
\]
where \(y_i \in \mathbb{Z}\) varies for \(x_i \geq y_i \geq z_i\) for all \(i\).

In order to assemble these \(GL_2\) pattern blocks to build \(GL_4\) to \(GL_2\) branching multiplicity spaces, we consider all the directed paths for \(P_4\).

Using down-right, up-right, harpoon-up and harpoon-down arrows, they can be represented as
\[
[ \rightarrow \uparrow \uparrow ] \quad [ \rightarrow \rightarrow \rightarrow ] \\
[ \downarrow \rightarrow \uparrow ] \quad [ \downarrow \downarrow \rightarrow ].
\]

Then, from Definition 4.2, we obtain the tilings
\[
\begin{bmatrix} 8 & 5 & 2 & 0 \\ y_1 & y_2 & y_3 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 0\end{bmatrix}
\]
\[
\begin{bmatrix} 8 & 4 & 2 & 0 \\ y_1 & y_2 & y_3 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 0\end{bmatrix}
\]
\[
\begin{bmatrix} 8 & 4 & 2 & 0 \\ y_1 & y_2 & y_3 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 1 & 0\end{bmatrix}
\]
corresponding to the branching multiplicity spaces
\[
\text{Hom}_{GL_2}(V_2^{(4,1)}, V_4^{(8,5,2,0)}), \quad \text{Hom}_{GL_2}(V_2^{(4,2)}, V_4^{(8,5,1,0)})
\]
\[
\text{Hom}_{GL_2}(V_2^{(5,1)}, V_4^{(8,4,2,0)}), \quad \text{Hom}_{GL_2}(V_2^{(5,2)}, V_4^{(8,4,1,0)})
\]
which are, by Theorem 3.5, as GL₂ representations, isomorphic to
\[ C \otimes V_{2}^{(8,5)} \otimes V_{2}^{(4,2)} \otimes V_{2}^{(1,0)} \]
where \( g \in \text{GL}_2 \) acts on \( C \) by \( \det(g)^{-5}, \det(g)^{-6}, \det(g)^{-6} \) and \( \det(g)^{-7} \), respectively.

We note that if some of the entries in the sequence \( (x_1, z_1, \ldots, x_{n-1}, z_{n-1}) \) are equal, then different paths may give the same tiling, and therefore the same branching multiplicity space.

5. Branching multiplicity spaces of other classical groups

As in the case of the general linear group, we can study the GL₂ module structure of branching multiplicity spaces for the symplectic group. We can also obtain similar results for the orthogonal group within certain stable ranges. For more about stable range conditions in branching rules for classical groups, we refer readers to [4].

5.1. We denote the complex symplectic group of rank \( n \) and the complex special orthogonal group of rank \( \lfloor m/2 \rfloor \) by \( \text{Sp}_{2n} \) and \( \text{SO}_{m} \), respectively. The dominant weights of \( \text{Sp}_{2n} \) are of the form \( \lambda \in \mathbb{Z}^n \) with \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \); and the dominant weights of \( \text{SO}_{2n} \) are of the same form with \( \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \).

We will state branching rules for individual cases (see, for example, [1, Section 25.3] or [3, Section 8.1]) with the convention of Gelfand–Tsetlin patterns, that is, the entries in each array are weakly decreasing along the diagonals from left to right.

5.2. Let \( W_{2n}^{\lambda} \) be the irreducible representation of \( \text{Sp}_{2n} \) with highest weight \( \lambda \). Then for a dominant weight \( \mu \) of \( \text{Sp}_{2n-2} \), the multiplicity of \( W_{2n-2}^{\mu} \) in \( W_{2n}^{\lambda} \) as a \( \text{Sp}_{2n} \) representation is equal to the number of \( \text{Sp}_{2n} \) dominant weights \( \kappa \) such that
\[
\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\
\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_n \\
\mu_1 & \mu_2 & \cdots & \mu_{n-1}
\end{bmatrix}
\]
Note that we can identify this \( \text{Sp}_{2n} \) to \( \text{Sp}_{2n-2} \) branching rule with the \( \text{GL}_{n+1} \) to \( \text{GL}_{n-1} \) branching rule in Proposition 3.2. Therefore, as GL₂ representations,
\[
\text{Hom}_{\text{Sp}_{2n-2}}(W_{2n-2}^{\mu}, W_{2n}^{\lambda}) \cong \text{Hom}_{\text{GL}_{n+1}}(V_{n-1}^{\mu}, V_{n+1}^{\lambda'})
\]
where \( \lambda' = (\lambda_1, \ldots, \lambda_n, 0) \) (see [12, Theorem 3.1]). Then we can apply Theorem 3.3 to tile the \( \text{Sp}_{2n} \) to \( \text{Sp}_{2n-2} \) branching multiplicity space with GL₂ pattern blocks. From Theorem 3.5, we can express the branching multiplicity space as a tensor product of GL₂ representations. Also, by restricting GL₂ to its subgroup SL₂ and using the explanation in Section 2.3, we can obtain the SL₂ module structure of the branching multiplicity space.

