Compositio Mathematica **138:** 113–124, 2003.

Almost Squares and Factorisations in Consecutive Integers

N. SARADHA and T. N. SHOREY
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India. e-mail: {saradha;shorey}@math.tifr.res.in

(Received: 29 May 2001; accepted in final form: 22 March 2002)

Abstract. We show that there is no square other than 12^2 and 720^2 such that it can be written as a product of $k-1$ integers out of k ($\geqslant 3$) consecutive positive integers. We give an extension of a theorem of Sylvester that a product of k consecutive integers each greater than k is divisible by a prime exceeding k.

Mathematics Subject Classification (2000). Primary: 11D61.

Key words. consecutive, diophantine equations, elliptic equations, factorisation, primes.

1. Introduction

Let $\ell \geqslant 2$ be a prime number. Erdős and Selfridge [3] proved that a product of $k \geqslant 2$ consecutive positive integers is never an ℓth power. Further Saradha and Shorey [12] showed that there is no ℓth power with $\ell > 2$ other than 8 such that it can be written as a product of $k-1$ integers out of $k \geqslant 3$ consecutive positive integers. This settled a question of Erdős and Selfridge [3, p. 300]. In this paper, we solve the analogous question for $\ell = 2$ of Erdős and Selfridge [3, p. 300] where they observed that $6!/5 = 12^2$, $10!/7 = 720^2$. We prove

THEOREM 1. *Let $k \geqslant 3$. There is no square other than 12^2 and 720^2 such that it can be written as a product of $k-1$ integers out of k consecutive positive integers.*

It is clear that the assumption $k \geqslant 3$ is necessary in Theorem 1. For an integer $v > 1$, we write $P(v)$ and $\omega(v)$ for the greatest prime factor of v and the number of distinct prime divisors of v, respectively. Further we put $P(1) = 1$ and $\omega(1) = 0$. We shall denote by b, n, k and y positive integers such that b is square free. We always write p for a prime number. We shall prove a more general result.

THEOREM 2. *Let $k \geqslant 4$, $P(b) \leqslant k$, $n > k^2$ and $0 < i < k-1$ be an integer. Suppose that*

$$n(n+1)\cdots(n+i-1)(n+i+1)\cdots(n+k-1) = by^2. \tag{1}$$

Then

$$(n, k, y, b, i) = (24, 4, 90, 2, 2).$$

It is clear that we should assume in Theorem 2 that $P(y) > k$ otherwise (1) has infinitely many solutions and (1) with $P(y) > k$ implies that $n > k^2$. Further, we see that $k \geqslant 3$ since $0 < i < k - 1$. We rewrite $n(n+2) = 2y^2$ as Pell's equations $|\, y_1^2 - 2y_2^2 \,| = 1$ to observe that it has infinitely many solutions. Thus the assumption $k \geqslant 4$ in Theorem 2 is necessary. If $i = 0$ or $k - 1$, the left-hand side of (1) is a product of $k - 1$ consecutive positive integers. Then we have

THEOREM A. *Let $k \geqslant 3, n > k^2$ and $P(b) \leqslant k$. The equation*

$$n(n+1)\cdots(n+k-1) = by^2 \tag{2}$$

has no solution in positive integers n, k and y such that $k \geqslant 4$. If $k = 3$, then (2) *implies that* $(n, y, b) = (48, 140, 6)$.

Theorem A with $k \geqslant 4$ and $P(b) < k$ was proved by Erdős and Selfridge [3]. The assumption $P(b) < k$ has been relaxed to $P(b) \leqslant k$ by Saradha [10]. Theorem A with $k = 3$ is a consequence of some old diophantine results. Now we combine Theorems A and 2 to obtain

COROLLARY 1. *Let $k \geqslant 4$ and $n > k^2$. Assume that (n, k) is different from* $(24, 4), (47, 4), (48, 4)$. *Then there exist distinct primes $p_1 > k$ and $p_2 > k$ such that the maximal power of each of p_1 and p_2 dividing $n(n+1)\cdots(n+k-1)$ is odd.*

It is not known whether there are infinitely many primes p such that $p^2 - 1 = 2y^2$. Thus the case $k = 3$ remain open. In the case $k = 2$, if (u, v) is a solution of the Pell's equation $u^2 - 2v^2 = 1$ and $n = 2v^2$, then $n(n+1) = 2(uv)^2$. Hence, the corollary does not hold for $k = 2$. An immediate consequence of Corollary 1 is the following improvement of Theorem A.

COROLLARY 2. *Let $k \geqslant 3, n > k^2$ and $P(b) \leqslant p_k$ where p_k denotes the least prime exceeding k. Then* (2) *implies that* $(n, k, y, b) = (48, 3, 140, 6)$.

The assumption $k \geqslant 3$ in Corollary 2 is necessary since the Pell's equations $|\, y_1^2 - 3y_2^2 \,| = 1$ has infinitely many solutions.

