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ZETA FUNCTIONS OF PREHOMOGENEOUS AFFINE SPACES

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§0. Introduction

Let ρ be an algebraic homomorphism of a linear algebraic group G into the affine transformation group $\mathrm{Aff}(V)$ of a finite dimensional vector space V. We say that a triplet (G, V, ρ) is a *prehomogeneous affine space,* if there exists a proper algebraic subset S of V such that $V - S$ is a single $\rho(G)$-orbit. In particular, (G, V, ρ) is a usual prehomogeneous vector space (PV, briefly) in the case where $\rho(G) \subset GL(V)$ (cf. [5], [7]). In the preceding paper [2], we defined zeta functions associated with certain prehomogeneous affine spaces and proved their analytic continuation and functional equations.

In the case of the PV's, M. Sato and Shintani [8] and F. Sato [3] established the theory of zeta functions associated with *regular* PV's (for the definition of a regular PV, see [5, Ch. 1, §1] or [7, §4, Definition 7]). Thus it is desirable to expand a similar theory for general prehomogeneous affine spaces. However there seems to be no appropriate definition of the *dual* of a prehomogeneous affine space and this causes a serious difficulty in studying zeta functions in the framework of prehomogeneous affine spaces.

In the present paper, we introduce the notion of an affine datum $\mathbf{D} = (G, V, \rho, \alpha)$ and its dual $\mathbf{D}^*$, where $\rho : G \rightarrow \mathrm{Aff}(V)$ is an algebraic homomorphism and $\alpha : V \times G \rightarrow \mathbf{G}_a$ is an affine 1-cocycle with respect to ρ. We say that $\mathbf{D}$ is a *prehomogeneous affine datum* (briefly, PAD) if (G, V, ρ) is prehomogeneous. As in the case of the PV's, the dual of a PAD is not necessarily prehomogeneous and we are led to introduce the notion of a *refular* PAD (for definition, see §2). In fact, we show that the dual of a regular PAD is also prehomogeneous and regular (Proposition 2.4). The object of the paper is to define zeta functions associated with regular PAD's and prove their analytic continuation and functional equations

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under certain mild assumptions.

We now explain a brief account of each section. In the first section, we state several elementary properties of a PAD without proofs since they are to be shown by the standard arguments in the theory of PV's. In §2, regular PAD's are defined and their fundamental properties are proved. The next two sections are devoted to the study of a-functions and b-functions of a regular PAD. The proofs are done by the arguments used in [5] with a slight modification (see also [6]). In §5, we consider (modified) complex powers of relative invariants of a regular PAD and study their Fourier transforms. In §6, we introduce zeta functions associated with a regular PAD and prove their functional equations by using the Poisson summation formula together with the results of §5. In the last §7, we explain several example of PAD's, one of which is closely related to the classical Hurwitz-Lerch zeta functions.

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Notation. As usual, we denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. We put $\mathbf{e}[x] = \exp(2\pi ix)$ for $x \in \mathbf{C}$. For a finite dimensional vector space V over $\mathbf{R}$, $\mathcal{S}(V)$ stands for the space of rapidly decreasing smooth functions on V. Denote by V^* the dual of V. The contragredient $A^{\sim} \in GL(V^*)$ of $A \in GL(V)$ is defined to be $\langle Ax, A^{\sim}x^* \rangle = \langle x, x^* \rangle$ $(x \in V,\ x^* \in V^*)$, where $\langle\ ,\ \rangle$ is the natural pairing of V and V^*

§1. Prehomogeneous affine datum

Let G be a connected linear algebraic group and V a finite dimensional vector space with a right G-action ρ, all defined over C. Assume that ρ is an affine action (that is to say, ρ defines an algebraic homomorphism of G into the group $\mathrm{Aff}(V)$ of affine transformations of V). A regular rational function α on $V \times G$ is called an *affine* 1-*cocycle* with respect to ρ if α satisfies the cocycle condition

$$\alpha(x, gg') = \alpha(x\rho(g), g') + \alpha(x, g) \qquad (x \in V,\ g,\ g' \in G) \tag{1.1}$$

and if the mapping $x \to \alpha(x, g) - \alpha(0, g)$ is linear on V for every $g \in G$. We call a quartet $\mathbf{D} = (G, V, \rho, \alpha)$ an *affine datum* if α is an affine cocycle with respect to ρ.

Denote by $\langle x, x^* \rangle$ the natural pairing of V and its dual V^*. The *dual* $\mathbf{D}^*$ of an affine datum $\mathbf{D} = (G, V, \rho, \alpha)$ is defined to be an affine datum $(G, V^*, \rho^*, \alpha^*)$ which satisfies

$$\langle x\rho(g), x^* \rangle + \alpha(x, g) = \langle x, x^*\rho^*(g^{-1}) \rangle + \alpha^*(x^*, g^{-1}) \tag{1.2}$$

for any $x \in V$, $x^* \in V^*$ and $g \in G$. It is easy to see that the dual $\mathbf{D}^*$ always exists and is uniquely determined by the condition (1.2). To describe $\mathbf{D}^*$ in an explicit manner, we define $a(g) \in GL(V)$, $b(g) \in V$ and $c^*(g) \in V^*$ to be

$$x\rho(g) = a(g)(x) + b(g) \tag{1.3}$$

$$\alpha(x, g) = \langle x, c^*(g) \rangle + \alpha(0, g). \tag{1.4}$$

Then ρ^* and α^* are given by

$$x^*\rho^*(g) = a(g)^{\tilde{}}x^* + c^*(g^{-1}) \tag{1.5}$$

$$\alpha^*(x^*, g) = \langle b(g^{-1}), x^* \rangle + \alpha(0, g^{-1}) \tag{1.6}$$

for $x^* \in V^*$ and $g \in G$. Note that the dual of $\mathbf{D}^*$ is $\mathbf{D}$.

An affine datum $\mathbf{D} = (G, V, \rho, \alpha)$ is called a *prehomogeneous affine datum* (briefly a PAD) if there exists a proper algebraic subset S of V such that $V - S$ is a single G-orbit under ρ. We call S the *singular set* of $\mathbf{D}$. Note that the dual of a PAD is not necessarily prehomogeneous.

Let $\mathbf{D} = (G, V, \rho, \alpha)$ be a PAD. Let $X(G)$ be the group of rational characters of G. A non-zero rational function P on V is called a *relative invariant* of $\mathbf{D}$ corresponding to $\chi \in X(G)$ if $P(x\rho(g)) = \chi(g)P(x)\,(x \in V, g \in G)$. As in the case of PV's, a relative invariant is uniquely determined by the corresponding character up to a constant multiple and any prime divisor of a relative invariant is also a relative invariant. Note that relative invariants are not necessarily homogeneous in our case (compare with [7]; see §7 for examples of non-homogeneous relative invariants).

For any subfield K of $\mathbf{C}$, an affine datum $\mathbf{D} = (G, V, \rho, \alpha)$ is said to be *defined over* K if G and V admit K-structures such that ρ and α are defined over K. It is obvious that the dual $\mathbf{D}^*$ of $\mathbf{D}$ is defined over K if so is $\mathbf{D}$. In the remaining part of this section, we assume that $\mathbf{D} = (G, V, \rho, \alpha)$ is a PAD defined over a fixed subfield K of $\mathbf{C}$. Denote by G_1 the normal closed subgroup of G generated by the commutator subgroup $[G, G]$ of G and the stabilizer $G(x) = \{g \in G \mid x\rho(g) = x\}$ for a generic point $x \in V - S$. The group G_1 does not depend on the choice of x. Put $X_\rho(G) = \{\chi \in X(G) \mid \chi$ is trivial on $G_1\}$. It is shown that $X_\rho(G)$ coincides with the group of rational characters of G corresponding to relative invariants of $\mathbf{D}$ (cf. [7, §4 Prop. 19]). Denote by $X_\rho(G)_K$ the subgroup of

$X_\rho(G)$ consisting of rational characters in $X_\rho(G)$ defined over K. The next two lemmas are proved in the same manner as Lemma 1.1 and Lemma 1.2 of [3].

