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CRITICAL EXTINCTION AND BLOW-UP EXPONENTS FOR FAST DIFFUSIVE POLYTROPIC FILTRATION EQUATION WITH SOURCES

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Abstract This paper is mainly concerned with the critical extinction and blow-up exponents for the homogeneous Dirichlet boundary-value problem of the fast diffusive polytropic filtration equation with reaction sources.

Keywords: critical exponents; extinction; blow-up; fast diffusive polytropic filtration equation

2000 *Mathematics subject classification:* Primary 35B33; 35K55
Secondary 35K15; 35K20

1. Introduction

This paper is devoted to the critical extinction and blow-up exponents for the fast diffusive polytropic filtration equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \lambda u^q, \quad (x,t) \in \Omega \times (0,+\infty), \tag{1.1}$$

subject to the initial- and boundary-value conditions

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,+\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where $0 < m(p-1) < 1$, $q > 0$, $\lambda > 0$, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N > 2$, and $u_0(x)$ is a non-negative and bounded function with $u_0^m(x) \in W_0^{1,p}(\Omega)$.

As two of the important features of many evolutionary equations, blow-up and extinction properties of solutions have been the subject of intensive study during the past few decades. Among these investigations, it was Fujita [**4**] who first established the so-called theory of critical blow-up exponents for the heat equation with reaction sources in 1966; this can, of course, be regarded as the elegant description for either blow-up or global existence of solutions. Since then, there has been increasing interest in the study of critical Fujita exponents for different kinds of evolutionary equations (see [**1**, **10**] for a survey of the literature). In recent years, special attention has been paid to the blow-up property to nonlinear degenerate or singular diffusion equations with different nonlinear sources, including the inner sources, boundary flux or multiple sources (see, for example, [**5**, **15**–**18**]). Besides the blow-up, the extinction phenomenon has also received much attention (see [**2**, **3**, **6**, **7**, **9**, **11**, **14**]). Our interest lies in whether or not there exist both a critical blow-up exponent and a critical extinction exponent for (1.1). In a recent paper [**13**], we showed that this indeed happens for the fast diffusive p-Laplacian equation, namely the special case $m = 1$ of (1.1).

In this paper, we focus our attention on the fast diffusive polytropic filtration equation with sources, and to reveal the fact that problem (1.1)–(1.3) admits two critical exponents $q_1, q_2 \in (0, +\infty)$ with $q_1 < q_2$, which will be called the critical extinction exponent and critical blow-up exponent, respectively. More precisely, q_2 is called the blow-up exponent since for $q \in (0, q_2]$ the problem admits global solutions for any non-negative initial data, while for $q \in (q_2, +\infty)$ there exist both global solutions and blow-up solutions. On the other hand, q_1 is called the extinction exponent, since the extinction can always occur for q in the interval $(q_1, +\infty)$, whereas for $q \in (0, q_1)$ there always exists a non-extinction bounded solution for any non-negative initial data. Moreover, where the critical case $q = q_1$ is concerned, the other parameter λ is also found to have a critical value. In fact, such a critical value is just equal to λ_1, the first eigenvalue of the p-Laplacian equation with a homogeneous Dirichlet boundary-value condition, and the solution has completely different properties for λ belonging to $(0, \lambda_1)$ or $(\lambda_1, +\infty)$.

This paper is organized as follows: in § 2, we firstly give some definitions and notations. Thereafter, some auxiliary lemmas and the basic existence proposition are given. Because it is standard and rather lengthy, for the convenience of the reader the proof of the existence proposition is given in the appendix. Furthermore, in § 3, we discuss the blow-up exponent of solutions; a global existence result is also given. Finally, § 4 is devoted to the critical extinction exponent.

2. Preliminaries

In this section, we establish the global existence, uniqueness and boundedness of non-negative weak solutions of problem (1.1)–(1.3). First, we introduce some notation which will be used throughout the paper:

$$
\begin{gathered}
Q = \Omega \times (0, \infty), \qquad Q_T = \Omega \times (0, T), \qquad Q_{(t_1,t_2)} = \Omega \times (t_1, t_2), \\
E = \{u \in L^{2q}(Q_T) \cap L^2(Q_T);\ u \in C([0, T]; L^1(\Omega)),\ \nabla u^m \in L^p(Q_T)\}, \\
\tilde{E} = \left\{u \in L^2(Q_T);\ u(\cdot\,, t) \in C([0, T]; L^2(\Omega)),\ \frac{\partial u}{\partial t} \in L^2(Q_T),\ \nabla u \in L^p(Q_T)\right\}, \\
E_0 = \{u \in E;\ u|_{\partial\Omega} = 0\}, \qquad \tilde{E}_0 = \{u \in \tilde{E};\ u|_{\partial\Omega} = 0\}.
\end{gathered}
$$

Because of the degeneracy and the singularity, equation (1.1) might not have classical solutions in general, and hence we consider the non-negative solution of (1.1) in some weak sense.

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Definition 2.1. A function $u \in E$ is called a weak upper solution of problem (1.1)–(1.3) provided that, for any $T > 0$ and any $0 \leqslant \varphi \in \tilde{E}_0$,

$$\int_\Omega u(x,T)\varphi(x,T)\,\mathrm{d}x - \int_\Omega u_0(x)\varphi(x,0)\,\mathrm{d}x - \iint_{Q_T} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} |\nabla u^m|^{p-2}\nabla u^m \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \geqslant \lambda \iint_{Q_T} u^q\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\begin{aligned} u(x,t) &\geqslant 0, && (x,t) \in \partial\Omega \times (0,+\infty),\\ u(x,0) &\geqslant u_0(x), && x \in \Omega. \end{aligned}$$

Replacing '$\geqslant$' by '$\leqslant$' in the above inequalities yields the definition of a weak lower solution. Furthermore, if u is a weak upper solution as well as a weak lower solution, then we call it a weak solution of problem (1.1)–(1.3).

In order to show the existence of non-negative weak solutions, the following fixed-point theorem is necessary; this can be found in [**12**].

Lemma 2.2 (Leray–Schauder fixed-point theorem). *Let X denote a real Banach space and let $\Gamma(u,\sigma)$ be a mapping from $X \times [0,1]$ to X, satisfying the following conditions:*

(i) *Γ is compact;*

(ii) *$\Gamma(u,0) = 0$ for any $u \in X$;*

(iii) *there exists a positive constant M independent of u and σ, such that, for any $u \in X$, if $u = \Gamma(u,\sigma)$ for some $\sigma \in [0,1]$, then $\|u\|_X \leqslant M$.*

Then $\Gamma(\cdot,1)$ has a fixed point, namely, there exists an $u \in X$, such that $u = \Gamma(u,1)$.

Employing the above Leray–Schauder fixed-point theorem, we can prove the following existence results.

Proposition 2.3. *Assume that $u_0(x) \geqslant 0$ with $u_0^m \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. Then if $q \leqslant 1$, problem (1.1)–(1.3) admits at least a non-negative weak solution. In addition, if $q = 1$, the non-negative weak solution is unique; if $q > 1$, for any non-negative initial data $u_0(x)$, problem (1.1)–(1.3) is uniquely solvable locally in time and, furthermore, if $\lambda \leqslant \lambda_1$, $u_0^m(x) \leqslant \varphi_1(x)$ for $x \in \Omega$, where λ_1 is the first eigenvalue of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition, and $\varphi_1(x) \geqslant 0$ with $\|\varphi_1\|_{L^\infty(\Omega)} = 1$ is the eigenfunction corresponding to the eigenvalue λ_1, then the solution exists globally and is bounded.*

The proof is classical and lengthy and is therefore deferred to the appendix.

After establishing the basic existence, we turn to the boundedness of weak solutions. As is shown later, there exist blow-up weak solutions for the case $q > 1$. Therefore, we only consider the case $0 < q \leqslant 1$.

Proposition 2.4. *Let $0 < q \leqslant 1$. Then the weak solution of problem (1.1)–(1.3) is locally bounded. In particular, when $q < m(p-1)$, the weak solution is globally uniformly bounded.*

Proof. Suppose that u is a weak solution of problem (1.1)–(1.3), then by the weak maximum principle, we conclude that

$$\|u\|_{L^\infty(Q_t)} \leqslant \|u_0\|_{L^\infty(\Omega)} + \lambda t \|u\|^q_{L^\infty(Q_t)}.$$

When $q < 1$, it is not difficult to see that u is bounded locally uniformly, while if $q = 1$, then we have

$$\|u\|_{L^\infty(Q_{(t_n,t_{n-1})})} \leqslant \|u(\cdot\,, t_{n-1})\|_{L^\infty(\Omega)} + \lambda(t_n - t_{n-1})\|u\|_{L^\infty(Q_{(t_n,t_{n-1})})},$$

that is

$$\|u\|_{L^\infty(Q_{(t_n,t_{n-1})})} \leqslant 2\|u(\cdot\,, t_{n-1})\|_{L^\infty(\Omega)} \leqslant 2^n \|u_0\|_{L^\infty(\Omega)},$$

where $t_n = t_{n-1} + 1/2\lambda$, $t_0 = 0$. Then it is not difficult to see that u is bounded locally uniformly.

