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A SIMPLE PROOF OF PITMAN'S $2M-X$ THEOREM

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Abstract

Pitman has shown that if X is Brownian motion with maximum process M, then $2M-X$ is a $\mathrm{BES}_0(3)$ process. We show that this can be seen by looking at finite-dimensional densities.

BROWNIAN MOTION; BESSEL PROCESS; DENSITY FACTORIZATIONS

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1. Introduction

Pitman (1975) discovered the following striking fact: if $X=\{X_t, t\geqq 0\}$ is a BM_0 (one-dimensional Brownian motion starting at $X_0=0$) and $M=\{M_t, t\geqq 0\}$ where $M_t=\sup\{X_s: s\leqq t\}$, then '$X$ reflected with respect to M' namely $2M-X=\{2M_t-X_t, t\geqq 0\}$ is a BES_0 (3) process, i.e. has the law of the distance to the origin of a three-dimensional Brownian motion. An extension of this result to spectrally positive Lévy processes has been given by Bertoin (1991). Pitman's proof, via discrete approximations, is quite involved. Tanaka (1989) gives another approach which applies also in the case of drift first considered by Rogers and Pitman (1981). For zero drift a slightly more elaborate form of the result, stated below, is proved by Ikeda and Watanabe ((1981), Chap. III Sect. 4.3) and found also in Revuz and Yor ((1991), Chap. VI, (3.5)), where further approaches are mentioned (top of p. 258). All of those use non-elementary tools from stochastic analysis. It seems therefore useful to show that a simple proof can be given by obtaining and comparing finite-dimensional densities of the pair (X, M) with those of (Z, F) where Z is a $\mathrm{BES}_0(3)$ process and $F=\{F_t, t\geqq 0\}$ its future minimum process, $F_t=\inf\{Z_s: s\geqq t\}$. The conclusion is then the following.

Theorem. The processes $(2M-X, M)$ and (Z, F) have the same law.

As already mentioned, this holds also when X is $\mathrm{BM}_0(\delta)$ (that is, when $\{X_t-\delta T, t\geqq 0\}$ is BM_0) with $\delta>0$ and simultaneously Z is $\mathrm{BES}(3, \delta)$. We shall write $\boldsymbol{P}_\delta(\cdot)$ for probabilities relating to those processes and $\boldsymbol{P}^x_\delta(\cdot)$ when they are conditioned to initial value $x\neq 0$. When $\delta=0$, we only need a few known densities which we recapitulate. For $t>0$ and $w\in\mathbb{R}$, $p(t, w)=:(2\pi t)^{-1/2}\exp\{-w^2/2t\}$ and $g(t; w)=:(|w|/t)p(t; w)$. Then (Karatzas and Shreve (1988), (8.2) p. 95 and (8.9) p. 97)

$$\boldsymbol{P}^x(X_t\in dx', M_t\in dy)=2g(t; 2y-x-x')\,dx'\,dy, \qquad y>x\vee x', \tag{1}$$

$$\boldsymbol{P}^x(X_t\in dx', X_s>0 \text{ for } 0<s<t)/dx'=q(t; x, x')=:p(t; x-x')-p(t; x+x'), \; x\wedge x'>0. \tag{2}$$

For Z one has correspondingly (e.g. using (3.1) in Imhof (1984)) when $z>0$ and $m_t=\min\{Z_s, s\leqq t\}$,

$$\boldsymbol{P}^z(Z_t\in dz', m_t\in df)-\frac{z'}{z}2g(t; z+z'-2f)\,dz'\,df, \qquad 0<f<z\vee z', \tag{3}$$

$$\boldsymbol{P}^z(Z_t\in dz', m_t>f)=\frac{z'}{z}q(t; z-f, z'-f)\,dz', \qquad 0<f<z\wedge z'. \tag{4}$$

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The following are also well known:

(5) $$P(Z_t \in dz') = z'2g(t, z')\,dz', \qquad z' > 0,$$

(6) $$P^z(F_0 \in dz') = (1/z)\,dz', \qquad 0 < z' < z.$$

2. Proof

To obtain the joint density for (X, M) at times $0 < t_1 < \cdots < t_n$, one uses the Markov property at times $t_1, \cdots, t_{n-1}$ and must distinguish between the cases where the excursion of X below M at time t_i extends beyond t_{i+1} (so $M_{t_i} = M_{t_{i+1}}$) and those where it does not (so $M_{t_i} < M_{t_{i+1}}$). For (Z, F) the corresponding distinction is between $F_{t_i} = F_{t_{i+1}}$ (so F_{t_i} is achieved after t_{i+1}) and $F_{t_i} < F_{t_{i+1}}$ (so F_{t_i} is the minimum achieved during (t_i, t_{i+1})). A last application of the Markov property at time t_n, using (6), then takes care of F_{t_n} and ensures that all initial factors coming from (5), then repeatedly (3) and (4) and finally (6) cancel two by two, which in this elementary approach is the key to the theorem. This is illustrated well enough if one considers only the case $n = 2$. Writing for brevity dx^2 instead of $dx_1\,dx_2$, and similarly in $y, \cdots$, and agreeing that j always takes values 1 and 2, one has according to (1), for $x_j < y_j$ and $0 < y_1 < y_2$,