LEMMA 1.1. *Let S be the singular set of $\mathbf{D}$ and S' the union of the irreducible components of S of codimension one. Then both of S and S' are defined over K.*

LEMMA 1.2. (i) *There exists a finite Galois extension L of K such that every relative invariant of $\mathbf{D}$ is expressed as a product of a complex number and a rational function with coefficients in L.*
(ii) *Let $P(x)$ be a relative invariant of $\mathbf{D}$ corresponding to $\chi \in X_\rho(G)$. Then $P(x)$ is expressed as a product of a complex number and a rational function with coefficients in K if and only if $\chi \in X_\rho(G)_K$.*

Let $S_1, \ldots, S_n$ be the K-irreducible components of S of codimension one and $P_1, \ldots, P_n$ be K-irreducible polynomials that define $S_1, \ldots, S_n$ respectively. The following results are proved in quite a similar manner as in the case of the PV's (see [3, §1]).

LEMMA 1.3. *The polynomials $P_1, \ldots, P_n$ are algebraically independent relative invariants corresponding to $\chi_1, \ldots, \chi_n \in X_\rho(G)_K$ respectively. Furthermore any relative invariant $P(x)$ with coefficients in K is of the form $P(x) = c \cdot P_1(x)^{m_1} \cdots P_n(x)^{m_n}$ $(c \in K,\ m_1, \ldots, m_n \in \mathbf{Z})$.*

LEMMA 1.4. *The group $X_\rho(G)_K$ is a free $\mathbf{Z}$-module of rank n generated by $\chi_1, \ldots, \chi_n$.*

§2. K-regular PAD

In this section we let $\mathbf{D}$ be a PAD defined over a fixed subfield K of $\mathbf{C}$. The next lemma is easily verified.

LEMMA 2.1. *The following conditions are equivalent:*
(i) $\alpha(x_0, G(x_0)) = 0$ *for some* $x_0 \in V - S$.
(ii) $\alpha(x, G(x)) = 0$ *for any* $x \in V - S$.
(iii) *There exists a regular rational function β on $V - S$ that is defined over K and satisfies*

$$\alpha(x, g) = \beta(x\rho(g)) - \beta(x) \quad (x \in V - S,\ g \in G). \tag{2.1}$$

(*Recall that $G(x)$ is the stabilizer subgroup of x in G.*)

From now on, we always assume that $\mathbf{D}$ satisfies the above equivalent conditions and fix a function β satisfying (2.1) once and for all. Let $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}(x)$ $(x \in V - S)$ be the Lie algebras of G, G_1 and $G(x)$, respectively. For $x \in V$ and $A \in \mathfrak{g}$, put

$$xd\rho(A) = \frac{d}{dt}\{x\rho(\exp(tA))\}\,|_{t=0} \in V \tag{2.2}$$

$$d\alpha(x, A) = \frac{d}{dt}\{\alpha(x, \exp(tA))\}\,|_{t=0} \in \mathbf{C}. \tag{2.3}$$

Our assumption implies

$$d\alpha(x, \mathfrak{g}(x)) = 0 \quad (x \in V - S). \tag{2.4}$$

The next lemma follows from a straightforward calculation.

LEMMA 2.2. *For $x \in V$, $g \in G$ and $A \in \mathfrak{g}$, we have*

$$xd\rho(\mathrm{Ad}(g)A) = x\rho(g)d\rho(A)\rho(g^{-1}) - 0\cdot\rho(g^{-1}), \tag{2.5}$$

$$d\alpha(x, \mathrm{Ad}(g)A) = \alpha(x\rho(g)d\rho(A), g^{-1}) + d\alpha(x\rho(g), A) - \alpha(0, g^{-1}), \tag{2.6}$$

where Ad *stands for the adjoint representation of G on $\mathfrak{g}$.*

A rational mapping ϕ of a G-stable Zariski open subset V' of V into V^* is said to be *G-equivariant* if $\phi(x\rho(g)) = \phi(x)\rho^*(g)$ $(x \in V', g \in G)$. We denote by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$.

PROPOSITION 2.3. *For $\omega \in \mathfrak{g}^*$, the following two assertions are equivalent.*
(i) *There exists a unique G-equivariant rational mapping $\phi_\omega : V - S \to V^*$ satisfying*

$$\langle xd\rho(A), \phi_\omega(x)\rangle + d\alpha(x, A) = \omega(A) \quad (x \in V - S, A \in \mathfrak{g}). \tag{2.7}$$

(ii) *ω vanishes on $\mathfrak{g}_1$.*

Proof. (i)$\Rightarrow$(ii): Take an $x \in V - S$. Since $\mathfrak{g}(x) = \{A \in \mathfrak{g} \mid xd\rho(A) = 0\}$, (2.7) and (2.4) imply that $\omega(\mathfrak{g}(x)) = 0$. It remains to prove that $\omega([\mathfrak{g}, \mathfrak{g}]) = 0$, since $\mathfrak{g}_1$ is generated by $\mathfrak{g}(x)$ and $[\mathfrak{g}, \mathfrak{g}]$. Chaging x (resp. A) into $x\rho(g^{-1})$ (resp. $\mathrm{Ad}(g)A$) in (2.7) and applying Lemma 2.2, we have

$$\omega(\mathrm{Ad}(g)A) = \langle xd\rho(A)\rho(g^{-1}) - 0\cdot\rho(g^{-1}), \phi_\omega(x)\rho^*(g^{-1})\rangle$$

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$$+ \alpha(xd\rho(A), g^{-1}) + d\alpha(x, A) - \alpha(0, g^{-1}).$$

It follows from (1.3), (1.4) and (1.5) that

$$\begin{aligned}&\langle xd\rho(A)\rho(g^{-1}) - 0\cdot\rho(g^{-1}), \phi_\omega(x)\rho^*(g^{-1})\rangle \\ &= \langle xd\rho(A), \phi_\omega(x)\rangle + \alpha(xd\rho(A)\rho(g^{-1}), g) - \alpha(0\cdot\rho(g^{-1}), g).\end{aligned}$$

Since $\alpha(xd\rho(A), g^{-1}) + \alpha(xd\rho(A)\rho(g^{-1}), g) - \alpha(0, g^{-1}) - \alpha(0\cdot\rho(g^{-1}), g) = 0$ by the cocycle condition (1.1), we obtain $\omega(\mathrm{Ad}(g)A) = \omega(A)$. This implies that $\omega([\mathfrak{g}, \mathfrak{g}]) = 0$.