In the following, we show the global boundedness for the case when $q < m(p-1)$. Assume that u is a non-negative weak solution of problem (1.1)–(1.3). Let

$$k_0 = \|u_0\|_{L^\infty(\Omega)}, \qquad l = \max\left\{k_0,\ \left(\frac{m\lambda(mp-m+1)^p}{\lambda_1 (mp)^p}\right)^{1/(mp-m-q)}\right\},$$

and set

$$A_h(t) = \{x \in \Omega;\ u(x,t) \geqslant h\}.$$

In the following, we shall show that

$$\|u(\cdot\,, t)\|_{L^\infty(\Omega)} \leqslant l \quad \text{for any } t > 0. \tag{2.1}$$

Multiplying equation (1.1) by $(u-l)_+$ and integrating in Ω yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l)_+^2 \,\mathrm{d}x + \int_\Omega (|\nabla u^m|^{p-2}\nabla u^m)\nabla(u-l)_+ \,\mathrm{d}x = \lambda \int_\Omega u^q (u-l)_+ \,\mathrm{d}x,$$

that is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l)_+^2 \,\mathrm{d}x + \frac{(mp)^p}{m(mp-m+1)^p}\int_{A_l(t)} |\nabla u^{(mp-m+1)/p}|^p \,\mathrm{d}x = \lambda \int_\Omega u^q (u-l)_+ \,\mathrm{d}x. \tag{2.2}$$

Let λ_1 be the first eigenvalue of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition, in which case we have

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x \geqslant \lambda_1 \int_\Omega u^p \,\mathrm{d}x \quad \text{for any } u \in W_0^{1,p}(\Omega).$$

Furthermore, we also have

$$\int_{A_l(t)} |\nabla u|^p \,\mathrm{d}x \geqslant \lambda_1 \int_{A_l(t)} u^p \,\mathrm{d}x \quad \text{for any } u \in W_0^{1,p}(\Omega). \tag{2.3}$$

Indeed, if we take

$$\tilde{u} = \begin{cases} u & \text{if } u \geqslant l, \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$\int_\Omega |\nabla \tilde{u}|^p \,\mathrm{d}x \geqslant \lambda_1 \int_\Omega \tilde{u}^p \,\mathrm{d}x;$$

that is, (2.3) holds. Taking (2.2) into account, we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l)_+^2 \,\mathrm{d}x + \lambda_1 \frac{(mp)^p}{m(mp-m+1)^p}\int_{A_l(t)} u^{mp-m+1} \,\mathrm{d}x \leqslant \lambda \int_\Omega u^q (u-l)_+ \,\mathrm{d}x.$$

We further have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l)_+^2 \,\mathrm{d}x + \lambda_1 \frac{(mp)^p}{m(mp-m+1)^p}\int_\Omega u^{mp-m}(u-l)_+ \,\mathrm{d}x \leqslant \lambda \int_\Omega u^q (u-l)_+ \,\mathrm{d}x.$$

Recalling the definition of l, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l)_+^2 \,\mathrm{d}x \leqslant \int_\Omega u^q (u-l)_+ \left(\lambda - \frac{\lambda_1 (mp)^p}{m(mp-m+1)^p} u^{mp-m-q}\right) \mathrm{d}x \leqslant 0. \tag{2.4}$$

Noting that $\int_\Omega (u_0 - l)_+^2 \,\mathrm{d}x = 0$, and combining this with the above inequality, we conclude that

$$\int_\Omega (u-l)_+^2 \,\mathrm{d}x = 0,$$

which implies (2.1). The conclusion of the proposition follows immediately. □

3. Critical blow-up exponent

We are now in a position to discuss the critical blow-up exponent of problem (1.1)–(1.3). By using the method of Levine [**8**], we define

$$F(u) = \frac{1}{p}\int_\Omega |\nabla u^m|^p \,\mathrm{d}x - \frac{\lambda m}{q+m}\int_\Omega u^{q+m} \,\mathrm{d}x, \qquad H(u) = \frac{1}{m+1}\int_\Omega u^{m+1} \,\mathrm{d}x.$$

The following theorem shows that the blow-up exponent $q_2 = 1$.

Theorem 3.1. *Assume that $u_0 \geqslant 0$, $F(u_0) \leqslant 0$ and $H(u_0) > 0$. If $q > 1$, then there exists T^* with $0 < T^* < \infty$, such that*

$$\lim_{t \to T^*} H(u(t)) = +\infty;$$

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and if $0 < q \leqslant 1$, *then the weak solution exists globally. In particular, if* $q = 1$, *then*

$$\lim_{t\to+\infty} H(u(t)) = +\infty;$$

if $m(p-1) < q < 1$,

$$\lim_{t\to+\infty} \|u(\cdot\,,t)\|_{L^\infty(\Omega)} = +\infty;$$

if $0 < q < m(p-1)$, *then the weak solution is uniformly bounded.*

Proof. By the definition of $F(u)$ and $H(u)$, and a simple calculation, we have

$$\frac{\mathrm{d}F(u)}{\mathrm{d}t} = -\frac{4m}{(m+1)^2}\int_\Omega \left(\frac{\partial u^{(m+1)/2}}{\partial t}\right)^2 \mathrm{d}x \leqslant 0, \tag{3.1}$$

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_\Omega \frac{\partial u}{\partial t} u^m \,\mathrm{d}x \\ &= -\int_\Omega |\nabla u^m|^p \,\mathrm{d}x + \int_\Omega \lambda u^{q+m}\,\mathrm{d}x \\ &= \left(1 - \frac{mp}{q+m}\right)\int_\Omega \lambda u^{q+m}\,\mathrm{d}x - pF(u).\end{aligned} \tag{3.2}$$

According to (3.1), we see that $F(u_0) \leqslant 0$ implies that $F(u) \leqslant 0$. Therefore, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t} \geqslant \lambda\left(1 - \frac{mp}{q+m}\right)\int_\Omega u^{q+m}\,\mathrm{d}x. \tag{3.3}$$

Case 1. When $q > 1$, recalling Hölder's inequality, we obtain

$$\int_\Omega u^{q+m}\,\mathrm{d}x \geqslant C\left(\frac{1}{m+1}\int_\Omega u^{m+1}\,\mathrm{d}x\right)^{(m+q)/(1+m)}.$$

Thus, we have

$$\frac{\mathrm{d}H(u)}{\mathrm{d}t} \geqslant C\lambda\left(1 - \frac{mp}{q+m}\right)(H(u))^{(m+q)/(1+m)}.$$

Then there must exist a positive constant $T^* < \infty$ such that

$$\lim_{t\to T^*} H(u(t)) = +\infty. \tag{3.4}$$

Otherwise, we would have

$$\frac{1+m}{1-q}\frac{\mathrm{d}H^{(1-q)/(1+m)}(u)}{\mathrm{d}t} \geqslant C\lambda\left(1 - \frac{mp}{q+m}\right).$$

Integrating from 0 to t yields

$$H^{(1-q)/(m+1)}(u_0) - H^{(1-q)/(m+1)}(u(t)) \geqslant \frac{q-1}{m+1}C\lambda\left(1 - \frac{mp}{q+m}\right)t;$$

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that is,

$$H^{(1-q)/(m+1)}(u(t)) \leqslant H^{(1-q)/(m+1)}(u_0) - \frac{q-1}{m+1}C\lambda\left(1 - \frac{mp}{q+m}\right)t.$$

Since $H(u_0) > 0$ implies that $H^{(1-q)/(m+1)}(u_0) < \infty$, there exists a positive constant $T^* > 0$ such that $H^{(1-q)/(m+1)}(u(t)) < 0$ for all $t > T^*$, which is a contradiction.

Case 2. Since the global existence for $0 < q \leqslant 1$ or the boundedness for $0 < q < m(p-1)$ is a direct consequence of Propositions 2.3 and 2.4, it suffices to prove the remaining part.

When $q = 1$, from (3.3) it follows that

$$\frac{\mathrm{d}H(u(t))}{\mathrm{d}t} \geqslant \lambda(m + 1 - mp)H(u(t)).$$

By a direct calculation, we obtain

$$H(u(t)) \geqslant H(u_0)\mathrm{e}^{\lambda(m+1-mp)t}.$$

Therefore, we have

$$\lim_{t\to+\infty} H(u(t)) = +\infty, \tag{3.5}$$

while, for the case when $m(p-1) < q < 1$, if u is a weak solution, by means of the weak maximum principle, for any $t > 0$ we have

$$\|u\|_{L^\infty(Q_t)} \leqslant \|u_0\|_{L^\infty(\Omega)} + t\lambda\|u\|^q_{L^\infty(Q_t)},$$

which implies that u is bounded in any finite time, namely the weak solution could not blow up in finite time. However, we have

$$\lim_{t\to\infty} \|u(\cdot\,, t)\|_{L^\infty(\Omega)} = +\infty. \tag{3.6}$$

Suppose to the contrary that there exists a positive constant M such that

$$\|u(\cdot\,, t)\|_{L^\infty(\Omega)} \leqslant M.$$

Then

$$\int_\Omega u^{m+1}\,\mathrm{d}x \leqslant M^{1-q}\int_\Omega u^{q+m}\,\mathrm{d}x.$$

From (3.3), we infer that

$$\begin{aligned}\frac{\mathrm{d}H(u(t))}{\mathrm{d}t} &\geqslant \lambda\left(1 - \frac{mp}{q+m}\right)M^{q-1}\int_\Omega u^{1+m}\,\mathrm{d}x \\ &\geqslant \lambda\left(m + 1 - \frac{mp(m+1)}{q+m}\right)M^{q-1}H(u(t)),\end{aligned}$$

which implies that

$$\lim_{t\to+\infty} H(u(t)) = +\infty.$$

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Taking this into account and noting that

$$H(u(t)) \leqslant \frac{1}{m+1}|\Omega|\,\|u\|_{L^\infty(\Omega)}^{m+1},$$

we conclude that $\|u(\cdot\,,t)\|_{L^\infty(\Omega)}$ goes to infinity as $t \to \infty$, which is a contradiction. The proof of the theorem is complete. □

4. Critical extinction exponent

Now, we turn to the discussion of the critical extinction exponent for problem (1.1)–(1.3). The following two theorems exhibit the details of such characteristics.