Sylvester [15] proved that $P(n(n+1)\cdots(n+k-1)) > k$ if $n > k$. For deriving Theorem 1 from Theorem 2, we show that the product on the left-hand side of (1) with $n > k$ is divisible by a prime exceeding k. In fact, we show that the proof of Erdős [1] of Sylvester's theorem allows to prove the following stronger version.

THEOREM 3. *Let* $k \geqslant 3, n > k$ *and* μ *a positive integer such that*

$$\mu \geqslant k - \left[\frac{1}{3}\pi(k)\right] - 1 \tag{3}$$

where $\pi(k)$ *denotes the number of primes* $\leqslant k$. *Let* $e_1 < \cdots < e_\mu$ *be integers in* $[0, k)$ *such that*

$$P((n+e_1)\cdots(n+e_\mu)) \leqslant k.$$

Then $k \leqslant 17$ *and*

$$\begin{aligned} &n \in \{4,6,7,8,16\} \quad \textit{if } k=3;\ n \in \{6\} \textit{ if } k=4;\\ &n \in \{6,7,8,9,12,14,15,16,23,24\} \quad \textit{if } k=5;\\ &n \in \{7,8,15\} \quad \textit{if } k=6;\ n \in \{8,9,10,12,14,15,24\} \textit{ if } k=7;\\ &n \in \{9,14\} \quad \textit{if } k=8. \end{aligned} \tag{4}^*$$

We observe that there are infinitely many n satisfying the assertion of Theorem 3 if $k = 2$ and it is necessary to exclude the cases given in (4). Further we see from Prime Number Theorem that $\frac{1}{3}$ in Theorem 3 cannot be replaced by a number larger than 1. An immediate consequence of Theorem 3 is the following result.

COROLLARY 3. *Let* $n > k \geqslant 3$.

(i) $\omega(n(n+1)\cdots(n+k-1)) \geqslant \pi(k) + [\frac{1}{3}\pi(k)] + 2$, *except when* n, k *take values as in* (4).

(ii) $\omega(n(n+1)\cdots(n+k-1)) \geqslant \pi(k) + 2$, *unless* $n \in \{4, 6, 7, 8, 16\}$ *if* $k = 3$; $n \in \{6\}$ *if* $k = 4$; $n \in \{6, 8\}$ *if* $k = 5$.

In view of the result of Sylvester stated above, the assumption $n > k^2$ in Theorem A can be replaced by $n > k$. Further analogues of Theorem 2 and Corollaries 1,2 follow immediately from Corollary 3(ii) whenever $k < n \leqslant k^2$. These results continue to be valid for $n \leqslant k$ if $\pi(n+k-1) - \pi(k) \geqslant 2$ and we refer to studies on number of primes in short intervals for the latter inequality. Corollaries 1 and 3(ii) have been applied in [13] that a product of four or more positive integers in arithmetic progression with common difference a prime power is never a square. For convenience, we shall prove the following equivalent version of Theorem 3.

THEOREM 3′. *Let* $k \geqslant 3$, x *and* μ *be positive integers satisfying* $x \geqslant 2k$ *and* (3). *Let* $f_1 < \cdots < f_\mu$ *be integers in* $[0, k)$ *such that*

$$P((x-f_1)\cdots(x-f_\mu)) \leqslant k. \tag{5}$$

Then $k \leqslant 8$ *and*

$$\begin{aligned} &x \in \{6,8,9,10,18\} \quad \textit{if } k=3;\ x \in \{9\} \textit{ if } k=4;\\ &x \in \{10,11,12,13,16,18,19,20,27,28\} \quad \textit{if } k=5;\\ &x \in \{12,13,20\} \quad \textit{if } k=6;\ x \in \{14,15,16,18,20,21,30\} \textit{ if } k=7;\\ &x \in \{16,21\} \quad \textit{if } k=8. \end{aligned} \tag{6}$$

*For exceptions when $8 < k \leqslant 17$, see Remark at the end.

https://doi.org/10.1023/A:1025480729778 Published online by Cambridge University Press

The equivalence of Theorems 3 and 3′ can be easily seen by taking $n+k-1=x$ and $e_i = k-1-f_{\mu+1-i}$ for $1 \leqslant i \leqslant \mu$.

For a survey of results related to (1) and (2), we refer to [14].

2. Lemmas

We begin with a lemma for the proof of Theorem 2.

LEMMA 1. *Let* (1) *be satisfied with* b, n, k, y *and* i *as given in Theorem* 2. *Then* $k \leqslant 9$ *or* $k \in \{11, 13, 19\}$.