(ii)$\Rightarrow$(i): Let $x \in V - S$. Since $\omega(A) - \mathrm{d}\alpha(x, A) = 0$ for any $A \in \mathfrak{g}(x)$ and since $A \to xd\rho(A)$ induces a linear isomorphism of $\mathfrak{g}/\mathfrak{g}(x)$ onto V, there uniquely exists $\phi_\omega(x) \in V^*$ with $\langle xd\rho(A), \phi_\omega(x)\rangle = \omega(A) - d\alpha(x, A)$ for $A \in \mathfrak{g}$. Then $x \to \phi_\omega(x)$ defines a regular rational mapping of $V - S$ to V^*. We now prove the G-equivariance of ϕ_ω. Since $\omega(\mathrm{Ad}(g)A) = \omega(A)$ $(A \in \mathfrak{g}, g \in G)$, we have

$$\begin{aligned}&\langle x\rho(g)d\rho(A), \phi_\omega(x\rho(g))\rangle + d\alpha(x\rho(g), A) \\ &\qquad = \langle xd\rho(\mathrm{Ad}(g)A), \phi_\omega(x)\rangle + d\alpha(x, \mathrm{Ad}(g)A).\end{aligned}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned}&\langle x\rho(g)d\rho(A), \phi_\omega(x\rho(g))\rangle \\ &= \langle x\rho(g)d\rho(A)\rho(g^{-1}) - 0\cdot\rho(g^{-1}), \phi_\omega(x)\rangle + \alpha(x\rho(g)d\rho(A), g^{-1}) - \alpha(0, g^{-1}) \\ &= \langle x\rho(g)d\rho(A)\rho(g^{-1}), \phi_\omega(x)\rangle + \alpha(x\rho(g)d\rho(A), g^{-1}) - \alpha^*(\phi_\omega(x), g).\end{aligned}$$

(Note that $\langle 0\cdot\rho(g^{-1}), \phi_\omega(x)\rangle + \alpha(0, g^{-1}) = \alpha^*(\phi_\omega(x), g)$.) In view of the relation (1.2), we have $\langle x\rho(g)d\rho(A), \phi_\omega(x\rho(g))\rangle = \langle x\rho(g)d\rho(A), \phi_\omega(x)\rho^*(g)\rangle$ for $A \in \mathfrak{g}$, which proves the assertion. q.e.d.

A PAD $\mathbf{D}$ is said to be *quasi-regular* if $\mathbf{D}$ satisfies the conditions in Lemma 2.1 and if there exists $\omega \in X_1^* = \{\omega \in \mathfrak{g}^* \mid \omega$ vanishes on $\mathfrak{g}_1\}$ such that ϕ_ω is dominant (that is, the image of ϕ_ω is Zariski dense in V^*). In this case ω is said to be *non-degenerate*. Let $\chi_1,\ldots,\chi_n$ be the K-rational characters of G defined by $P_i(x\rho(g)) = \chi_i(g)P_i(x)$ as in §1. Since their infinitesimal characters $d\chi_1,\ldots,d\chi_n$ vanish on $\mathfrak{g}_1$, $X_0^* = \mathbf{C}d\chi_1 + \cdots + \mathbf{C}d\chi_n$ is a subspace of X_1^*. A quasi-regular PAD defined over K is said to be *K-regular* if there exists a non-degenerate element $\omega \in X_0^*$.

PROPOSITION 2.4. *If $\mathbf{D}$ is a quasi-regular (resp. K-regular) PAD, then its dual $\mathbf{D}^*$ is also a quasi-regular (resp. K-regular) PAD. Furthermore if ω is a non-degenerate element, then ϕ_ω gives a one-to-one biregular rational mapping of $V - S$*

onto $V^* - S^*$ (S^* *is the singular set of* $\mathbf{D}^*$) *and* $G^*(\phi_\omega(x)) = \{g \in G \mid \phi_\omega(x)\rho^*(g) = \phi_\omega(x)\}$ *coincides with* $G(x)$ *for any* $x \in V - S$.

Proof. Let $\omega \in X_1^*$ (resp. $\omega \in X_0^*$) be a non-degenerate element. Then $\phi_\omega(V - S)$ is a Zariski dense G-orbit under the action ρ^*. Hence the affine datum $\mathbf{D}^* = (G, V^*, \rho^*, \alpha^*)$ is prehomogeneous and $\phi_\omega(V - S) = V^* - S^*$ where S^* is the singular set of $\mathbf{D}^*$. For $x \in V - S$, put $x^* = \phi_\omega(x)$ and $G^*(x^*) = \{g \in G \mid x^*\rho^*(g) = x^*\}$. Then we have $G(x) \subset G^*(x^*)$ and $\dim G^*(x^*) = \dim G - \dim V = \dim G(x)$. This implies that $\mathfrak{g}_1$ coincides with the Lie algebra of G_1^*, the group generated by $G^*(x^*)$ and the commutator subgroup $[G, G]$. By Proposition 2.3, there exists a rational mapping $\psi_\omega : V^* - S^* \to V$ satisfying $\psi_\omega(x^*\rho^*(g)) = \psi_\omega(g^*)\rho(g)$ and $\langle \psi_\omega(x^*), x^*d\rho^*(A)\rangle + d\alpha^*(x^*, A) = -\omega(A)$ ($x^* \in V^* - S^*$, $g \in G$, $A \in \mathfrak{g}$). Therefore we have

$$\begin{aligned}\langle x, x^*d\rho^*(A)\rangle &= \langle -xd\rho(A), \phi_\omega(x)\rangle - d\alpha(x, A) - d\alpha^*(x^*, A)\\ &= -\omega(A) - d\alpha^*(x^*, A) = \langle \psi_\omega(x^*), x^*d\rho^*(A)\rangle\end{aligned}$$

for any $A \in \mathfrak{g}$. This implies $\psi_\omega(\phi_\omega(x)) = x$ for $x \in V - S$. A similar argument shows that $\phi_\omega \circ \psi_\omega$ is the identity mapping on $V^* - S^*$. Thus ϕ_ω is a one-to-one biregular mapping of $V - S$ onto $V^* - S^*$ and ψ_ω is its inverse. The remaining part of the proposition follows from this fact. q.e.d.

COROLLARY 2.5. *Let* $\mathbf{D} = (G, V, \rho, \alpha)$ *be a quasi-regular PAD defined over* K *and* $\mathbf{D}^* = (G, V^*, \rho^*, \alpha^*)$ *its dual. Then*
(i) $G_1 = G_1^*$.
(ii) *The number* n^* *of* K*-irreducible components of* S^* *of codimension one is equal to that of* S.
(iii) $X_\rho(G)_K = X_{\rho^*}(G)_K$.

Proof. These are easily deduced from Proposition 2.4 if we observe $n^* = \operatorname{rank} X_{\rho^*}(G)_K$ and $X_{\rho^*}(G)_K = X(G/G_1^*)_K$. q.e.d.

Fix a basis of V and let $(x_1, \ldots, x_N)$ be the coordinate of $x \in V$ ($N = \dim V$). From now on, coordinates of elements of V^* are taken to be with respect to the dual basis of the above one. For a smooth function f on V, we define $\operatorname{grad} f : V \to V^*$ to be $\operatorname{grad} f(x) = \left(\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_N}\right)$. It is easy to see that this definition does not depend upon the choice of a basis of V. If f does not vanish on $V - S$, we put $\operatorname{grad} \log f(x) = f(x)^{-1} \operatorname{grad} f(x)$ ($x \in V - S$). For $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, put

$$\omega_s = \sum_{1 \leq i \leq n} \left(-\frac{s_i}{2\pi i}\right) d\chi_i \in X_0^*. \tag{2.8}$$

LEMMA 2.6. *We have*

$$\phi_{\omega_s}(x) = -\frac{1}{2\pi i} \sum_{1 \leq i \leq n} s_i \cdot \operatorname{grad} \log P_i(x) - \operatorname{grad} \beta(x). \tag{2.9}$$

Proof. Differentiating the equalities $\alpha(x, \exp(tA)) = \beta(x\rho(\exp(tA))) - \beta(x)$ and $P_i(x\rho(\exp(tA))) = \chi_i(\exp(tA)) \cdot P_i(x)$ $(x \in V - S, A \in \mathfrak{g})$ at $t = 0$, we obtain $d\alpha(x, A) = \langle xd\rho(A), \operatorname{grad} \beta(x) \rangle$ and $d\chi_i(A) = \langle xd\rho(A), \operatorname{grad} \log P_i(x) \rangle$. These prove the assertion of the lemma. q.e.d.