$$P(X_{t_j} \in dx_j,\ M_{t_j} \in dy_j) = 2g(t_1; 2y_1 - x_1)2g(t_2 - t_1; 2y_2 - x_1 - x_2)\,dx^2\,dy^2.$$

If on the other hand $0 < y = y_1 = y_2$, (1) and (2) give

$$P(X_{t_j} \in dx_j,\ M_{t_1} = M_{t_2} \in dy) = 2g(t_1; 2y_1 - x_1)q(t_2 - t_1; y - x_1, y - x_2)\,dx^2\,dy.$$

Changing variables from X_{t_j} to $X^*_{t_j} = 2M_{t_j} - X_{t_j}$ gives

(7) $$P(X^*_{t_j} \in dx^*_j,\ M_{t_j} \in dy_j) = 2g(t_1; x^*_1)2g(t_2 - t_1; x^*_1 + x^*_2 - 2y_1)\,dx^{*2}\,dy^2$$

where $0 < y_j < x^*_j$ and $0 < y_1 < y_2$, while if $y_1 = y_2 = y$

(8) $$P(X^*_{t_j} \in dx^*_j,\ M_{t_1} = M_{t_2} \in dy) = 2g(t_1; x^*_1)q(t_2 - t_1; x^*_1 - y, x^*_2 - y)\,dx^{*2}\,dy.$$

One can mention here that if marginalization with respect to y_1 is done in (7) and the result added to (8), one easily deduces from the joint density for $X^*_{t_j}$, M_{t_2} thus obtained the uniform conditional law of M_{t_2} over $(0, X^*_{t_2})$, given the $X^*_{t_j}$ (and more generally given $\{X^*_t, 0 \leqq t \leqq t_2\}$). This is a characteristic feature also resulting from (6) via the theorem.

We want to compare (7) and (8) with $P(Z_{t_j} \in dz_j, F_{t_j} \in df_j)$ and $P(Z_{t_j} \in dz_j,\ F_{t_1} = F_{t_2} \in df)$. Those densities are obtained by using, for the first, (5) over the time interval $[0, t_1)$, (3) over $[t_1, t_2]$ and finally (6) over $[t_2, \infty]$. For the second, one must use (4) over $[t_1, t_2]$. After simplification of the initial factors (successively z_1, z_2/z_1 and $1/z_2$) the densities obtained are precisely (7) and (8) written in terms of z_j, f_j instead of x^*_j, y_j. The same routine extended to arbitrary n proves the theorem when $\delta = 0$.

It is now easy to take care of the case $\delta > 0$. For (X, M) considered over $[0, t_2]$, there is a Radon–Nikodym factor which, replacing x_2 with $2y_2 - x^{**}_2$, gives on the right of (7) and (8) an additional factor $\exp\{-\frac{1}{2}\delta^2 t_2 + \delta(2y_2 - x^*_2)\}$. For Z, the passage from P^z to P^z_δ-densities in (3) and (4) amounts to replacing the initial factor z'/z with $\exp\{-\frac{1}{2}\delta^2 t\} \sinh \delta z'/\sinh \delta z$, while in (5) the initial z' must be replaced with $\exp\{-\frac{1}{2}\delta^2 t\} \sinh \delta z'/\delta$ (Rogers and Pitman (1981), formulas (10) and (12)). In addition, one computes easily that one has instead of (6)

$$P^{z_2}_\delta(F_0 \in df_2) = (\delta/\sinh \delta z_2)\exp\{\delta(2f_2 - z_2)\}, \qquad 0 < f_2 < z_2.$$

Thus after simplification we have

$$P_\delta(Z_{t_j} \in dz_j,\ F_{t_j} \in df_j) = \exp\{-\tfrac{1}{2}\delta^2 t_2 + \delta(2f_2 - z_2)\}P(Z_{t_j} \in dz_j,\ F_{t_j} \in df_j),$$

and a corresponding formula when $F_{t_1} = F_{t_2} \in df$. The finite-dimensional densities for (X^*, M) and (Z, F) are thus again the same.

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