Theorem 4.1. *Assume that $u_0(x) \geqslant 0$ with $u_0^m(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$. If $q > m(p-1)$, then the weak solution of problem (1.1)–(1.3) vanishes in finite time for appropriately small initial data u_0. In addition, if $q = m(p-1)$ with $\lambda < \lambda_1$, then the weak solution goes to zero in the sense of the norm $L^{m+1}(\Omega)$ as $t \to \infty$ and, in particular, if*

$$\frac{(m+1)N}{mN+m+1} \leqslant p \quad \text{or} \quad 1 < p < \frac{(m+1)N}{mN+m+1}$$

with

$$\lambda < \lambda_1 \biggl(\frac{mp}{mp - Nm + (m+1)(N/p-1)}\biggr)^p \frac{N + Nm - p - Nmp}{mp},$$

then u vanishes in finite time too.

Remark 4.2. In fact, from the proof of Theorem 4.1, we see that when $q = 1$ the boundedness restriction on the solution u is unnecessary.

Theorem 4.3. *For any non-negative function $u_0(x)$ with $u_0^m(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, problem (1.1)–(1.3) admits at least one bounded non-negative and non-extinction weak solution for the case $q = m(p-1)$ with $\lambda > \lambda_1$ or $0 < q < m(p-1)$.*

For convenience, in the following proof, we assume that the weak solution is appropriately smooth; otherwise, we can consider the corresponding regularized problem, and the same result can also be obtained through an approximate process.

Proof of Theorem 4.1. We divide the proof into three steps according to the different intervals of q: that is,

(i) $q \geqslant 1$,

(ii) $m(p-1) < q < 1$,

(iii) $q = m(p-1)$ with $\lambda < \lambda_1$.

(i) In view of Proposition 2.3, we see that the existence of bounded solutions is possible for a suitable initial datum u_0. Set $M = \|u\|_{L^\infty(Q)}$. Multiplying equation (1.1) by u^r, where $r > 0$ satisfies

$$\frac{p(r+1)}{mp-m+r} \leqslant \frac{Np}{N-p} \quad \text{if } N > p$$

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(namely $r > \max\{N(m+1)/p-1-Nm, 0\}$), otherwise $r > 0$ is arbitrary, and integrating over Ω we conclude that

$$\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + \left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x \leqslant \lambda M^{q-1}\int_\Omega u^{r+1}\,\mathrm{d}x. \qquad (4.1)$$

Moreover, recalling the imbedding theorem, we also have

$$\left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x \geqslant C\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(mp-m+r)/(r+1)}.$$

Combining this result with (4.1), we conclude that

$$\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + C\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(mp-m+r)/(r+1)} \leqslant \lambda M^{q-1}\int_\Omega u^{r+1}\,\mathrm{d}x. \quad (4.2)$$

Let

$$f(t) = \int_\Omega u^{r+1}\,\mathrm{d}x, \qquad \alpha = \frac{mp-m+r}{r+1} < 1.$$

It follows that

$$f'(t) + C(r+1)f^\alpha(t) \leqslant \lambda(r+1)M^{q-1}f(t).$$

If there exists a $t_0 > 0$ such that $f(t_0) = 0$, then

$$f(t) \leqslant \lambda(r+1)M^{q-1}\int_{t_0}^t f(\tau)\,\mathrm{d}\tau.$$

Recalling Grönwall's inequality, we obtain

$$f(t) \equiv 0 \quad \text{for any } t > t_0.$$

Otherwise, $f(t) > 0$ holds for all t. Then we have

$$(f^{1-\alpha})' - \lambda(1-\alpha)(r+1)M^{q-1}f^{1-\alpha}(t) \leqslant -C(r+1)(1-\alpha).$$

By a simple calculation, we arrive at

$$\begin{aligned} f^{1-\alpha}(t) &\leqslant f^{1-\alpha}(0)\mathrm{e}^{\lambda(1-\alpha)(r+1)M^{q-1}t} - \frac{C}{\lambda M^{q-1}}(\mathrm{e}^{\lambda(1-\alpha)(r+1)M^{q-1}t} - 1) \\ &\leqslant \left(\left(\int_\Omega u_0^{r+1}\,\mathrm{d}x\right)^{1-\alpha} - \frac{C}{\lambda M^{q-1}}\right)\mathrm{e}^{\lambda(1-\alpha)(r+1)M^{q-1}t} + \frac{C}{\lambda M^{q-1}}. \end{aligned}$$

Choose a sufficiently small $u_0(x)$ such that

$$\left(\int_\Omega u_0^{r+1}\,\mathrm{d}x\right)^{1-\alpha} \leqslant \frac{C}{2\lambda M^{q-1}}.$$

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Then the two inequalities above give

$$f^{1-\alpha}(t) \leqslant -\frac{C}{2\lambda M^{q-1}}(\mathrm{e}^{\lambda(1-\alpha)(ps+2)M^{q-1}t} - 2).$$

From the inequality above, we see that there must exist a $T > 0$ such that $f^{1-\alpha}(t) \leqslant 0$ for any $t \geqslant T$. Obviously, this is a contradiction. In conclusion, there exists a positive constant T^* such that

$$\int_\Omega u^{r+1}\,\mathrm{d}x \equiv 0 \quad \text{for any } t \geqslant T^*,$$

which implies that u vanishes in finite time. This completes the proof of (i).

(ii) Multiplying equation (1.1) by u^r, and integrating over Ω, we deduce

$$\begin{aligned}\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + \left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x \\ = \lambda\int_\Omega u^{r+q}\,\mathrm{d}x \leqslant \lambda|\Omega|^{(1-q)/(r+1)}\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(r+q)/(r+1)}. \qquad (4.3)\end{aligned}$$

Choose an appropriately large $r > 0$, such that

$$\frac{p(r+1)}{mp-m+r} \leqslant \frac{Np}{N-p} \quad \text{if } N > p;$$

otherwise, $r > 0$ is arbitrary. Then, according to the imbedding theorem, we have

$$\left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x \geqslant C\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(mp-m+r)/(r+1)}.$$

Substituting into (4.3), we obtain

$$\begin{aligned}\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + C\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(mp-m+r)/(r+1)} \\ \leqslant \lambda|\Omega|^{(1-q)/(r+1)}\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(r+q)/(r+1)}. \qquad (4.4)\end{aligned}$$

By a direct calculation, we further have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(1-m(p-1))/(r+1)} \\ &\quad \leqslant (1-m(p-1))\left(\lambda|\Omega|^{(1-q)/(r+1)}\left(\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x\right)^{(q-m(p-1))/(r+1)} - C\right). \qquad (4.5)\end{aligned}$$

For simplicity, we set

$$\mathcal{M}(u(t)) = \lambda|\Omega|^{(1-q)/(r+1)}\left(\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x\right)^{(q-m(p-1))/(r+1)} - C.$$

If $\mathcal{M}(u_0) < 0$, combining this with (4.5), we obtain that $\mathcal{M}(u(t))$ is decreasing on t. Hence, we have

$$\mathcal{M}(u(t)) \leqslant \mathcal{M}(u_0).$$

We further have

$$\left(\int_\Omega u^{r+1}\,\mathrm{d}x\right)^{(1-m(p-1))/(r+1)} \leqslant (1-m(p-1))\mathcal{M}(u_0)t + \|u_0\|_{L^{r+1}(\Omega)}^{1-m(p-1)}.$$

It is evident that there exists a positive constant T_0, such that

$$\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x \equiv 0 \quad \text{for all } t \geqslant T_0,$$

which implies $u = 0$ a.e. in Ω for $t \geqslant T_0$.