For $x \in V - S$, we define the differential mapping $d\phi_\omega(x) : V \to V^*$ of ϕ_ω to be $d\phi_\omega(x)(y) = \frac{d}{dt}(\phi_\omega(x + ty))|_{t=0}$ for $y \in V$. A straightforward calculation shows $d\phi_\omega(x\rho(g))(a(g)y) = a(g)^{\sim} d\phi_\omega(x)(y)$ $(x \in V - S, y \in V, g \in G)$, which implies that $\det(d\phi_\omega(x\rho(g))) = \chi_0(g)^{-1} \det(d\phi_\omega(x))$ $(x \in V - S, g \in G)$ with

$$\chi_0(g) = (\det a(g))^2. \tag{2.10}$$

Thus we have proved

LEMMA 2.7. *If $\mathbf{D} = (G, V, \rho, \alpha)$ is defined over K and quasi-regular, then $\chi_0 \in X_\rho(G)_K$.*

LEMMA 2.8. *Let $\mathbf{D}$ be a PAD and $\omega \in X_1^*$. Then*

$$\frac{\partial \phi_\omega(x)_j}{\partial x_i} = \frac{\partial \phi_\omega(x)_i}{\partial x_j} \qquad (1 \leq i, j \leq N)$$

where $\phi_\omega(x) = (\phi_\omega(x)_i)_{1 \leq i \leq N} \in V^*$.

Proof. For $x = (x_1, \ldots, x_N) \in V$, $A \in \mathfrak{g}$ and $1 \leq i \leq N$, we put

$$(x \cdot d\rho(A))_i = \sum_{1 \leq j \leq N} x_j a_{ji}(A) + b_i(A), \; d\alpha(x, A) = \sum_{1 \leq i \leq N} \alpha_i(A) x_i + d\alpha(0, A)$$

$(a_{ji}(A), b_i(A), \alpha_i(A) \in \mathbf{C})$. Then it is easy to see that $(x^* d\rho^*(A))_i = -\sum_{1 \leq j \leq N} x_j^* a_{ij}(A) - \alpha_i(A)$ for $x^* = (x_1^*, \ldots, x_N^*) \in V^*$. It follows from (2.7) that

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$$\sum_{1\leq i,j\leq N}(x_j a_{ji}(A)+b_i(A))\phi_\omega(x)_i+\sum_{1\leq i\leq N}x_i\alpha_i(A)+\alpha(0,A)=\omega(A).$$

Differentiating the above formula in x_k, we obtain

$$\text{(2.11)}\qquad \sum_{1\leq i\leq N}a_{ki}(A)\phi_\omega(x)_i+\sum_{1\leq i,j\leq N}(x_j a_{ji}(A)+b_i(A))\frac{\partial\phi_\omega(x)_i}{\partial x_k}+\alpha_k(A)=0.$$

On the other hand, the G-equivariance of ϕ_ω implies

$$\text{(2.12)}\qquad \phi_\omega(x\rho(\exp(tA)))_k=(\phi_\omega(x)\rho^*(\exp(tA)))_k.$$

Differentiating (2.12) at $t=0$, we get

$$\text{(2.13)}\qquad \sum_{1\leq i,j\leq N}\frac{\partial\phi_\omega(x)_k}{\partial x_i}(x_j a_{ji}(A)+b_i(A))$$
$$=-\sum_{1\leq i\leq N}+\alpha_{ki}(A)\phi_\omega(x)_i-\alpha_k(A)\quad(1\leq k\leq N).$$

Comparing (2.11) and (2.13), we prove the lemma. q.e.d.

§3. The a-functions

Let $\mathbf{D}=(G,V,\rho,\alpha)$ be a K-regular PAD and $\mathbf{D}^*=(G,V^*,\rho^*,\alpha^*)$ its dual. Let $S_1,\ldots,S_n$ (resp. $S_1^*,\ldots,S_n^*$) be the K-irreducible components of the singular set S (resp. S^*) of $\mathbf{D}$ (resp. $\mathbf{D}^*$) of codimension one and $P_1,\ldots,P_n$ (resp. $P_1^*,\ldots,P_n^*$) be their defining equations in $K[V]$ (resp. $K[V^*]$). Let χ_i (resp. χ_i^*) be the corresponding character to the relative invariant P_i (resp. P_i^*) of $\mathbf{D}$ (resp. $\mathbf{D}^*$). By Lemma 1.4 and Corollary 2.5, we see that the group generated by χ_i $(1\leq i\leq n)$ coincides with the one generated by χ_i^* $(1\leq i\leq n)$. This implies

$$\text{(3.1)}\qquad \chi_i=\prod_{1\leq j\leq n}\chi_j^{*u_{ij}}\quad(1\leq i\leq n)$$

with $U=(u_{ij})\in GL_n(\mathbf{Z})$. Take K-rational functions β and β^* on $V-S$ and V^*-S^* so that $\alpha(x,g)=\beta(x\rho(g))-\beta(x)$ $(x\in V-S,g\in G)$ and that $\alpha^*(x^*,g)=\beta^*(x^*\rho^*(g))-\beta^*(x^*)$ $(x^*\in V^*-S^*,g\in G)$, respectively.

For $s=(s_1,\ldots,s_n)\in\mathbf{C}^n$, we define rational mapping $\phi_s:V-S\rightarrow V^*$ and $\psi_s:V^*-S^*\rightarrow V$ as follows:

$$\text{(3.2)}\qquad \phi_s(x)=\phi_{\omega_s}(x)=-\frac{1}{2\pi i}\sum_{1\leq i\leq n}s_i\cdot\operatorname{grad}\log P_i(x)-\operatorname{grad}\beta(x)$$

$$\text{(3.2)}\qquad \psi_s(x^*)=\psi_{\omega_s}(x^*)=\frac{1}{2\pi i}\sum_{1\leq i\leq n}s_i'\cdot\operatorname{grad}\log P_i^*(x^*)-\operatorname{grad}\beta^*(x^*)$$

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where $s' = (s'_1, \ldots, s'_n) = sU$ (cf. proof of Proposition 2.4 and (2.8)). The following two lemmas are easily verified.

LEMMA 3.1. *We have*

$$\phi_s(x\rho(g)) = \phi_s(x)\rho^*(g),\ \psi_s(x^*\rho^*(g)) = \psi_s(x^*)\rho(g) \qquad (x \in V - S,\ x^* \in V^* - S^*,\ g \in G).$$

LEMMA 3.2. *The set $\Omega = \{s \in \mathbf{C}^n \mid \phi_s$ is dominant$\}$ is Zariski dense in $\mathbf{C}^n$. If $s \in \Omega$ then ψ_s is dominant and we have $\phi_s \circ \psi_s = \mathrm{id}_{V^*-S^*}$, $\psi_s \circ \phi_s = \mathrm{id}_{V-S}$.*

For simplicity, we write X_K for $X_\rho(G)_K = X_{\rho^*}(G)_K$. For $\chi \in X_K$, we define $d(\chi) = (d(\chi)_1, \ldots, d(\chi)_n)$, $d^*(\chi) = (d^*(\chi)_1, \ldots, d^*(\chi)_n) \in \mathbf{Z}^n$ to be

$$\chi = \prod_{1 \le i \le n} \chi_i^{d(\chi)_i} = \prod_{1 \le i \le n} \chi_i^{*d^*(\chi)_i}. \tag{3.4}$$

By (3,1), we have $d^*(\chi) = d(\chi)U$. Put $P^\chi = \Pi_{1 \le i \le n} P_i^{d(\chi)_i}$ and $P^{*\chi} = \Pi_{1 \le i \le n} (P_i^*)^{d^*(\chi)_i}$, which are relative invariants of $\mathbf{D}$ and $\mathbf{D}^*$ respectively, both corresponding to χ.