Proof. Let

$$k \geqslant 10 \quad \text{and} \quad k \notin \{11, 13, 19\}. \tag{7}$$

By (1), we have

$$n+j = a_j x_j^2, a_j \text{ square free}, \quad P(a_j) \leqslant k \text{ for } 0 \leqslant j < k, \; j \neq i.$$

Suppose $a_{j_1} = a_{j_2}$ for some $j_2 > j_1$. Then $x_{j_2} > x_{j_1}$ and

$$\begin{aligned} k-1 &\geqslant j_2 - j_1 = (n+j_2) - (n+j_1) = a_{j_1}(x_{j_2}^2 - x_{j_1}^2) \\ &> 2a_{j_1} x_{j_1} \geqslant 2(a_{j_1} x_{j_1}^2)^{\frac{1}{2}} \geqslant 2n^{\frac{1}{2}} > 2k, \end{aligned}$$

a contradiction. Hence all the a_j's for $0 \leqslant j < k, j \neq i$ are distinct. For any integer $m \geqslant 1$, we denote by $f(k, m)$ the number of a_j's with $0 \leqslant j < k$ and $j \neq i$ composed of the first m primes $2 = p_1 < p_2 < \cdots < p_m$. Then

$$f(k,m) \geqslant f_0(k,m) =: k-1-\sum_{i \geqslant m+1}\left(\left[\frac{k}{p_i}\right]+\epsilon_i\right),$$

where $\epsilon_i = 0$ if either $p_i > k$ or if $p_i \mid k$ and $\epsilon_i = 1$ otherwise. Since a_j's are square free we see that $f(k, m) \leqslant 2^m$ and hence

$$f_0(k,m) \leqslant 2^m. \tag{8}$$

We observe that

$$\prod a_j \geqslant \prod_{j=1}^{k-1} s_j \geqslant (1.5)^{k-1}(k-1)! \quad \text{for } k \geqslant 64, \tag{9}$$

where $\prod a_j$ is the product over all a_j's and s_j denotes the jth square free integer. We refer to [3] and [11, p. 32] for the above inequaliy. On the other hand, for a prime $p \leqslant k$, the number of a_j's divisible by p does not exceed $[(k-1)/p]+1$ and thus $\prod a_j \mid \prod_{p \leqslant k} p^{[\frac{k-1}{p}]+1}$ implying that $\prod a_j \mid (k-1)! \prod_{p \leqslant k} p$. Now we follow the argument in [11, Lemma 3] to get

$$\prod a_j \leqslant 153819970\, k^{16} (2.78)^k (2.8819)^{-k} (k-1)!. \tag{10}$$

Comparing (9) and (10), we get $(1.5549)^k \leqslant 230729960\, k^{16}$ which implies that $k \leqslant 250$. Now we check that

$$f_0(k,3) \geqslant 9 \text{ for } 25 \leqslant k \leqslant 78; \quad f_0(k,4) \geqslant 17 \text{ for } 79 \leqslant k \leqslant 250;$$
$$f_0(10,2) \geqslant 5; f_0(21,3) \geqslant 9; \quad f_0(22,3) \geqslant 9$$

and we conclude by (8) that

$$k \leqslant 24, \quad k \neq 10, 21, 22. \tag{11}$$

Further we find that

$$f_0(k,3) \geqslant 8 \text{ if } k \in \{23, 24\}, \quad f_0(k,2) \geqslant 4 \text{ if } k \leqslant 20.$$

Let $k = 24$. Then by (8), we have $f_0(k,3) = 8$. Hence, the primes 23, 19, 17, 13, 11, 7 divide exactly 2, 2, 2, 2, 3, 4 distinct a_j's and none of these a_j's is divisible by more than one of these primes. Therefore 23 divides a_0, a_{23}; 7 divides a_1, a_8, a_{15}, a_{22}. Then 11 does not divide three other a_j's. This is a contradiction. In this way, we exclude all k given by (7) and (11) except $k = 17$. When $k = 17$, we have $f_0(k,2) = 4$ and the primes 17, 13, 11,7, 5 divide exaclty 1, 2, 2, 3, 4 distinct a_j's. Hence, 5 divides either a_0, a_5, a_{10}, a_{15} or a_1, a_6, a_{11}, a_{16}. In the former case we see that 7 divides a_2, a_9, a_{16} and 13 divides a_1, a_{14}. In the latter case, 7 divides a_0, a_7, a_{14} and 13 divides a_2, a_{15}. Then 11 does not divide two other a_j's in both the cases, a contradiction. □

The following lemmas are used in the proof of Theorem 3′.

LEMMA 2. *For* $x > 1$,

(i) $\pi(x) < \dfrac{x}{\log x} + \dfrac{1.5x}{\log^2 x}$

(ii) $\pi(2x) - \pi(x) \geqslant \dfrac{3x}{5\log x}$ *for* $x \geqslant 20.5$

(iii) $\prod_{p^a \leqslant x} p^a < (2.83)^x$.

See [9, p. 69–71] for the above assertions.