(iii) First we show that

$$\|u(\cdot\,,t)\|_{L^\infty(\Omega)} \leqslant M_0, \tag{4.6}$$

where

$$M_0 = \begin{cases} \|u_0\|_{L^\infty(\Omega)}, & \alpha_0 \geqslant 1, \\ \dfrac{1}{1-\alpha_0}\|u_0\|_{L^\infty(\Omega)}, & \alpha_0 < 1. \end{cases}$$

Set

$$l_0 = \|u_0\|_{L^\infty(\Omega)}, \qquad \alpha_0 = \frac{\lambda_1 (mp)^p}{\lambda m(mp-m+1)^p}.$$

Multiplying equation (1.1) by $(u-l_0)_+$ and integrating over Ω we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l_0)_+^2\,\mathrm{d}x + \frac{(mp)^p}{m(mp-m+1)^p}\int_{A_{l_0}(t)} |\nabla u^{(mp-m+1)/p}|^p\,\mathrm{d}x = \lambda\int_\Omega u^{m(p-1)}(u-l_0)_+\,\mathrm{d}x,$$

where $A_{l_0}(t) = \{x \in \Omega;\ u(x,t) > l_0\}$. Since λ_1 is the first eigenvalue of p-Laplacian equation with homogeneous Dirichlet boundary-value condition, combining this with (2.3), we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l_0)_+^2\,\mathrm{d}x + \lambda_1\frac{(mp)^p}{m(mp-m+1)^p}\int_{A_{l_0}(t)} u^{mp-m+1}\,\mathrm{d}x \leqslant \lambda\int_\Omega u^{m(p-1)}(u-l_0)_+\,\mathrm{d}x.$$

We further have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l_0)_+^2\,\mathrm{d}x \leqslant \lambda\int_{A_{l_0}(t)} u^{m(p-1)}\left(u - l_0 - \frac{\lambda_1(mp)^p}{\lambda m(mp-m+1)^p}u\right)\mathrm{d}x. \tag{4.7}$$

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If $\alpha_0 \geqslant 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l_0)_+^2\,\mathrm{d}x \leqslant 0.$$

Therefore, we have

$$\int_\Omega (u-l_0)_+^2\,\mathrm{d}x = 0 \quad \text{a.e. in } \Omega,$$

which implies that $\|u(\cdot\,,t)\|_{L^\infty(\Omega)} \leqslant l_0$, while, if $\alpha_0 < 1$, let

$$T_0 = \inf\{t > 0;\ \|u(\cdot\,,t)\|_{L^\infty(\Omega)} \geqslant l_0/(1-\alpha_0)\}.$$

Since

$$\|u(\cdot\,,t)\|_{L^\infty(\Omega)} \leqslant \|u_0\|_{L^\infty(\Omega)} + t\lambda\|u\|_{L^\infty(Q_t)}^{m(p-1)},$$

we see that $T_0 > 0$. Suppose to the contrary that $T_0 = +\infty$. Then we obtain

$$\|u\|_{L^\infty(Q)} \leqslant \|u_0\|_{L^\infty(\Omega)}/(1-\alpha_0).$$

Otherwise, $T_0 < +\infty$. Taking (4.7) into account, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u-l_0)_+^2\,\mathrm{d}x \leqslant 0 \quad \text{for all } t \leqslant T_0.$$

Thus, we have $\|u(\cdot\,,T_0)\|_{L^\infty(\Omega)} \leqslant l_0$. Clearly, it is a contradiction. Now (4.6) is a direct consequence of what we have proved.

Multiplying equation (1.1) by u^m and integrating over Ω, we conclude that

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m+1}\,\mathrm{d}x + \int_\Omega |\nabla u^m|^p\,\mathrm{d}x \leqslant \lambda\int_\Omega u^{mp}\,\mathrm{d}x. \tag{4.8}$$

Noting that

$$\lambda_1\int_\Omega u^{mp}\,\mathrm{d}x \leqslant \int_\Omega |\nabla u^m|^p\,\mathrm{d}x,$$

we have

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m+1}\,\mathrm{d}x \leqslant -(\lambda_1-\lambda)\int_\Omega u^{mp}\,\mathrm{d}x.$$

Let $v = u/M_0$. Then we further have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^{m+1}\,\mathrm{d}x &\leqslant -(m+1)M_0^{mp-m-1}(\lambda_1-\lambda)\int_\Omega v^{mp}\,\mathrm{d}x\\ &\leqslant -(m+1)M_0^{mp-m-1}(\lambda_1-\lambda)\int_\Omega v^{m+1}\,\mathrm{d}x,\end{aligned}$$

which implies that

$$\int_\Omega v^{m+1}\,\mathrm{d}x \leqslant \mathrm{e}^{-(m+1)M_0^{mp-m-1}(\lambda_1-\lambda)t}\int_\Omega v_0^{m+1}\,\mathrm{d}x\,\mathrm{d}x.$$

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Furthermore, we have

$$\int_\Omega u^{m+1}\,\mathrm{d}x \leqslant \mathrm{e}^{-(m+1)M_0^{mp-m-1}(\lambda_1-\lambda)t}\int_\Omega u_0^{m+1}\,\mathrm{d}x\,\mathrm{d}x.$$

Therefore, we conclude that $\|u(\cdot\,,t)\|_{L^{m+1}(\Omega)} \to 0$ as $t\to\infty$.

In addition, by (4.8), we have

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m+1}\,\mathrm{d}x + \int_\Omega |\nabla u^m|^p\,\mathrm{d}x \leqslant \frac{\lambda}{\lambda_1}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x.$$

Using the imbedding theorem we obtain that if $p \geqslant (m+1)N/(mN+m+1)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{m+1}\,\mathrm{d}x &\leqslant -(m+1)\biggl(1-\frac{\lambda}{\lambda_1}\biggr)\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\\ &\leqslant -C_0(m+1)\biggl(1-\frac{\lambda}{\lambda_1}\biggr)\biggl(\int_\Omega u^{m+1}\,\mathrm{d}x\biggr)^{mp/(m+1)}.\end{aligned}$$

A similar argument to that above shows that there must exist a $T_0>0$, such that $\int_\Omega u^{m+1}(x,t)\,\mathrm{d}x \equiv 0$ for all $t \geqslant T_0$, which implies that u vanishes in a finite time.

The following argument is devoted to the discussion of the case when $1<p<(m+1)N/(mN+m+1)$. Noting (4.3), and taking $r=(N-p-Nmp+Nm)/p>m$, we obtain

$$\begin{aligned}\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + \biggl(\frac{mp}{mp-m+r}\biggr)^p\frac{r}{m}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x&\\ = \lambda\int_\Omega u^{r+m(p-1)}\,\mathrm{d}x \leqslant \frac{\lambda}{\lambda_1}\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x.&\qquad (4.9)\end{aligned}$$

Furthermore, we obtain

$$\frac{1}{r+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}\,\mathrm{d}x \leqslant -\biggl(\biggl(\frac{mp}{mp-m+r}\biggr)^p\frac{r}{m}-\frac{\lambda}{\lambda_1}\biggr)\int_\Omega |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x.$$

If

$$\lambda < \lambda_1\biggl(\frac{mp}{mp-m+r}\biggr)^p\frac{r}{m},$$

then according to the imbedding theorem we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{r+1}\,\mathrm{d}x \leqslant (r+1)\biggl(\frac{\lambda}{\lambda_1}-\biggl(\frac{mp}{mp-m+r}\biggr)^p\frac{r}{m}\biggr)\biggl(\int_\Omega u^{r+1}\,\mathrm{d}x\biggr)^{(mp-m+r)/(r+1)}.$$

A similar argument to that above shows that there exists a $T_1>0$ such that

$$\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x \equiv 0 \quad \text{for any } t\geqslant T_1.$$

We have thus proved the theorem. □

Remark 4.4. As for the case when $q = m(p-1)$ with $\lambda = \lambda_1$, $k\varphi_1^{1/m}(x)$ with $k > 0$ is a steady-state weak solution of (1.1)–(1.3). Then, for any non-trivial non-negative initial datum $u_0(x)$, the weak solution $u(x,t)$ of (1.1) satisfies $\int_\Omega u^p(x,t) > 0$ for all $t > 0$ or $u(x,t)$ is identically equal to zero.

Proof of Theorem 4.3. The result will be proved by using the weak upper and lower solutions method. Consider the following problem:

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \lambda(u_+ + 1)^q, &\qquad (x,t) \in \Omega \times (0,+\infty),\\ u = 0, &\qquad (x,t) \in \partial\Omega \times (0,+\infty),\\ u(x,0) = u_0(x) \geqslant 0, &\qquad x \in \Omega. \end{aligned}\right\} \tag{4.10}$$

Clearly, if u is a weak solution of the above problem, then by the weak maximum principle we have $u \geqslant 0$. By using the Leray–Schauder fixed-point theorem, as in the proof of Proposition 2.3, we may show that the problem (4.10) admits at least one non-negative weak solution u. In addition, the weak solution u is also a weak upper solution of problem (1.1)–(1.3). The following is devoted to the construction of weak lower solutions of problem (1.1)–(1.3) by using the first eigenvalue λ_1 and the first eigenfunction $\varphi_1(x)$ of the p-Laplacian equation with homogeneous Dirichlet boundary-value condition.