For $s \in \Omega$, the function $x \to P^{*\chi}(\phi_s(x)) \cdot P^{\chi^{-1}}(x)$ on $V - S$ is $\rho(G)$-invariant and hence constant on $V - S$. We denote its value by $a_\chi(s)$ and call this rational function on $\mathbf{C}^n$ the *a-function* of $\mathbf{D}$ corresponding to χ. The next lemma is deduced from the definition of $a_\chi(s)$.

LEMMA 3.3. *We have $a_{\chi\chi'}(s) = a_\chi(s) \cdot a_{\chi'}(s)$ for $\chi, \chi' \in X_K$.*

LEMMA 3.4. *If either $P^{*\chi}$ or $p^{\chi^{-1}}$ is a polynomial, then $a_\chi(s)$ is a polynomial of $s \in \mathbf{C}^n$.*

Proof. This immediately follows from the equality $a_\chi(s) = P^{*\chi}(\phi_s(x)) \cdot P^{\chi^{-1}}(x) = P_\chi^*(x^*) \cdot P^{\chi^{-1}}(\psi_s(x^*))$ for $x \in V - S$ and $x^* \in V^* - S^*$. q.e.d.

For $s \in \Omega$, let us consider the function on $V - S$ given by

$$x \to \langle x, \phi_s(x) \rangle + \beta(x) + \beta^*(\phi_s(x)). \tag{3.5}$$

This function is invariant under $\rho(G)$ and hence constant on $V - S$. If we write $F(s)$ for its value, then $s \to F(s)$ defines a rational function on $\mathbf{C}^n$. In the remaining part of the paper, we always assume the following condition:

(3.6) $F(s)$ *is a non-constant polynomial of* s.

Remark. Since $F(s) = \langle x^*, \phi_s(x^*)\rangle + \beta(\phi_s(x^*)) + \beta^*(x^*)$ for any $x^* \in V^* - S^*$, the assumption is satisfied if either α or α^* is zero.

PROPOSITION 3.5. *Under the assumption* (3.6), *there exist distinct linear forms* $e_1, \ldots, e_m$ *on* $\mathbf{C}^n$, *natural numbers* $M_1, \ldots, M_m$ *and* $c \in \mathrm{Hom}(X_K, \mathbf{C}^\times)$ *satisfying the following conditions:*

(i) $a_\chi(s) = c(\chi) \prod_{1 \le j \le m} (e_j(s))^{-M_j e_j(d(\chi))}$

(ii) *All the coefficients of* e_j *are non-negative integers and* $e_j(\mathbf{Z}^n) = \mathbf{Z}$ $(1 \le j \le m)$.

Proof. Let $g_1(s), \ldots, g_m(s)$ be the distinct prime divisors of $a_{\chi_1}(s)^{-1}, \ldots, a_{\chi_n}(s)^{-1}$. By Lemma 3.3, there uniquely exist $\varepsilon_1, \ldots, \varepsilon_m \in \mathrm{Hom}(X_K, \mathbf{Z})$ such that $a_\chi(s) = c'_\chi \prod_{1 \le j \le m} g_j(s)^{\varepsilon_j(\chi)}$ for $\chi \in X_K$, where c'_χ is a constant depending only on χ. Define linear forms $e'_1, \ldots, e'_m$ on $\mathbf{C}^n$ by $e'_j(s) = \prod_{1 \le i \le n} s_i \cdot \varepsilon_j(\chi_i)$ $(s = (s_1, \ldots, s_n) \in \mathbf{C}^n)$. Since $e'_j(d(\chi)) = \varepsilon_j(\chi)$, we have $e'_j(\mathbf{Z}^n) \subset \mathbf{Z}$ and hence we can find linear forms $e_1, \ldots, e_m$ and natural numbers $M_1, \ldots, M_m$ such that $e'_j = -M_j e_j$ and $e_j(\mathbf{Z}^n) = \mathbf{Z}$. Since $\varepsilon_j(\chi_j) \le 0$ $(1 \le i \le n,\ 1 \le j \le m)$, we see that all the coefficients of e_j are non-negative integers. Thus we have

$$P^{*\chi}(\phi_s(x)) \cdot P^{\chi^{-1}}(x) = c'_\chi \prod_{1 \le j \le m} g_j(s)^{-M_j e_j(d(\chi))} \quad ((x, s) \in (V - S) \times \Omega). \tag{3.7}$$

Recall that $\operatorname{grad} \log P^{\chi^{-1}}(x) = 2\pi i(\phi_{d(\chi)}(x) + \operatorname{grad} \beta(x))$ and $\operatorname{grad} \log P^{*\chi}(x^*) = 2\pi i(\phi_{d(\chi)}(x^*) + \operatorname{grad} \beta^*(x^*))$. Taking the logarithmic derivatives of the both sides of (3.7), we obtain

$$\begin{aligned} &\phi_{d(\chi)}(y)\, dy + \operatorname{grad} \beta^*(y)\, dy + \phi_{d(\chi)}(x)\, dx + \operatorname{grad} \beta(x)\, dx \\ &= -\frac{1}{2\pi i} \sum_{1 \le j \le m} M_j e_j(d(\chi)) \frac{dg_j(s)}{g_j(s)} \end{aligned}$$

for $\chi \in X_K$ and hence

$$\begin{aligned} &\phi_{s'}(y)\, dy + \operatorname{grad} \beta^*(y)\, dy + \phi_{s'}(x)\, dx + \operatorname{grad} \beta(x)\, dx \\ &= -\frac{1}{2\pi i} \sum_{1 \le j \le m} M_j e_j(s') \frac{dg_j(s)}{g_j(s)} \end{aligned}$$

for any $s, s' \in \mathbf{C}^n$, where we put $y = \phi_s(x)$. Putting $s' = s$ in the above formula, we get

(3.8) $$dF(s) = -\frac{1}{2\pi i} \sum_{1 \le i \le m} M_j e_j(s) \frac{dg_j(s)}{g_j(s)}.$$

By (3.8) and the assumption (3.6), $g_j(s)$ divides $e_j(s)$ and hence $g_j(s)$ is a constant multiple of $e_j(s)$. Thus the proof of the proposition is completed. q.e.d.

COROLLARY 3.6. *Under the assumption* (3.6), *we have* $F(s) = -\frac{1}{2\pi i} \sum_{1 \le i \le n} \delta_i s_i + c$ *with non-negative integers* $\delta_1, \dots, \delta_n$ *and* $c \in \mathbf{C}$. *In fact, we have* $\delta_i = \sum_{1 \le j \le m} M_j e_{ji}$ $(1 \le i \le n)$ *where* $e_{ij} \in \mathbf{Z}$ *is defined to be* $e_j(s) = \sum_{1 \le i \le n} e_{ji} s_i$ (*note that* $e_{ji} \ge 0$).