LEMMA 3. *Let* $k \geqslant 3$. *Suppose that the assumptions of Theorem* 3′ *hold and* $x < k^{3/2}$. *Then*

$$\binom{x}{k} \leqslant (2.83)^{k+\sqrt{x}}\, x^{k-\mu}. \tag{12}$$

Proof. We follow the argument of Erdős [1]. For any prime p with $p^a \| \binom{x}{k}$, one has $p^a \leqslant x$. Therefore

$$\prod_{p \leqslant k} p^{\mathrm{ord}_p \binom{x}{k}} \leqslant \prod_{\substack{p \leqslant k \\ p^a \leqslant x}} p^a \leqslant \prod_{p \leqslant k} p \prod_{p \leqslant x^{\frac{1}{2}}} p \prod_{p \leqslant x^{\frac{1}{3}}} p \cdots. \tag{13}$$

https://doi.org/10.1023/A:1025480729778 Published online by Cambridge University Press

From Lemma 2 (iii) with x replaced by $x^{\frac{1}{2}}$ and k, we get

$$\prod_{p\leqslant x^{\frac{1}{2}}} p \prod_{p\leqslant x^{\frac{1}{4}}} p \prod_{p\leqslant x^{\frac{1}{6}}} p \cdots < (2.83)^{\sqrt{x}} \tag{14}$$

and

$$\prod_{p\leqslant k} p \prod_{p\leqslant k^{\frac{1}{2}}} p \cdots < (2.83)^{k}.$$

In the latter inequality, we use the fact that $k^{\frac{1}{\ell}} > x^{\frac{1}{2\ell-1}}$ for $\ell \geqslant 2$ since $x < k^{3/2}$ to get

$$\prod_{p\leqslant k} p \prod_{p\leqslant x^{\frac{1}{3}}} p \prod_{p\leqslant x^{\frac{1}{5}}} p \cdots < (2.83)^{k}. \tag{15}$$

Now we combine (13), (14) and (15) to get

$$\prod_{p\leqslant k} p^{\mathrm{ord}_p\binom{x}{k}} \leqslant (2.83)^{k+\sqrt{x}}. \tag{16}$$

By (5), there are at most $k-\mu$ terms in $\binom{x}{k}$ divisible by a prime $> k$. Thus

$$\prod_{p>k} p^{\mathrm{ord}_p\binom{x}{k}} \leqslant x^{k-\mu}$$

which, together with (16), implies (12). □

LEMMA 4. *Suppose that the assumptions of Theorem* 3′ *hold. Then* $x < k^{3/2}$ *for* $k \geqslant 153$, $x < k^{7/4}$ *for* $k \geqslant 50$ *and* $x < k^{2}$ *for* $k \geqslant 27$.

Proof. Let $x \geqslant k^{3/2}$ and $k \geqslant 153$. By (5), we have

$$\begin{aligned}(x-f_1)\cdots(x-f_\mu) &\leqslant \left(\prod_{p\leqslant k} p^{\mathrm{ord}_p\binom{x}{k}}\right)k! \\ &\leqslant \left(\prod_{p\leqslant k} x\right)k! \leqslant x^{\pi(k)}k!.\end{aligned}$$

On the other hand,

$$(x-f_1)\cdots(x-f_\mu) \geqslant (x-f_\mu)^{\mu} \geqslant (x-k+1)^{\mu} > x^{\mu}\left(1-\frac{1}{\sqrt{k}}\right)^{\mu} \geqslant x^{\mu}\left(1-\frac{1}{\sqrt{k}}\right)^{k}.$$

By comparing the bounds obtained above for $(x-f_1)\cdots(x-f_\mu)$, we derive from (3) and $x \geqslant k^{3/2}$ that

$$k^{\frac{1}{2}-\frac{3}{2k}\pi(k)-\frac{3}{2k}[\frac{1}{3}\pi(k)]-\frac{3}{2k}}\left(1-\frac{1}{\sqrt{k}}\right) < 1 \tag{17}$$

which, by Lemma 2(i), implies that

$$k^{\frac{1}{2}-2\left(\frac{1}{\log k}+\frac{1.5}{\log^2 k}\right)-\frac{3}{2k}}\left(1-\frac{1}{\sqrt{k}}\right)<1.$$

This is not valid for $k=210$ and the left hand side is an increasing function of k. Therefore it is not valid for $k\geqslant 210$. Now we use the exact value of $\pi(k)$ to conclude that (17) is not possible since $k\geqslant 153$. The argument for other assertions is similar. □

LEMMA 5. *Suppose that the assumptions of Theorem* 3′ *hold and* $x<k^{3/2}$. *Then*

(i) $k\leqslant 116$ *if* $x\geqslant 4k$
(ii) $k\leqslant 762$ *if* $\frac{5}{2}k\leqslant x<4k$
(iii) $k\leqslant 789$ *if* $2k\leqslant x<\frac{5}{2}k$.

Proof. (i) If $x\geqslant 4k$, then $\binom{x}{k}\geqslant\binom{4k}{k}>8^k/2k$ which, together with (12), gives $k\leqslant 116$.