Case 1. When $q = m(p-1)$, $\lambda > \lambda_1$, let $v(x,t) = g(t)\varphi_1^{1/m}(x)$, where $g(t)$ satisfies

$$\begin{aligned} &g'(t) = (\lambda - \lambda_1)g^{m(p-1)}(t), && t > 0,\\ &g(t) > 0, && t > 0,\\ &g(0) = 0. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= (\lambda - \lambda_1)\varphi_1^{1/m}(x)g^{m(p-1)}(t)\\ &\leqslant (\lambda - \lambda_1)\varphi_1^{p-1}(x)g^{m(p-1)}(t)\\ &= \operatorname{div}(|\nabla v^m|^{p-2}\nabla v^m) + \lambda v^{m(p-1)}, \end{aligned}$$

i.e. v is a weak lower solution.

Case 2. When $q < m(p-1)$, let $v(x,t) = \mu g(t)\varphi_1^{1/m}(x)$, where $g(t)$ is a solution of the following problem:

$$\left.\begin{aligned} &g'(t) = -\lambda_1 g^{m(p-1)}(t) + \lambda g^q, && t > 0,\\ &g(t) > 0, && t > 0,\\ &g(0) = 0. \end{aligned}\right\} \tag{4.11}$$

Then we have

$$\frac{\partial v}{\partial t} = (-\lambda_1 g^{m(p-1)}(t) + \lambda g^q)\mu\varphi_1^{1/m}(x)$$

and

$$\operatorname{div}(|\nabla v^m|^{p-2}\nabla v^m) + \lambda v^q = -\lambda_1 \mu^{m(p-1)} g^{m(p-1)} \varphi_1^{p-1}(x) + \lambda \mu^q g^q \varphi_1^{q/m}(x).$$

If $g(t)$ is bounded in $\mathbb{R}^+$, let

$$M = \max_{t>0} \frac{\lambda_1}{\lambda} g^{m(p-1)-q}(t).$$

Then we can choose a $\mu > 0$ small enough such that

$$\lambda_1 g^{m(p-1)}(\mu^{m(p-1)}\varphi_1^{p-1} - \mu\varphi_1^{1/m}) \leqslant \lambda g^q(\mu^q \varphi_1^{q/m} - \mu\varphi_1^{1/m}). \tag{4.12}$$

Then v is a weak lower solution. Indeed, from (4.11), it is not difficult to see that $g(t)$ is a non-decreasing and bounded function. In addition, let $F(x) = (x^q - x)/(x^{m(p-1)} - x)$. It is easy to check that $F(x)$ is decreasing in $(0,1)$, and $\lim_{x\to 0^+} F(x) = +\infty$. Thus, we can choose a sufficiently small $\mu > 0$ such that (4.12) holds.

Up to now, we have constructed a pair of weak upper and lower solutions u, v. If $v \leqslant u$, then the problem admits a weak solution $v \leqslant \tilde{u} \leqslant u$. Next, we show that $v \leqslant u$.

From the definition of u, v, we deduce that

$$\begin{aligned}
\int_\Omega (v-u)\varphi(x,t)\,\mathrm{d}x - \int_\Omega (v_0 - u_0)\varphi(x,0)\,\mathrm{d}x - \iint_{Q_t} (v-u)\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau \\
+ \iint_{Q_t} (|\nabla v^m|^{p-2}\nabla v^m - |\nabla u^m|^{p-2}\nabla u^m)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}\tau \\
\leqslant \lambda \iint_{Q_t} (v^q - (u_+ + 1)^q)\varphi\,\mathrm{d}x\,\mathrm{d}\tau \quad \text{for all } \varphi \geqslant 0.
\end{aligned}$$

Take $\varphi_\varepsilon(x,t) = H_\varepsilon(v^m - u^m)$, where $H_\varepsilon(s)$ is defined as above. Letting $\varepsilon \to 0$ yields

$$\begin{aligned}
\int_\Omega (v(x,t) - u(x,t))_+\,\mathrm{d}x &\leqslant \lambda \iint_{Q_t} (v^q - (u_+ + 1)^q) H(v-u)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \lambda q \iint_{Q_t} (v - (u_+ + 1))_+\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \lambda q \iint_{Q_t} (v-u)_+\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Recalling Grönwall's inequality yields

$$\int_\Omega (v(x,t) - u(x,t))_+\,\mathrm{d}x = 0 \quad \text{for all } t > 0,$$

which implies that $v \leqslant u$ a.e. in Q. Since $v(x,t)$ does not vanish, neither does $\tilde{u}$. □

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Appendix A. Proof of Proposition 2.3

We divide the proof of Proposition 2.3 into two parts: for the cases when $m \geqslant 1$ and $m < 1$.

Case 1. We first consider the case when $m \geqslant 1$. The proof will again be divided into several steps. Namely, we first show the existence of non-negative weak solutions for the case when $q \leqslant 1$, and the second step is devoted to the case when $q > 1$.

Step 1. For any $T > 0$, let us consider the following problem:

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \lambda u^q, \qquad (x,t) \in \Omega \times (0,T), \tag{A 1}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{A 2}$$

$$u(x,0) = u_0(x) \geqslant 0, \qquad x \in \Omega. \tag{A 3}$$

In order to prove the existence of the above problem, we first consider the following auxiliary problem:

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \sigma\lambda v_+^q, \qquad (x,t) \in \Omega \times (0,T), \tag{A 4}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{A 5}$$

$$u(x,0) = \sigma u_0(x), \qquad x \in \Omega. \tag{A 6}$$

Clearly, according to the weak maximum principle, if u is a weak solution of the above problem, then $u \geqslant 0$ since $u_0 \geqslant 0$. Now, let us define an operator

$$\Gamma : L^{r^*+1}(Q_T) \to L^{r^*+1}(Q_T),$$
$$v \mapsto u,$$

where r^* is determined later, u is a weak solution of the above problem. In order to apply the Leray–Schauder fixed-point theorem, it is necessary to show that Γ is compact.

It is well known that there exists a weak solution u in the generalized sense for the above problem. In order to obtain some necessary estimates, we assume that the weak solution is appropriately smooth, since the same result can be obtained through an approximate process by considering a related regularized problem.

Multiplying equation (A 4) by u^r ($r^* \geqslant r > \max\{0, 2q+m-2\}$) and integrating over Q_t (for any $0 < t < T$), we arrive at

$$\begin{aligned}
&\frac{1}{r+1}\int_\Omega u^{r+1}(x,t)\,\mathrm{d}x + \left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m} \iint_{Q_t} |\nabla u^{(mp-m+r)/p}|^p \,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\leqslant \sigma\lambda \iint_{Q_t} v_+^q u^r \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{r+1}\int_\Omega u_0^{r+1}\,\mathrm{d}x \\
&\quad\leqslant \frac{\sigma\lambda}{r+1}\iint_{Q_t} v_+^{(r+1)q}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{r\sigma\lambda}{r+1}\iint_{Q_t} u^{r+1}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{r+1}\int_\Omega u_0^{r+1}\,\mathrm{d}x.
\end{aligned} \tag{A 7}$$

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By dropping the second term on the left-hand side, and according to Grönwall's inequality, it follows that

$$\int_{\Omega} u^{r+1}(x,t)\,\mathrm{d}x \leqslant \left(\sigma\lambda \iint_{Q_T} v_+^{(r+1)q}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Omega} u_0^{r+1}\,\mathrm{d}x\right)\mathrm{e}^{r\sigma\lambda t}. \tag{A 8}$$

Furthermore, we also have

$$\begin{aligned}\left(\frac{mp}{mp-m+r}\right)^p \frac{r}{m}\iint_{Q_T} |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant \frac{\sigma\lambda}{r+1}\iint_{Q_T} v_+^{(r+1)q}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{r+1}\int_{\Omega} u_0^{r+1}\,\mathrm{d}x \\ + \frac{\mathrm{e}^{r\sigma\lambda T}-1}{r+1}\left(\sigma\lambda \iint_{Q_T} v_+^{(r+1)q}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Omega} u_0^{r+1}\,\mathrm{d}x\right).\end{aligned} \tag{A 9}$$

Using the imbedding theorem, we further obtain

$$\left(\iint_{Q_T} u^l\,\mathrm{d}x\,\mathrm{d}\tau\right)^{(mp-m+r)/l} \leqslant C\iint_{Q_T} |\nabla u^{(mp-m+r)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau, \tag{A 10}$$

where $0 < l \leqslant N(mp-m+r)/(N-p)$ and C is a positive constant depending only on l and Ω. Moreover, by noting that

$$\iint_{Q_T} v_+^{q(r+1)}\,\mathrm{d}x\,\mathrm{d}t \leqslant \left(\iint_{Q_T} v_+^{r^*+1}\,\mathrm{d}x\,\mathrm{d}t\right)^{q(r+1)/(r^*+1)} |Q_T|^{1-(q(r+1))/(r^*+1)}$$

and taking (A 9), (A 10) into account, for any $l \leqslant N(mp-m+r^*)/(N-p)$, it follows that

$$\iint_{Q_T} u^l(x,t)\,\mathrm{d}x\,\mathrm{d}t \text{ is bounded uniformly.} \tag{A 11}$$

If we take $r^* > (N-p+Nm(1-p))/p$, then $N(mp-m+r^*)/(N-p) > r^*+1$.