Proof. In the proof of Proposition 3.5, we have obtained the formula $dF(s) = \sum_{1 \le j \le m} M_j de_j(s)$. Our assertion follows from this. q.e.d.

Note that the constant term of $F(s)$ depends upon the choice of β and β^*, though the linear term of $F(s)$ depends only upon $\mathbf{D}$. We say that (β, β^*) is a *normalized pair* for $(\mathbf{D}, \mathbf{D}^*)$ if the constant term of $F(s)$ is zero. The following is an immediate consequence of Proposition 3.5 and Corollary 3.6.

COROLLARY 3.7. *If either* $P^{\chi^{-1}}$ *or* $P^{*\chi}$ *is a polynomial, then* $a_\chi(s)$ *is a homogeneous polynomial of* $s \in \mathbf{C}^n$ *of degree* $-\sum_{1 \le i \le n} \delta_i d(\chi)_i$.

Remark. If $\mathbf{D} = (G, V, \rho, \alpha)$ is a K-regular PV, that is, if $\mathrm{Im}(\rho) \subset \mathrm{GL}(V)$ and $\alpha \equiv 0$, the assumption (3.6) is always satisfied and the integer δ_i is equal to the degree of P_i.

§4. The b-functions

We keep the notation and the assumptions of §3 and furthermore assume that K is a subfield of $\mathbf{R}$. For an integer i $(1 \le i \le N)$, let D_i be the differential operator of the first order on $V_{\mathbf{R}}$ given by

$$D_i f(x) = \frac{1}{2\pi i} \frac{\partial f}{\partial x_i}(x) - \frac{\partial \beta}{\partial x_i}(x) \cdot f(x) \quad (x \in V_{\mathbf{R}}, f \in C^\infty(V_{\mathbf{R}})).$$

Since the D_i's mutually commute, the differential operator

(4.1) $$R_\chi = P^{*\chi}(D_1, \dots, D_N)$$

on $V_{\mathbf{R}}$ is well-defined for $\chi \in X_K^+ = \{\chi \in X_K \mid P^{*\chi}$ is a polynomial$\}$. We write

$R_\chi = P^{*\chi}\Big(\frac{1}{2\pi i}\,\mathrm{grad} - \mathrm{grad}\,\beta\Big)$ symbolically.

LEMMA 4.1. *We have*

$$R_\chi(\mathbf{e}[\langle x, x^*\rangle + \beta(x)]) = P^{*\chi}(x^*)\cdot\mathbf{e}[\langle x, x^*\rangle + \beta(x)].$$

Proof. This follows from $D_i(\mathbf{e}[\langle x, x^*\rangle + \beta(x)]) = x_i^*\cdot\mathbf{e}[\langle x, x^*\rangle + \beta(x)]$. q.e.d.

Let $R_\chi^\sim$ be the adjoint operator of R_χ given by $R_\chi^\sim = P^{*\chi}\Big(-\frac{1}{2\pi i}\,\mathrm{grad} - \mathrm{grad}\,\beta\Big)$. For $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, we put $P^s = P_1^{s_1}\cdots P_n^{s_n}$.

LEMMA 4.2. *For $\chi \in X_K^+$, there exists a polynomial $b_\chi(s)$ of $s \in \mathbf{C}^n$ such that*

$$R_\chi^\sim(P^s) = b_\chi(s)\cdot P^{s+d(\chi)}. \tag{4.2}$$

Proof. For a function φ on $V_{\mathbf{R}}$, we write $\varphi^g(x) = \varphi(x\rho(g))$. Since $(R_\chi^\sim\varphi)(x\rho(g)) = \chi(g)\cdot R_\chi^\sim(\varphi^g)(x)$ $(x \in V_{\mathbf{R}}, g \in G_{\mathbf{R}})$, we have $R_\chi^\sim P^s(x\rho(g)) = \chi(g)\chi^s(g)\cdot P^s(g)$. Here χ^s stands for $\chi_1^{s_1}\cdots\chi_n^{s_n}$. The lemma easily follows from this formula. q.e.d.

LEMMA 4.3. *For $\chi, \chi' \in X_K^+$, we have*

$$b_{\chi\chi'}(s) = b_\chi(s)\cdot b_{\chi'}(s + d(\chi)). \tag{4.3}$$

Proof. This is deduced from the formula $R_{\chi\chi'}^\sim = R_{\chi'}^\sim\circ R_\chi^\sim$. q.e.d.

We can define a rational function $b_\chi(s)$ of $s \in \mathbf{C}^n$ for any $\chi \in X_K$ preserving the relation (4.3). We call $b_\chi(s)$ the *b-function* of $\mathbf{D}$ corresponding to $\chi \in X_K$. The next results are proved in a similar manner as in [5, Ch. I, §2].

LEMMA 4.4. *If $\chi \in X_K^+$, the highest homogeneous part of $b_\chi(s)$ is equal to $a_\chi(s)$ and* $\deg b_\chi(s) = -\sum_{1\le i\le n}\delta_i\cdot d(\chi)_i$.

PROPOSITION 4.5. *The notation and the assumption being the same as in Proposition 3.5, we have*

$$b_\chi(s) = c(\chi)\,\frac{\gamma(s)}{\gamma(s + d(\chi))},$$

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where $\gamma(s) = \Pi_{1\leq j\leq m}\Pi_{k\in\mathbf{Z}}\Gamma(e_j(s)+k-C_j)^{n_k}$ *with* $C_j \in \mathbf{C}$ *and* $n_k \in \mathbf{Z}$. *Here all but a finite number of* n_k *are equal to zero.*

§5. Fourier transforms of modified complex powers of relative invariants

In the rest of the paper, we let $\mathbf{D} = (G, V, \rho, \alpha)$ be a $\mathbf{Q}$-regular PAD and $\mathbf{D}^* = (G, V^*, \rho^*, \alpha^*)$ its dual. We further assume that the singular set S of $\mathbf{D}$ is a hypersurface of V. Then the singular set S^* of $\mathbf{D}^*$ is also a hypersurface of V^*. Thus S (resp. S^*) is the disjoint union of the $\mathbf{Q}$-irreducible components $S_1,\ldots, S_n$ (resp. $S_1^*,\ldots,S_n^*$). Each S_i (resp. S_i^*) is defined by a single equation $P_i(x) = 0$ (resp. $P_i^*(x^*) = 0$) with a $\mathbf{Q}$-irreducible polynomial $P_i \in \mathbf{Q}[V]$ (resp. $P_i^* \in \mathbf{Q}[V^*]$). Denote by $\chi_1,\ldots,\chi_n$ (resp. $\chi_1^*,\ldots,\chi_n^*$) be the corresponding characters to relative invariants $P_1,\ldots,P_n$ (resp. $P_1^*,\ldots,P_n^*$). For $s = (s_1,\ldots,s_n) \in \mathbf{C}^n$, we put

$$(5.1)\qquad |P(x)|^s = |P_1(x)|^{s_1}\cdots|P_n(x)|^{s_n} \qquad (x \in V_{\mathbf{R}} - S_{\mathbf{R}})$$

$$(5.2)\qquad |P^*(x^*)|^s = |P_1^*(x^*)|^{s_1}\cdots|P_n^*(x^*)|^{s_n} \qquad (x^* \in V_{\mathbf{R}}^* - S_{\mathbf{R}}^*).$$