(ii) If $\frac{5}{2}k\leqslant x<4k$, then $\binom{x}{k}>4^k/2k\left(\frac{5}{4}\right)^k$ which, together with (12), gives $k\leqslant 762$.

(iii) Let $2k\leqslant x<5/2\ k$. Then $\binom{x}{k}\geqslant 4^k/2k$. Further, we observe that $\binom{x}{k}$ is not divisible by any prime p with $\frac{x}{3}<p\leqslant k$. Now it is clear from the proof of Lemma 3 that $\binom{x}{k}\leqslant(2.83)^{\frac{x}{3}+\sqrt{x}}x^{k-\mu}$ for $x>27$ which we may assume. Hence $k\leqslant 789$. □

LEMMA 6. *Suppose* (5) *holds for some* x,k,μ *and* $f_1,\dots,f_\mu$ *as in Theorem* 3′. *Let* k_1 *be the largest prime such that* $k_1\leqslant k$. *Then there exists an integer* μ_1 *such that*

$$\mu_1\geqslant k_1-[\tfrac{1}{3}\pi(k_1)]-1$$

and

$$P((x-f_1)\cdots(x-f_{\mu_1}))\leqslant k_1$$

where $f_i\in[0,k_1)$ *for* $1\leqslant i\leqslant\mu_1$.

Proof. Since k_1 is the largest prime not exceeding k, we have

$$P((x-f_1)\cdots(x-f_\mu))\leqslant k_1. \tag{18}$$

Let $f_1,\dots,f_{\mu_1}\in[0,k_1)$ and $f_{\mu_1+1},\dots,f_\mu\in[k_1,k)$. We observe that $\mu_1\geqslant 1$, otherwise $\mu\leqslant k-k_1$ which is not possible by (3) since $\pi(k)=\pi(k_1)$. Then from (18) it follows that

$$P((x-f_1)\cdots(x-f_{\mu_1}))\leqslant k_1$$

and

$$\mu_1=\mu-(\mu-\mu_1)\geqslant k-\left[\frac{1}{3}\pi(k)\right]-1-(k-k_1)\geqslant k_1-\left[\frac{1}{3}\pi(k_1)\right]-1$$

since $\pi(k)=\pi(k_1)$. □

For positive integers a and b with $a<b$, we say that the property P_k holds for an interval $[a,b]$ if there exist $[\frac{1}{3}\pi(k)]+2$ integers in $[a,b]$ each having a prime factor $>k$.

We denote by $a_0 = a_0(k)$ the least integer greater than a such that the property P_k holds for $[a, a_0]$. We observe that $a_0(k)$ is a nondecreasing function of k.

LEMMA 7. *Suppose that the assumptions of Theorem* 3′ *hold. If* $x - k + 1 \leqslant a$, *then* $x < a_0(k)$.

Proof. Assume that $x - k + 1 \leqslant a$ and $x \geqslant a_0(k)$. Then we have $[x - k + 1, x] \supset [a, a_0(k)]$. Thus by the definition of $a_0(k)$, the interval $[x - k + 1, x]$ contains $[\frac{1}{3}\pi(k)] + 2$ integers each having a prime factor $> k$. This is not possible by (3) and (5). □

LEMMA 8. *Suppose that the assumptions of Theorem* 3′ *hold. Then* $k \leqslant 52$.

Proof. First let $k \geqslant 153$. Then by Lemma 4, we have $x < k^{3/2}$. Now we apply Lemma 5 to derive that $x < 4k$ and $k \leqslant 789$. We shall show that $k \notin [k_0, k_1]$ for suitable values of k_0, k_1. We begin by taking $k_0 = 400, k_1 = 789$ and $k \in [400, 789]$. Then $x - k + 1 \leqslant 2367$. We take $a = 2367$ and find that $a_0(789) = 2566$ implying $a_0(k) \leqslant 2566$ for $k \leqslant 789$. Then by Lemma 7, we may assume that $x < 2566$ which gives $x - k + 1 \leqslant 2166$. Now we repeat the above procedure several times. We give below the sequence of upper bounds for $x - k + 1$ obtained in this way:

$$2166 \to 1999 \to 1834 \to 1669 \to 1513 \to 1362 \to 1257 \to 1197 \to$$
$$1149 \to 1089 \to 1047 \to 1009 \to 921 \to 859 \to 793 \to 709.$$

Thus we may assume that $x - k + 1 \leqslant 709$ which implies that $k \leqslant 709$ since $x \geqslant 2k$. Now we take $k_0 = 400, k_1 = 709, x - k + 1 \leqslant 709$ and follow the above procedure to get the following smaller bounds for $x - k + 1$:

$$709 \to 619 \to 487 \to 301.$$

The last bound is not possible since $x - k + 1 \geqslant k + 1 \geqslant 401$. Hence $k \leqslant 399$. Next we take $k_0 = 250, k_1 = 399$ and $k \in [250, 399]$. Then $x - k + 1 \leqslant 1197$ since $x < 4k$. We follow the above procedure to get the following sequence of upper bounds for $x - k + 1$:

$$1197 \to 1043 \to 931 \to 801 \to 669 \to 589 \to 511 \to 441 \to 367 \to 273 \to 151$$

implying $k \leqslant 249$. Now we take $k \in [153, 249]$. Thus $x - k + 1 \leqslant 747$. Proceeding as above we get the following sequence of upper bounds for $x - k + 1$:

$$747 \to 660 \to 590 \to 509 \to 434 \to 385 \to 346 \to 296 \to 266 \to$$
$$226 \to 206 \to 154 \to 76$$

which is again not possible.