Now, multiplying equation (A 4) by $\partial u^m/\partial t$ and integrating the resulting relation over Q_T, and through a simple calculation, we derive

$$\begin{aligned}\frac{4m}{(m+1)^2}\iint_{Q_T}\left(\frac{\partial u^{(m+1)/2}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t + \frac{1}{p}\int_{\Omega}|\nabla u^m(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \frac{1}{p}\int_{\Omega}|\nabla u_0^m|^p\,\mathrm{d}x + \frac{2m}{(m+1)^2}\iint_{Q_T}\left(\frac{\partial u^{(m+1)/2}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t + C\iint_{Q_T} v_+^{2q}u^{m-1}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Furthermore, if $r^* > m$, then we have

$$\begin{aligned}\frac{2m}{(m+1)^2}\iint_{Q_T}\left(\frac{\partial u^{(m+1)/2}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t \\ \leqslant \frac{1}{p}\int_{\Omega}|\nabla u_0^m|^p\,\mathrm{d}x + C\|v\|_{L^{r^*+1}}^{2q}\|u\|_{L^{r^*+1}}^{(m-1)/(r^*+1)}|Q_T|^{(r^*+2-2q-m)/(r^*+1)}.\end{aligned} \tag{A 12}$$

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Recalling (A 7) and taking $r = m - p/2(m-1) > 1/m > 0$ (since $m(p-1) < 1$) yields

$$\frac{(2m)^{p-1}(p+2m-mp)}{(m+1)^p}\iint_{Q_T}|\nabla u^{(m+1)/2}|^p\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{r}\int_\Omega u^{r+1}\,\mathrm{d}x \leqslant \sigma\lambda\|v_+\|^q_{L^{r^*+1}}\|u\|^r_{L^{r^*+1}}|Q_T|^{(r^*+1-q-r)/(r^*+1)}+\frac{1}{r}\int_\Omega u_0^{r+1}\,\mathrm{d}x. \quad (\text{A }13)$$

Choose a suitable large r^* such that

$$r^* > \max\biggl\{m, \frac{N-p+Nm(1-p)}{p}, m-\tfrac{1}{2}p(m-1)\biggr\}.$$

Recalling (A 11)–(A 13), and combining these with the compact imbedding theorem, we conclude that Γ is a compact operator. Furthermore, it is easy to see that $\Gamma(u,0)=0$. In addition, if $\Gamma(u,\sigma)=u$, noting (A 7), we arrive at

$$\begin{aligned}\frac{1}{r^*+1}\int_\Omega u^{r^*+1}(x,t)\,\mathrm{d}x &\leqslant \sigma\lambda\iint_{Q_t}u^{r^*+q}\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{r^*+1}\int_\Omega u_0^{r^*+1}\,\mathrm{d}x\\ &\leqslant \sigma\lambda\frac{r^*+q}{r^*+1}\iint_{Q_t}u^{r^*+1}\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1-q}{r^*+1}\sigma\lambda|Q_T|+\frac{1}{r^*+1}\int_\Omega u_0^{r^*+1}\,\mathrm{d}x.\end{aligned}$$

Recalling Grönwall's inequality, we further obtain

$$\int_\Omega u^{r^*+1}(x,t)\,\mathrm{d}x \leqslant \biggl((1-q)\sigma\lambda|Q_T|+\int_\Omega u_0^{r^*+1}\,\mathrm{d}x\biggr)\mathrm{e}^{\sigma\lambda(r^*+q)T}, \quad (\text{A }14)$$

which implies that, for any $u\in L^{r^*+1}(Q_T)$, if it satisfies $\Gamma(u,\sigma)=u$, then u is bounded uniformly in $L^{r^*+1}(Q_T)$. By means of the Leray–Schauder fixed-point theorem, we conclude that $\Gamma(\cdot,1)$ admits a fixed point, which implies that the problem (A 1)–(A 3) admits a non-negative weak solution. In the following, we consider problem (1.1)–(1.3) in $Q_{(T,2T)}, Q_{(2T,3T)}, \dots, Q_{((n-1)T,nT)}$. Then, by an inductive argument, we infer that problem (1.1)–(1.3) admits a weak solution in Q. Furthermore, we assert that problem (A 1)–(A 3) admits a unique non-negative weak solution if $q=1$. Suppose by contradiction that u_1, u_2 are non-negative weak solutions of problem (A 1)–(A 3). Let $w=u_1-u_2$, and take the test function

$$\varphi_\varepsilon(x,t)=H_\varepsilon(u_1(x,t)-u_2(x,t)),$$

where $H_\varepsilon(s)$ is a monotone increasing smooth approximation of the function $H(s)$:

$$H(s)=\begin{cases}1, & s>0,\\ 0, & \text{otherwise}.\end{cases}$$

It is easy to see that $H'_\varepsilon(s)\to\delta(s)$ as $\varepsilon\to 0$. Letting $\varepsilon\to 0$ yields

$$\int_\Omega w_+\,\mathrm{d}x \leqslant \lambda\iint_{Q_t}w_+\,\mathrm{d}x\,\mathrm{d}t.$$

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Recalling Grönwall's inequality yields $w_+ = 0$ a.e., namely $u_1 \leqslant u_2$ a.e. Similarly, we also get $u_2 \leqslant u_1$. Summing up, we conclude that $u_1 = u_2$ a.e., which is a contradiction.

Step 2. In this step, we study the case when $q > 1$. Firstly, we are concerned with the local solvability in time. For any non-negative initial datum $u_0(x)$ with $u_0^m \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, there must exist a positive constant $T > 0$, such that

$$\left(\frac{1}{2\lambda T}\right)^{1/(q-1)} = 2\|u_0\|_{L^\infty(\Omega)}.$$

For simplicity, we set

$$R = \left(\frac{1}{2\lambda T}\right)^{1/(q-1)}.$$

Consider the following problem:

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \sigma\Psi(v), \quad (x,t) \in \Omega \times (0,T), \tag{A 15}$$

$$u = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{A 16}$$

$$u(x,0) = \sigma u_0(x) \geqslant 0, \qquad x \in \Omega, \tag{A 17}$$

where

$$\Psi(v) = \begin{cases} \lambda v_+^q, & |v| \leqslant R, \\ \lambda R^q, & |v| > R. \end{cases}$$

Define an operator

$$\Gamma : L^{r^*+1}(Q_T) \times [0,1] \to L^{r^*+1}(Q_T),$$
$$(v,\sigma) \mapsto u,$$

where r^* is the above-mentioned constant. If u is a fixed point of $\Gamma(\cdot\,,1)$ with $\|u\|_{L^\infty(Q_T)} \leqslant R$, then u is clearly a weak solution of problem (1.1)–(1.3) in Q_T. Similarly, we may show that Γ is compact. In addition, if $\Gamma(u,\sigma) = u$, according to the weakly maximum principle, we obtain that

$$\|u\|_{L^\infty(Q_T)} \leqslant \|u_0\|_{L^\infty(\Omega)} + T\sigma\|\Psi(u)\|_{L^\infty(Q_T)} \leqslant \tfrac{1}{2}R + T\lambda R^q \leqslant R$$

and

$$\|u\|_{L^{r^*+1}(Q_T)} \leqslant |Q_T|^{1/(r^*+1)}\|u\|_{L^\infty(Q_T)} \leqslant R|Q_T|^{1/(r^*+1)}.$$

Recalling the Leray–Schauder fixed-point theorem, we see that $\Gamma(\cdot\,,1)$ has a fixed point. By similar arguments to those above, we see that u is also a weak solution of problem (1.1)–(1.3) in Q_T.

The above assertion can then be extended to the maximum time-interval $(0,T^*)$, where, for any $T < T^*$, the solution remains bounded in $\Omega \times (0,T]$. Thus, the local-in-time solvability is established.

In what follows, we show the global existence of bounded solutions for the case when $u_0^m(x) \leqslant \varphi_1(x)$ and $\lambda \leqslant \lambda_1$. Clearly, we have $\|u_0\|_{L^\infty(\Omega)} \leqslant 1$. Let $T = 1/2^q\lambda$, and then consider problem (A 15)–(A 17) with

$$\Psi(v) = \begin{cases} \lambda v_+^q, & |v| \leqslant 2, \\ \lambda 2^q, & |v| > 2. \end{cases}$$

If u is a weak solution of the problem (A 15)–(A 17) with $\|u\|_{L^\infty(Q_T)} \leqslant 2$, then u is clearly a weak solution of problem (1.1)–(1.3) in Q_T. By a parallel argument, we get the local-in-time solvability and

$$\|u\|_{L^\infty(Q_T)} \leqslant \|u_0\|_{L^\infty(\Omega)} + \lambda T 2^q \leqslant 2,$$

which implies that u is also a weak solution of problem (1.1)–(1.3) in Q_T.