Let $V_1,\ldots,V_\nu$ (resp. $V_1^*,\ldots,V_\nu^*$) be the connected components of $V_{\mathbf{R}} - S_{\mathbf{R}}$ (resp. $V_{\mathbf{R}}^* - S_{\mathbf{R}}^*$). We here note that the the number of the connected components of $V_{\mathbf{R}} - S_{\mathbf{R}}$ coincides with that of $V_{\mathbf{R}}^* - S_{\mathbf{R}}^*$. Fix a normalized pair (β, β^*) for $(\mathbf{D}, \mathbf{D}^*)$ defined over $\mathbf{Q}$. For $s \in \mathbf{C}^n$ and $i = 1,\ldots,\nu$, we set

$$(5.3)\qquad \Phi_i(f, s) = \int_{V_i} |P(x)|^s \mathbf{e}[\beta(x)] f(x)\, dx \qquad (f \in \mathcal{S}(V_{\mathbf{R}}))$$

$$(5.4)\ \Phi_i^*(x^*) = \int_{V_i^*} |P^*(x^*)|^s \mathbf{e}[-\beta^*(x^*)] f^*(x^*)\, dx^* \qquad (f^* \in \mathcal{S}(V_{\mathbf{R}}^*)).$$

Here dx is a fixed Lebesgue measure on $V_{\mathbf{R}}$ and dx^* denotes its dual measure on $V_{\mathbf{R}}^*$. The integrals $\Phi_i(f, s)$ and $\Phi_i^*(f^*, s)$ are absolutely convergent and define holomorphic functions in the region $\{s \in \mathbf{C}^n \mid \operatorname{Re} s_1 > 0,\ldots,\operatorname{Re} s_n > 0\}$. Observe that, for a sufficiently large integer r, $(P_1\cdots P_n)^r\beta$ (resp. $(P_1^*\cdots P_n^*)^r\beta^*$) is a polynomial function on V (resp. V^*). Then it is straightforward to show that $u(x) = \mathbf{e}[\beta(x)]$ (resp. $u^*(x^*) = \mathbf{e}[\beta^*(x^*)]$) satisfies the assumptions of Theorem A.3 in the appendix of [1]. Applying this theorem to our situation, we get the following result:

PROPOSITION 5.1. *Assume that the condition* (3.6) *is satisfied.*
(i) *The integrals* $\Phi_i(f, s)$ *and* $\Phi_i^*(f, s)$ *are continued to meromorphic functions of* s

in $\mathbf{C}^n$.
(ii) *There exist gamma factors* $\Gamma_{\mathbf{D}}(s)$ *and* $\Gamma_{\mathbf{D}^*}(s)$ *independent of* f *and* f^* *of the form*

$$\Gamma_{\mathbf{D}}(s) = \prod_{1 \le i \le m} \Gamma(a_{i1}s_1 + \cdots + a_{in}s_n + b_i) \qquad (a_{ij},\, b_i \in \mathbf{C}),$$

$$\Gamma_{\mathbf{D}^*}(s) = \prod_{1 \le i \le m} \Gamma(a^*_{i1}s_1 + \cdots + a^*_{in}s_n + b^*_i) \qquad (a^*_{ij},\, b^*_i \in \mathbf{C}),$$

such that $\Gamma_{\mathbf{D}}(s)^{-1}\Phi_i(f, s)$ *and* $\Gamma_{\mathbf{D}^*}(s)^{-1}\Phi^*_i(f^*, s)$ *are entire functions.*
(ii) *The mappings* $f \to \Phi_i(f, s)$ *and* $f^* \to \Phi^*_i(f^*, s)$ *define tempered distributions depending meromorphically on* $s \in \mathbf{C}^n$. *If* Υ_0 *is a bounded domain in* $\mathbf{R}^n$ *such that* $\Phi_i(f, s)$ *and* $\Phi^*_i(f^*, s)$ *are holomorphic in the tube domain* $\Upsilon = \Upsilon_0 + i\mathbf{R}^n$, *then the orders of these tempered distributions are bounded for* $s \in \Upsilon$.

Let $F : \mathcal{S}(V_{\mathbf{R}}) \to \mathcal{S}(V^*_{\mathbf{R}})$ and $F^* : \mathcal{S}(V^*_{\mathbf{R}}) \to \mathcal{S}(V_{\mathbf{R}})$ be the Fourier transforms given by

$$Ff(x^*) = \int_{V_{\mathbf{R}}} f(x)\,\mathbf{e}[-\langle x, x^*\rangle]\,dx \qquad (f \in \mathcal{S}(V_{\mathbf{R}}),\ x^* \in V^*_{\mathbf{R}}) \tag{5.5}$$

$$F^*f^*(x) = \int_{V^*_{\mathbf{R}}} f^*(x^*)\,\mathbf{e}[\langle x, x^*\rangle]\,dx^* \qquad (f^* \in \mathcal{S}(V^*_{\mathbf{R}}),\ x \in V_{\mathbf{R}}). \tag{5.6}$$

Recall that $\chi_0(g) = (\det a(g))^2 \in X_\rho(G)_{\mathbf{Q}}$ (see Lemma 2.7). Put

$$\lambda = \frac{1}{2}d(\chi_0) \in \left(\frac{1}{2}\mathbf{Z}\right)^n,\ \lambda^* = \frac{1}{2}d^*(\chi_0^{-1}) \in \left(\frac{1}{2}\mathbf{Z}\right)^n, \tag{5.7}$$

Then we have $\lambda^* = -\lambda U$ (cf. (3.1)). Let $\gamma(s)$ and $c(\chi)$ be as in Proposition 4.5. We set

$$c(s) = c(\chi_1)^{s_1}\cdots c(\chi_n)^{s_n} \tag{5.8}$$

$$\Phi(f, s) = {}^t(\Phi_1(f, s), \ldots, \Phi_\nu(f, s)) \tag{5.9}$$

$$\Phi^*(f^*, s) = {}^t(\Phi^*_1(f^*, s), \ldots, \Phi^*_\nu(f^*, s)) \tag{5.10}$$

for $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, $f \in \mathcal{S}(V_{\mathbf{R}})$ and $f^* \in \mathcal{S}(V^*_{\mathbf{R}})$. By modifying the argument in [3, §5] in a suitable manner, we can prove

THEOREM 5.2. *The following functional equation holds:*

$$\Phi(F^*f^*, s) = c(-s)\gamma(s)A(s)\Phi^*(f^*, (s + \lambda)U),$$

where $A(s) = (A_{ij}(s))$ *is a* $\nu \times \nu$ *matrix whose entries are polynomials in* $\exp(\pm \pi i s_1), \ldots, \exp(\pm \pi i s_n)$.

§6. Zeta functions attached to a Q-regular PAD

We keep the notation and the assumptions in the previous section. In this section, we further assume that

$$x \in V_{\mathbf{Q}} - S_{\mathbf{Q}} \Rightarrow X(G(x)^0)_{\mathbf{Q}} = \{1\}, \tag{6.1}$$

where $G(x)^0$ is the identity component of $G(x)$. This assumption implies $X(G(x^*)^0) = \{1\}$ for $x^* \in V^*_{\mathbf{Q}} - S^*_{\mathbf{Q}}$.

We take a basis of V and a matrix expression of G so that they are compatible with the $\mathbf{Q}$-structure of $\mathbf{D}$. Let $\Gamma = G_{\mathbf{Z}} \cap G^+_{\mathbf{R}}$ be a discrete subgroup of $G^+_{\mathbf{R}}$. A lattice L in $V_{\mathbf{Q}}$ is said to be *Γ-admissible* with respect to (ρ, α) if $L \cdot \rho(\Gamma) \subset L$ and if $\alpha(L, \Gamma) \subset \mathbf{Z}$. The following lemma is easily verified.