Hence, $k < 153$ and we may assume that $k \geqslant 53$. Then we see from Lemma 4 that $x < k^{7/4}$. We apply the procedure given above to the intervals [101, 151], [61, 97] and [53, 59] to conclude that k does not belong to any of these intervals. Now we use Lemma 6 to arrive at a contradiction. □

https://doi.org/10.1023/A:1025480729778 Published online by Cambridge University Press

LEMMA 9. (i) *The solutions of* $3^m - 2^n = \pm 1$ *in nonnegative integers* m, n *are given by* $(m, n) = (0, 1), (1, 1), (2, 3), (1, 2)$.

(ii) *The solutions of* $3^m - 2^n = \pm 5$ *in nonnegative integers* m, n *are given by* $(m, n) = (1, 3), (2, 2), (3, 5)$.

The first result is a well known result of Leo Hebrews and Levi Ben Gerson (See Ribenboim [7]) and the second result is due to Herschfeld [6]. We apply this lemma to show

LEMMA 10. *The assertion of Theorem* 3′ *is valid for* $3 \leqslant k \leqslant 7$.

Proof. We observe that $\mu \geqslant k-1$ for $k = 3, 4$ and $\mu \geqslant k-2$ for $5 \leqslant k \leqslant 7$. Hence there are at least two terms of the product in (5) composed of 2 and 3. Thus we get $2^{a_1}3^{b_1} - 2^{a_2}3^{b_2} = r$ for some nonnegative integers a_1, b_1, a_2, b_2, r with $1 \leqslant r \leqslant 6$. Further we observe that $r < k$ and $2^{a_1}3^{b_1} > k, 2^{a_2}3^{b_2} > k$ by $x \geqslant 2k$. We see that the above equation can be reduced to an equation of the type mentioned in Lemma 9. Hence, we find that (a_1, a_2, b_1, b_2) equals $(0, 3, 2, 0)$ if $r = 1$; $(3, 1, 0, 1), (1, 2, 1, 0), (1, 4, 2, 0)$ if $r = 2$; $(2, 0, 1, 2)$, $(0, 1, 2, 1), (0, 3, 3, 1)$ if $r = 3$; $(4, 2, 0, 1)$, $(2, 3, 1, 0), (2, 5, 2, 0)$ if $r = 4$; $(5, 0, 0, 3)$ if $r = 5$; $(3, 1, 1, 2)$, $(1, 2, 2, 1)$, $(1, 4, 3, 1)$ if $r = 6$. Let $(a_1, a_2, b_1, b_2) = (0, 3, 2, 0)$. Then 8 and 9 are two terms of the product in (5). Hence, by (3), we find that $x = 9, 10$ if $k = 3$; $x = 9$ if $k = 4$; $x = 10, 11, 12$ if $k = 5$; $x = 12, 13$ if $k = 6$; $x = 14$ if $k = 7$. All other solutions are obtained similarly. □

3. Proofs of the Theorems

Proof of Theorem 3′. By Lemmas 8 and 10, we have $8 \leqslant k \leqslant 52$. Let $k = 8$. Then we see from (3) and (5) that there are at least four terms which are composed of 2, 3 and 5. By deleting a term corresponding to 2, 3 and 5 in which they appear to the maximum power, we find that $x - k + 1 \leqslant 60$. Similarly $x - k + 1 \leqslant 360$ if $k = 13$ and 720 if $k = 17, 19, 23$. Further we argue with the primes 2 and 3 to get $x - k + 1 \leqslant 24$ if $k = 9$ and 72 if $k = 10, 11$. Also $x - k + 1 \leqslant k^2 - k$ for $29 \leqslant k \leqslant 52$ by Lemma 4. Now we follow the procedure as illustrated in Lemma 8 and use Lemma 6 to get the assertion. For this, we also need bounds for $x - k + 1$ with $12 \leqslant k \leqslant 18$ which can be obtained as above. □