Recall the first eigenfunction $\varphi_1(x)$ ($\|\varphi_1(x)\|_{L^\infty(\Omega)} = 1$) of p-Laplacian equation with homogeneous Dirichlet boundary-value condition. Let $\Phi(x) = \varphi_1^{1/m}(x)$. Then we obtain

$$-\operatorname{div} |\nabla \Phi^m|^{p-2} \nabla \Phi^m = \lambda_1 |\Phi^m|^{p-2} \Phi^m \geqslant \lambda \Phi^q.$$

Then, for any $u_0(x) \leqslant \Phi(x)$, $\Phi(x)$ is clearly a weak upper solution.

In what follows, we shall show that $u \leqslant \Phi(x)$. Since u is a weak solution, and Φ is a weak upper solution, for any non-negative function $\varphi(x) \in \tilde{E}_0$, we have

$$\begin{aligned} &\int_\Omega (u(x,t) - \Phi(x,t))\varphi(x,t)\,\mathrm{d}x - \int_\Omega (u_0(x) - \Phi(x))\varphi(x,0)\,\mathrm{d}x \\ &\quad - \iint_{Q_t} (u - \Phi)\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_t} (|\nabla u^m|^{p-2}\nabla u^m - |\nabla \Phi^m|^{p-2}\nabla \Phi^m)\nabla \varphi\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad \leqslant \lambda \iint_{Q_t} (u^q - \Phi^q)\varphi\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

For any $t > 0$, noting that (A 12), we can choose $\varphi_\varepsilon(x) = H_\varepsilon(u^m - \Phi^m)$, where $H_\varepsilon(s)$ is a monotone increasing smooth approximation of the function $H(s)$ with $H'_\varepsilon(s) \to \delta(s)$. Then we have

$$\begin{aligned} &\int_\Omega (u(x,t) - \Phi(x))H_\varepsilon(u^m - \Phi^m)\,\mathrm{d}x - \int_\Omega (u_0 - \Phi)H_\varepsilon(u_0^m - \Phi^m)\,\mathrm{d}x \\ &\quad + \iint_{Q_t} H'_\varepsilon(u^m - \Phi^m)(|\nabla u^m|^{p-2}\nabla u^m - |\nabla \Phi^m|^{p-2}\nabla \Phi^m)\nabla(u^m - \Phi^m)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad - \iint_{Q_t} (u - \Phi)\frac{\partial H_\varepsilon(u^m - \Phi^m)}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad\quad \leqslant \lambda \iint_{Q_t} (u^q - \Phi^q)H_\varepsilon(u^m - \Phi^m)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

Letting $\varepsilon \to 0$ and noting that

$$\iint_{Q_t} (u - \Phi)\frac{\partial H_\varepsilon(u^m - \Phi^m)}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau \to 0,$$

we arrive at

$$\int_\Omega (u(x,t)-\Phi(x))_+\,\mathrm{d}x \leqslant \lambda \iint_{Q_t} (u^q-\Phi^q)_+\,\mathrm{d}x\,\mathrm{d}\tau \leqslant R^{q-1}\lambda \iint_{Q_t} (u-\Phi)_+\,\mathrm{d}x\,\mathrm{d}\tau.$$

Grönwall's inequality then yields

$$\int_\Omega (u-\Phi)_+\,\mathrm{d}x = 0,$$

which implies that $u(x,t) \leqslant \Phi(x)$ a.e. in Ω.

In what follows, we consider problem (1.1)–(1.3) in $Q_{(T,2T)}, Q_{(2T,3T)}, \dots, Q_{((n-1)T,nT)}$ in turn. Then by inductive argument, we infer that u is a weak solution in Q satisfying $u(x,t) \leqslant \Phi(x)$.

The following argument is devoted to the uniqueness of non-negative weak solutions. Suppose to the contrary that u, v, $v \neq u$, are two non-negative weak solutions of problem (1.1)–(1.3). It suffices to consider the case when both u and v are bounded in Q_{T^*}.

Indeed, suppose that we have proved the above conclusion for any bounded u, v with $u \not\equiv v$. Consider the following cases.

(i) Both u and v exist globally. Then for any $T > 0$, u, v are both bounded in Q_T. According to the arguments above, we conclude that $u(\cdot\,,t) = v(\cdot\,,t)$, a.e. in Ω.

(ii) u exists globally, while v does not. We assume that the maximum time-interval where v exists is $(0,T)$. Then there exist M, $\hat{M}$, T_1 with $0 < M < M+1 < \hat{M}$, $T_1 < T$, such that $M+1 < \|v\|_{L^\infty(Q_T)} < \hat{M}$ and $\|u\|_{L^\infty(Q_T)} < M$, similar to the arguments above. We also get that $u(\cdot\,,t) = v(\cdot\,,t)$ a.e. in Ω for any $t \in (0,T)$, which contradicts the fact that $\|v\|_{L^\infty(Q_T)} > M+1$, $\|u\|_{L^\infty(Q_T)} < M$.

Furthermore, for the case when u, v both exist locally but not globally (which includes the fact that the maximum time intervals of u, v may or may not be uniform), the arguments are all similar to those above, so we omit them.

Thus, we may assume that $\|v\|_{L^\infty(Q_{T^*})}$, $\|u\|_{L^\infty(Q_{T^*})} \leqslant M$, where M is a positive constant.

Take the test function as above: $\varphi_\varepsilon(x,t) = H_\varepsilon(u(x,t)-v(x,t))$. Then we have

$$\begin{aligned}\int_\Omega (u-v)H_\varepsilon(u^m-v^m)\,\mathrm{d}x - \iint_{Q_t} (u-v)\frac{\partial H_\varepsilon(u^m-v^m)}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau \\ + \iint_{Q_t} H_\varepsilon'(u^m-v^m)(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v^m|^{p-2}\nabla v^m)\nabla(u^m-v^m)\,\mathrm{d}x\,\mathrm{d}\tau \\ \leqslant \lambda q M^{q-1} \iint_{Q_t} (u-v)H_\varepsilon(u^m-v^m)\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

Letting $\varepsilon \to 0$, and discarding the third term on the left-hand side yields

$$\int_\Omega (u(x,t)-v(x,t))_+\,\mathrm{d}x \leqslant \lambda q M^{q-1} \iint_{Q_t} (u(x,\tau)-v(x,\tau))_+\,\mathrm{d}x\,\mathrm{d}\tau.$$

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Employing Grönwall's inequality yields

$$\int_\Omega (u(x,t)-v(x,t))_+ \,\mathrm{d}x = 0 \quad \text{for all } t \leqslant \delta,$$

which implies that $v \geqslant u$ a.e. in Ω for any $t \leqslant \delta$. Similarly, we also have $v \leqslant u$, which means that $v = u$ a.e. This is clearly a contradiction.

Case 2. For simplicity, we set $u^m = v$. Then problem (1.1)–(1.3) is transformed into

$$\frac{\partial v^\alpha}{\partial t} = \operatorname{div}(|\nabla v|^{p-2}\nabla v) + \lambda v^{\hat q}, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{A 18}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}, \tag{A 19}$$

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{A 20}$$

where $\alpha = 1/m$, $\hat q = q/m$. In order to show the existence of non-negative weak solutions for the above problem, let us first consider the corresponding regularized problem in a bounded domain Q_T:

$$\alpha(v_+^{\alpha-1}+\varepsilon)\frac{\partial v}{\partial t} = \operatorname{div}((|\nabla v|^2+\varepsilon)^{(p-2)/2}\nabla v) + \lambda(v^2+\varepsilon)^{(\hat q-1)/2} v \quad \text{in } Q_T, \tag{A 21}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{A 22}$$

$$v(x,0) = v_{\varepsilon 0}(x), \quad x \in \Omega, \tag{A 23}$$

where $v_{\varepsilon 0}(x) \in C_0^{2+\alpha}(\Omega)$ and

$$v_{\varepsilon 0}(x) \to v_0(x) \quad \text{in } L^\infty(\Omega), \qquad \nabla v_{\varepsilon 0}(x) \to \nabla v_0(x) \quad \text{in } L^p(\Omega)$$

as $\varepsilon \to 0$.

In the following, we shall show the existence of classical solutions of the above problem. For this purpose, we define a mapping

$$\Gamma : C^{\alpha,\alpha/2}(Q_T) \times [0,1] \to C^{\alpha,\alpha/2}(Q_T),$$
$$\Gamma : (u,\sigma) \to v.$$

Here, v is a solution of the following problem:

$$\alpha(u_+^{\alpha-1}+\varepsilon)\frac{\partial v}{\partial t} = \operatorname{div}((|\nabla v|^2+\varepsilon)^{(p-2)/2}\nabla v) + \sigma\lambda(u^2+\varepsilon)^{(\hat q-1)/2} v \quad \text{in } Q_T, \tag{A 24}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{A 25}$$

$$v(x,0) = \sigma v_{\varepsilon 0}(x), \quad x \in \Omega. \tag{A 26}$$

It is well known that $v = \Gamma(u,\sigma) \in C^{2+\alpha,1+\alpha/2}(\bar Q_T)$. By means of the compact imbedding theorem, we see that Γ is compact. Furthermore, it is easy to see that $\Gamma(u,0) = 0$. In addition, if $\Gamma(v,\sigma) = v$, then by the classical theory of parabolic equations we have

$$\|v\|_{2+\alpha,1+\alpha/2} \leqslant M\|v_{\varepsilon 0}\|_{2+\alpha,1+\alpha/2},$$

where M depends only on ε. According to the Leray–Schauder fixed-point theorem, we see that $\Gamma(\cdot\,,1)$ admits a fixed point, which implies that (A 21)–(A 23) admits a classical solution, v_ε. Combining this with the weakly maximum principle, we further have $v_\varepsilon \geqslant 0$.