We remark that Wallach and Yacobi studied \( \text{Sp}_{2n} \) to \( \text{Sp}_{2n-2} \) branching multiplicity spaces with \( \text{Sp}_2 = \text{SL}_2 \) and \( n \) copies of \( \text{SL}_2 \) in [10, 12], and Yacobi and the present author studied their algebraic and combinatorial properties in [5].
5.3. Let $W^l_{2n+1}$ be the irreducible representation of $SO_{2n+1}$ with highest weight $\lambda$. Then for a dominant weight $\mu$ of $SO_{2n-1}$, the multiplicity of $W^\mu_{2n-1}$ in $W^l_{2n+1}$ as a $SO_{2n-1}$ representation is equal to the number of dominant weights $\kappa$ of $SO_{2n}$ such that

$$\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\
\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_n \\
\mu_1 & \mu_2 & \cdots & \mu_{n-1}
\end{bmatrix} = \kappa_n,$$

Note that if $\mu_{n-1} = 0$, then the interlacing condition makes $\kappa_n = 0$, and this branching rule becomes exactly the same as the $GL_n$ to $GL_{n-2}$ branching rule in Proposition 3.2. Therefore, if $\mu_{n-1} = 0$, as $GL_2$ representations,

$$\text{Hom}_{SO_{2n-1}}(W^\mu_{2n-1}, W^l_{2n+1}) \cong \text{Hom}_{GL_{n-2}}(V^\mu_{n-2}, V^l_n)$$

where $\mu' = (\mu_1, \ldots, \mu_{n-2})$. Similarly, if $\lambda_n = 0$, as $GL_2$ representations,

$$\text{Hom}_{SO_{2n-1}}(W^\mu_{2n-1}, W^l_{2n+1}) \cong \text{Hom}_{GL_{n-1}}(V^\mu_{n-1}, V^l_{n+1})$$

where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1}, 0, 0)$. Then, we can apply Theorems 3.3 and 3.5 to tile the $SO_{2n+1}$ to $SO_{2n-1}$ branching multiplicity space with $GL_2$ pattern blocks and to factor it into $GL_2$ representations or $SL_2$ representations.

5.4. Let $W^l_{2n}$ be the irreducible representation of $SO_{2n}$ with highest weight $\lambda$. Then for a dominant weight $\mu$ of $SO_{2n-2}$, the multiplicity of $W^\mu_{2n-2}$ in $W^l_{2n}$ as a $SO_{2n-2}$ representation is equal to the number of dominant weights $\kappa$ of $SO_{2n-1}$ such that

$$\begin{bmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n-1} \\
\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_{n-2} \\
\mu_1 & \mu_2 & \cdots & \mu_{n-2}
\end{bmatrix} = \kappa_{n-1}.$$

If $\mu_{n-2} = 0$, then the interlacing condition makes $\kappa_{n-1} = \lambda_n = \mu_{n-1} = 0$ and this branching rule becomes exactly the same as the $GL_{n-1}$ to $GL_{n-3}$ branching rule in Proposition 3.2. Therefore, if $\mu_{n-2} = 0$, then, as $GL_2$ representations,

$$\text{Hom}_{SO_{2n-2}}(W^\mu_{2n-2}, W^l_{2n}) \cong \text{Hom}_{GL_{n-3}}(V^\mu_{n-3}, V^l_{n-1})$$

where $\mu' = (\mu_1, \ldots, \mu_{n-3})$ and $\lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. Similarly, if $\lambda_{n-1} = 0$, then $\kappa_{n-1} = \lambda_n = \mu_{n-1} = 0$ and as $GL_2$ representations,

$$\text{Hom}_{SO_{2n-2}}(W^\mu_{2n-2}, W^l_{2n}) \cong \text{Hom}_{GL_{n-2}}(V^\mu_{n-2}, V^l_n)$$

where $\mu'' = (\mu_1, \ldots, \mu_{n-2})$ and $\lambda'' = (\lambda_1, \ldots, \lambda_{n-2}, 0, 0)$. Then we can apply Theorems 3.3 and 3.5 to tile the $SO_{2n}$ to $SO_{2n-2}$ branching multiplicity space with $GL_2$ pattern blocks and to factor it into $GL_2$ representations or $SL_2$ representations.

Acknowledgements

The author thanks the late Professor Daya-Nand Verma for inspiring discussions regarding several aspects of this work. He also thanks Piu Andamiro and Mishima Heihachi for their helpful suggestions and comments.
References


SANGJIB KIM, Department of Mathematics, Ewha Womens University, Seoul 120-750, South Korea
e-mail: sangib@gmail.com