Proof of Theorem 2. We denote by b_1, b_2, b_3, b_4 and Y_1, Y_2, Y_3, Y_4 positive integers. We shall be using SIMATH together with some combinatorial arguments for solving several elliptic equations. By Lemma 1, we have $k \leqslant 9$ or $k \in \{11, 13, 19\}$. We first take $4 \leqslant k \leqslant 9$. Let $k = 4, 5$. Then we see that there exist $0 < j_1 < j_2 \leqslant j_3 \leqslant k - 1$ such that

$$(n + j_1)(n + j_2)(n + j_3) = b_1 Y_1^2$$

https://doi.org/10.1023/A:1025480729778 Published online by Cambridge University Press

with $P(b_1) \leqslant 3$. Putting $n + j_1 = X_1$, we re-write the above equation as

$$X_1(X_1 + p)(X_1 + q) = b_1 Y_1^2$$

where $p = j_2 - j_1$, $q = j_3 - j_1$. Thus $0 < p < q \leqslant k - 1$. We solve these elliptic equations using SIMATH. Then $n = X_1 - j_1$ with $0 \leqslant j_1 \leqslant k - 3$. We find that the only solution for (1) is $(n, k, y, b, i) = (24, 4, 90, 2, 2)$.

Next we take $6 \leqslant k \leqslant 9$. Then we find that either

$$n(n+1)(n+2) = b_2 Y_2^2 \quad \text{or} \quad (n+k-3)(n+k-2)(n+k-1) = b_3 Y_3^2$$

with $P(b_2) \leqslant 7$ and $P(b_3) \leqslant 7$. Thus we solve the elliptic equation

$$X_2(X_2 + 1)(X_2 + 2) = b_4 Y_4^2$$

where $X_2 = n$ or $n + k - 3$ and $P(b_4) \leqslant 7$. There are 16 such equations which are solved using SIMATH. We find that none of the solution yields any solution to (1).

Let $k = 19$. Then we see that either

17 divides a_0, a_{17}; 13 divides a_1, a_{14}; 11 divides a_4, a_{15};
7 divides a_2, a_9, a_{16}; 5 divides a_3, a_8, a_{13}, a_{18}

or

17 divides a_1, a_{18}; 13 divides a_4, a_{17}; 11 divides a_3, a_{14};
7 divides a_2, a_9, a_{16}; 5 divides a_0, a_5, a_{10}, a_{15}.

Thus there are four a_i's from either $\{a_5, a_6, a_7, a_{10}, a_{11}, a_{12}\}$ or $\{a_6, a_7, a_8, a_{11}, a_{12}, a_{13}\}$ which are distinct and take the values 1, 2, 3, 6. Hence, there exist three terms $n + j_1, n + j_2, n + j_3$ with $j_1 < j_2 < j_3$ and taking values from either $\{5, 6, 7, 10, 11, 12\}$ or $\{6, 7, 8, 11, 12, 13\}$ such that their product is a square. Writing $X = n + j_1$, we find that there exist integers r and s with $1 \leqslant r < s \leqslant 7$ such that

$$X(X + r)(X + s) = Y^2 \tag{19}$$

for some positive integer Y. This is an elliptic equation and we use SIMATH to exclude the case $k = 19$. The above combinatorial argument reduces the number of elliptic curves from $2^{\pi(19)} = 2^8 = 256$ to 21. We proceed as above to find that there are four distinct a_i's taking values 1, 2, 3, 6 with

$i \in \{3, 4, 5, 6, 9, 10\}$ or $\{1, 4, 5, 6, 8, 9\}$ or $\{3, 4, 6, 7, 8, 11\}$ or $\{2, 3, 6, 7, 8, 9\}$ if $k = 13$;
$i \in \{2, 3, 4, 6, 7, 9\}$ or $\{1, 3, 4, 6, 7, 8\}$ if $k = 11$

which give rise to elliptic equations of the form (19). Now we use SIMATH again to exclude the cases $k = 11, 13$. $\square$

Proof of Theorem 1. If $k = 3$, the assertion follows immediately and we suppose that $k \geqslant 4$. Erdős [2] and Rigge [8], independently, proved that a product of two or more consecutive positive integers is never a square. Therefore we may assume (1) with $b = 1$ for some i with $0 < i < k - 1$. By Theorem 2, we derive that $n \leqslant k^2$.

https://doi.org/10.1023/A:1025480729778 Published online by Cambridge University Press

Consequently, none of the factors on the left-hand side of (1) is divisible by a prime exceeding k. Further we check that the assertion of Theorem 1 is valid for any of the tuples (n, k) given by (4). Now we apply Theorem 3 with $\mu = k - 1$ to conclude that $n \leqslant k$. Then we observe that $n \leqslant (n+k)/2 < n+k-1$. Further we derive from Lemma 2 (ii) that $\pi(n+k-1) - \pi((n+k)/2) \geqslant 2$ for $n+k \geqslant 12$. Since every prime between $(n+k)/2$ and $n+k-1$ occurs only to the first power, we conclude that $n+k \leqslant 11$. For these values of n and k, we directly check that the assertion of Theorem 1 is valid. □