In what follows, we show the existence of weak solutions of the problem (A 18)–(A 20). We split the proof into two steps. In the first step, we are concerned with the case when $\hat{q} \leqslant \alpha$, that is $q \leqslant 1$. The second step is devoted to the case when $\hat{q} > \alpha$, that is $q > 1$.

Step 1. Assume that v_ε is a solution of problem (A 21)–(A 23). Replacing v by v_ε in equation (A 21), multiplying the equation by v_ε^r, $r \geqslant 1$, and then integrating in Q_t yields

$$\iint_{Q_t} \alpha(v_\varepsilon^{\alpha-1} + \varepsilon) v_\varepsilon^r \frac{\partial v_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau + r \iint_{Q_t} (|\nabla v_\varepsilon|^2 + \varepsilon)^{(p-2)/2} |\nabla v_\varepsilon|^2 v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \iint_{Q_t} \lambda(v_\varepsilon^2 + \varepsilon)^{(\hat{q}-1)/2} v_\varepsilon^{r+1}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{A 27}$$

If $p \geqslant 2$, then we have

$$\iint_{Q_t} (|\nabla v_\varepsilon|^2 + \varepsilon)^{(p-2)/2} |\nabla v_\varepsilon|^2 v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau \geqslant \iint_{Q_t} |\nabla v_\varepsilon|^p v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau,$$

while if $p < 2$,

$$\iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} (|\nabla v_\varepsilon|^2 + \varepsilon)^{(p-2)/2} |\nabla v_\varepsilon|^2 v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau \geqslant \frac{1}{2} \iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} |\nabla v_\varepsilon|^p v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau.$$

Combining this with the above two inequalities, a simple calculation yields

$$\frac{\alpha}{r+\alpha} \int_\Omega v_\varepsilon^{r+\alpha}\,\mathrm{d}x + \frac{\varepsilon\alpha}{r+1} \int_\Omega v_\varepsilon^{r+1}\,\mathrm{d}x + C_0 \iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau \leqslant \iint_{Q_t} \lambda v_\varepsilon^{r+\alpha}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\alpha}{r+\alpha} \int_\Omega v_{\varepsilon 0}^{r+\alpha}\,\mathrm{d}x + \frac{\varepsilon\alpha}{r+1} \int_\Omega v_{\varepsilon 0}^{r+1}\,\mathrm{d}x + M_0, \tag{A 28}$$

where M_0 is independent of ε. By using Grönwall's inequality, we obtain

$$\int_\Omega v_\varepsilon^{r+\alpha}(x,t)\,\mathrm{d}x \leqslant M_1 \quad \text{for any } t \leqslant T, \tag{A 29}$$

where M_1 depends only on r, T and $\|v_{\varepsilon 0}\|_{L^\infty(\Omega)}$. Furthermore, we have

$$\begin{aligned}
\iint_{Q_t} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau
&\leqslant \iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau + \iint_{Q_t;|\nabla v_\varepsilon|^2 < \varepsilon} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \left(\frac{r+p-1}{p}\right)^p \iint_{Q_t;|\nabla v_\varepsilon|^2 < \varepsilon} v_\varepsilon^{r-1} |\nabla v_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leqslant \iint_{Q_t;|\nabla v_\varepsilon|^2 \geqslant \varepsilon} |\nabla v_\varepsilon^{(r+p-1)/p}|^p\,\mathrm{d}x\,\mathrm{d}\tau + \left(\frac{r+p-1}{p}\right)^p \varepsilon^{p/2} \iint_{Q_t} v_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Combining this with (A 28)–(A 29), we obtain

$$\iint_{Q_T} |\nabla v_\varepsilon^{(r+p-1)/p}|^p \,\mathrm{d}x\,\mathrm{d}\tau \leqslant M_2.$$

In particular, we take $r = 1 + (\alpha - 1)p/2$. Then we arrive at

$$\iint_{Q_T} |\nabla v_\varepsilon^{(1+\alpha)/2}|^p \,\mathrm{d}x\,\mathrm{d}\tau \leqslant M_2. \tag{A 30}$$

Multiplying (A 21) by $\partial v_\varepsilon/\partial t$ yields

$$\iint_{Q_t} \alpha(v_\varepsilon^{\alpha-1} + \varepsilon)\left|\frac{\partial v_\varepsilon}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}\tau + \frac{1}{p}\int_\Omega (|\nabla v_\varepsilon|^2 + \varepsilon)^{p/2}\,\mathrm{d}x \\ \leqslant \frac{1}{p}\int_\Omega (|\nabla v_{\varepsilon 0}|^2 + \varepsilon)^{p/2}\,\mathrm{d}x + \frac{\lambda}{\hat{q}+1}\int_\Omega (v_\varepsilon^2 + \varepsilon)^{(\hat{q}+1)/2}\,\mathrm{d}x.$$

Furthermore, we have

$$\frac{4\alpha}{(1+\alpha)^2}\iint_{Q_t}\left|\frac{\partial v_\varepsilon^{(1+\alpha)/2}}{\partial t}\right|^2 + \frac{1}{p}\int_\Omega |\nabla v_\varepsilon|^p \\ \leqslant \frac{1}{p}\int_\Omega (|\nabla v_{\varepsilon 0}|^2 + \varepsilon)^{p/2} + \frac{\lambda}{\hat{q}+1}\int_\Omega (v_\varepsilon^2 + \varepsilon)^{(\hat{q}+1)/2}.$$

Combining this with (A 29), we conclude that

$$\left\|\frac{\partial v_\varepsilon^{(1+\alpha)/2}}{\partial t}\right\|_{L^2(Q_T)} \leqslant M^*, \tag{A 31}$$

$$\|\nabla v_\varepsilon\|_{L^p(\Omega)} \leqslant M^*, \tag{A 32}$$

where M^* is independent of ε. Noting the arbitrariness of $r > 1$, combining with (A 29), (A 30)–(A 32) and recalling the compact imbedding theorem, we conclude that there exists a $v \in L^\infty((0,T); L^r(\Omega)) \cap W_0^{1,p}(Q_T)$ and $\partial v^{(1+\alpha)/2}/\partial t \in L^2(Q_T)$ such that (up to the subsequence)

$$v_\varepsilon \to v \quad \text{in } L^r(\Omega), \qquad \nabla v_\varepsilon \rightharpoonup v \quad \text{in } L^p(Q_T),$$

and

$$\frac{\partial v_\varepsilon^{(1+\alpha)/2}}{\partial t} \rightharpoonup v \quad \text{in } L^2(Q_T)$$

as ε goes to zero, which implies that problem (A 18)–(A 20) admits a weak solution in Q_T. Then, consider problem (A 18)–(A 20) in $Q_{(T,2T)}, Q_{(2T,3T)}, \dots, Q_{((n-1)T,nT)}$ in turn. By induction, we infer that problem (A 18)–(A 20) admits a weak solution in Q. As for the uniqueness of solutions for the case $q = 1$ (i.e. $\hat{q} = 1/m$), this is similar to the case when $m \geqslant 1$ and we omit it.

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Step 2. When $\hat{q} > \alpha$, the proof is similar to that in step 1 and the case when $m \geqslant 1$, so we just give the outline of the proof. To prove the local-in-time existence, we first find a $T > 0$ with

$$\left(\frac{1}{2\lambda T}\right)^{1/(q-1)} = 2\|u_0\|_{L^\infty(\Omega)}.$$

For simplicity, set $R = (1/(2\lambda T))^{1/(q-1)}$. Then set

$$\Psi(v) = \begin{cases} \lambda v^{\hat{q}} & \text{if } |v| \leqslant R^m, \\ \lambda R^q & \text{otherwise.} \end{cases}$$

Similarly to the case when $m \geqslant 1$, we obtain the local-in-time solvability for any non-negative $v_0(x)$. Then we also obtain the global solvability by comparing it with the first eigenfunction's powers. The uniqueness is also similar to the case when $m \geqslant 1$, and thus we omit it.

Remark A 1. In Proposition 2.3 we obtained the weak solution satisfying $\partial u/\partial t \in L^2(Q_T)$ when $m \leqslant 1$. Indeed, since $\partial u^{(m+1)/2}/\partial t \in L^2(Q_T)$ and $u \in L^\infty(Q_T)$ for any given $0 < T < \infty$, by using

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q_T)} = \frac{2}{m+1}\left\|u^{(1-m)/2}\frac{\partial u^{(m+1)/2}}{\partial t}\right\|_{L^2(Q_T)},$$

we have

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2(Q_T)} \leqslant \frac{2}{m+1}\left\|\frac{\partial u^{(m+1)/2}}{\partial t}\right\|_{L^2(Q_T)}\|u\|_{L^\infty(Q_T)}^{(1-m)/2}.$$

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