4. Proofs of the Corollaries

Proof of Corollary 3. Let (n, k) be a pair distinct from the ones given by (4). If the inequality of Corollary 3(i) is not valid, then there are at most $[\frac{1}{3}\pi(k)] + 1$ terms in $n(n+1)\cdots(n+k-1)$ divisible by a prime exceeding k contradicting Theorem 3. Next we delete the cases in (4) satisfying the inequality of Corollary 3(ii) which, then, follows from Corollary 3(i). □

Proof of Corollary 1. We denote by b_5, b_6, b_7 and Y_5, Y_6, Y_7 positive integers. Let $k \geqslant 4$ and $n > k^2$. Suppose that the assertion of Corollary 1 is not valid. Then there exist prime $p > k$ and $\delta \in \{0, 1\}$ such that the left hand side of (2) is equal to $p^\delta b_5 Y_5^2$ with $P(b_5) \leqslant k$. By Theorem A, we may assume that $\delta = 1$. Suppose p divides $n + id$ with $0 < i < k-1$. We delete this term to obtain an equation as in (1) and we apply Theorem 2 to get $(n, k) = (24, 4)$. Suppose p divides either n or $n+k-1$. Then we get a product of $k-1$ consecutive positive integers equal to $b_6 Y_6^2$ with $P(b_6) \leqslant k$. If $P(b_6) < k$, then we apply Theorem A to get $(n, k) = (47, 4), (48, 4)$. Thus we may assume that $P(b_6) = k$ and in particular k is prime. Then from the $k-1$ consecutive positive integers, we remove the term divisible by k. By Theorem 2, we see that the removed term is either the first or the last. Thus we get a product of $k-2$ consecutive positive integers which equals $b_7 Y_7^2$ with $P(b_7) \leqslant k-2$. This is not possible by Theorem A. □

Proof of Corollary 2. By Corollary 1, we may assume (2) with $k = 3$. Further we may suppose that $b \in \{5, 10, 15, 30\}$ by Theorem A. Now we use SIMATH to find that these equations have no solution. □

Remark. The exceptions $n \in \{14, 15, 16, 18, 20, 21, 24\}$ if $k = 13$; $n \in \{15, 20\}$ if $k = 14$; $n = 20$ if $k = 17$ in Theorem 3 and the corresponding exceptions in Theorem 3′ and Corollary 3(i) should be added. We thank Shanta Laishram for pointing out these exceptions.

Acknowledgements

We thank Dr P. Gastesi for acquiring the SIMATH package which we use in the proof of Theorem 2 for solving several elliptic equations and helping us in using it. The SIMATH package has already been used in Filakovszky and Hajdu [4] and Hajdu and Pinter [5]. We also thank the referee for his comments and suggestions on an earlier draft of this paper.

References

1. Erdős, P.: A Theorem of Sylvester and Schur, *J. London Math. Soc.* **9** (1934), 282–288.
2. Erdős, P.: Note on the product of consecutive integers (I), *J. London Math. Soc.* **14** (1939), 194–198.
3. Erdős, P. and Selfridge, J. L.: The product of consecutive integers is never a power, *Illinois J. Math.* **19** (1975), 292–301.
4. Filakovszky, P. and Hajdu, L.: The resolution of the diophantine equation $x(x+d)\cdots(x+(k-1)d) = by^2$ for fixed d, *Acta Arith.* **98** (2001), 151–154.
5. Hajdu, L. and Pinter, Á.: Square product of three integers in short intervals, *Math. Comp.* **68**(227) (1999), 1299–1301.
6. Herschfeld, A.: The equation $2^x - 3^y = d$, *Bull. Amer. Math. Soc.* **42** (1936), 231–234.
7. Ribenboim, P.: *Catalan's Conjecture*, Academic Press, New York, 1994.
8. Rigge, O.: Über ein diophantisches Problem, In: *9th Congress Math. Scand., Helsingfors, 1938*, Mercator (1939), pp. 155–160.
9. Rosser, B. and Schoenfeld, L.: Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.
10. Saradha, N.: On perfect powers in products with terms from arithmetic progressions, *Acta Arith.* **82** (1997), 147–172.
11. Saradha, N.: Squares in products with terms in an arithmetic progression, *Acta Arith.* **86** (1998), 27–43.
12. Saradha, N. and Shorey, T. N.: Almost perfect powers in arithmetic progression, *Acta Arith.*, to appear.
13. Saradha, N. and Shorey, T. N.: Almost squares in arithmetic progression, *Compositio Math.* **138** (2003), 73–111.
14. Shorey, T. N.: Exponential diophantine equations involving products of consecutive integers and related equations, In: R. P.Bambah, V. C. Dumir and R. J. Hans-Gill (eds), *Number Theory*, Hindustan Book Agency, 1999, pp. 463–495.
15. Sylvester, J. J.: On arithmetical series, *Messenger of Math.* **21** (1892), 1–19, 87–120; and *Collected Mathematical Papers* **4** (1912), 687–731.