LEMMA 6.1. *Let L be a lattice of $V_{\mathbf{Q}}$ and $L^* = \{x^* \in V^*_{\mathbf{Q}} \mid \langle x, x^* \rangle \in \mathbf{Z}$ for any $x \in L\}$ be its dual. Then L is Γ-admissible with respect to (ρ, α) if and only if L^* is Γ-admissible with respect to (ρ^*, α^*).*

Define $G^+_{\mathbf{R}}$-invariant measures ω and ω^* on $V_{\mathbf{R}} - S_{\mathbf{R}}$ and $V^*_{\mathbf{R}} - S^*_{\mathbf{R}}$ by $\omega(x) = |P(x)|^{-\lambda} dx$, $\omega^*(x^*) = |P^*(x^*)|^{-\lambda^*} dx^*$ (for the definition of λ and λ^*, see (5.7)). We put

$$\begin{aligned} G(x)^+ &= G(x) \cap G^+_{\mathbf{R}},\ \Gamma(x) = G(x)^+ \cap \Gamma && (x \in V_{\mathbf{Q}} - S_{\mathbf{Q}}) \\ G^*(x^*)^+ &= G^*(x^*) \cap G^+_{\mathbf{R}},\ \Gamma^*(x^*) = G^*(x^*)^+ \cap \Gamma && (x^* \in V^*_{\mathbf{Q}} - S^*_{\mathbf{Q}}). \end{aligned}$$

Under the assumption (6.1), $\mathrm{vol}(\Gamma(x) \backslash G(x)^+)$ and $\mathrm{vol}(\Gamma^*(x^*) \backslash G^*(x^*)^+)$ are finite. From now on we fix a right invariant Haar measure dg on $G^+_{\mathbf{R}}$. For $x \in V_{\mathbf{Q}} - S_{\mathbf{Q}}$ (resp. $x^* \in V^*_{\mathbf{Q}} - S^*_{\mathbf{Q}}$), we normalize the Haar measure dm_x (resp. dm_{x^*}) on $G(x)^+$ (resp. $G^*(x^*)^+$) by

$$\begin{aligned} \int_{G^+_{\mathbf{R}}} \varphi(g)\, dg &= \int_{G(x)^+ \backslash G^+_{\mathbf{R}}} \omega(x\rho(g)) \int_{G(x)^+} \varphi(hg)\, dm_x(h) \\ &= \int_{G^*(x^*)^+ \backslash G^+_{\mathbf{R}}} \omega^*(x^*\rho^*(g)) \int_{G^*(x^*)^+} \varphi(hg)\, dm_{x^*}(h) \quad (\varphi \in C^\infty_c(G^+_{\mathbf{R}})). \end{aligned}$$

We define the *density* $\mu(x)$ (resp. $\mu^*(x^*)$) at $x \in V_{\mathbf{Q}} - S_{\mathbf{Q}}$ (resp. $x^* \in V^*_{\mathbf{Q}} - S^*_{\mathbf{Q}}$) to be

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$$(6.2)\qquad \mu(x) = \int_{\Gamma(x)\backslash G(x)^+} dm_x(h),\ \mu^*(x^*) = \int_{\Gamma^*(x^*)\backslash G^*(x^*)^+} dm_{x^*}(h).$$

Let L be a Γ-admissible lattice of $V_{\mathbf{R}}$ with respect to (ρ, α) and L^* its dual. Put $L' = L \cap (V_{\mathbf{R}} - S_{\mathbf{R}})$, $L^{*\prime} = L^* \cap (V_{\mathbf{R}}^* - S_{\mathbf{R}}^*)$, $L_i = L \cap V_i$ and $L_i^* = L^* \cap V_i^*$ $(1 \le i \le \nu)$. Then $L' = \cup_{1\le i\le\nu} L_i$ and $L^{*\prime} = \cup_{1\le i\le\nu} L_i^*$ (disjoint union). We now define the zeta functions $\xi_i(s, L)$ and $\xi_i^*(s, L^*)$ by the following Dirichlet series:

$$(6.3)\qquad \xi_i(s, L) = \sum_{x\in L_i/\rho(\Gamma)} \mu(x)\, \mathbf{e}[-\beta(x)]\, |P(x)|^{-s}$$

$$(6.4)\qquad \xi_i^*(s, L^*) = \sum_{x\in L_i^*/\rho^*(\Gamma)} \mu^*(x^*)\, \mathbf{e}[\beta^*(x^*)]\, |P^*(x^*)|^{-s}.$$

Henceforth we always assume

(6.5) *The Dirichlet series* $\xi_1(s, L), \ldots, \xi_\nu(s, L)$ *(resp.* $\xi_1^*(s, L^*), \ldots, \xi_\nu^*(s, L^*)$*) are absolutely convergent for* $\mathrm{Re}(s_1) > a_1, \ldots, \mathrm{Re}(s_n) > a_n$ *(resp.* $\mathrm{Re}(s_1) > a_1^*, \ldots, \mathrm{Re}(s_n) > a_n^*$*), where* a_i, a_i^* $(1 \le i \le n)$ *are certain positive real numbers.*

Set $B = \{s \in \mathbf{C}^n \mid \mathrm{Re}\, s_i > \mathrm{Max}(a_i, \lambda_i)\ (1 \le i \le n)\}$ and $B^* = \{s \in \mathbf{C}^n \mid \mathrm{Re}\, s_i > \mathrm{Max}(a_i^*, \lambda_i^*)\ (1 \le i \le n)\}$. For $f \in \mathcal{S}(V_{\mathbf{R}})$ and $f^* \in \mathcal{S}(V_{\mathbf{R}}^*)$, we set

$$(6.6)\qquad Z(s, f, L) = \int_{\Gamma\backslash G_{\mathbf{R}}^+} |\chi(g)|^{s-\lambda/2} \sum_{x\in L'} r(g) f(x)\, dg$$

$$(6.7)\qquad Z^*(s, f^*, L^*) = \int_{\Gamma\backslash G_{\mathbf{R}}^+} |\chi^*(g)|^{s-\lambda^*/2} \sum_{x^*\in L^{*\prime}} r^*(g) f^*(x^*)\, dg,$$

where the representations r and r^* of $G_{\mathbf{R}}^+$ on $\mathcal{S}(V_{\mathbf{R}})$ and $\mathcal{S}(V_{\mathbf{R}}^*)$ are given by

$$(6.8)\qquad r(g) f(x) = f(x\rho(g))\, \mathbf{e}[\alpha(x, g)] \left(\frac{d(x\rho(g))}{dx}\right)^{1/2}$$

$$(6.9)\qquad r^*(g) f^*(x^*) = f^*(x^*\rho^*(g))\, \mathbf{e}[-\alpha^*(x^*, g)] \left(\frac{d(x^*\rho^*(g))}{dx^*}\right)^{1/2}$$

$(x \in V_{\mathbf{R}},\ x^* \in V_{\mathbf{R}}^*,\ f \in \mathcal{S}(V_{\mathbf{R}}),\ f^* \in \mathcal{S}(V_{\mathbf{R}}^*))$. It is easy to see that

$$(6.10)\qquad F\cdot r(g) = r^*(g)\cdot F,\ r(g)\cdot F^* = F^*\cdot r^*(g) \qquad (g \in G_{\mathbf{R}}^+).$$

The following lemma is easily verified.

LEMMA 6.2. *The integrals* $Z(s, f, L)$ *and* $Z^*(s, f^*, L^*)$ *are absolutely convergent for* $s \in B$ *and* $s \in B^*$*, respectively. Furthermore we have*

$$Z(s, f, L) = \sum_{1 \leq i \leq \nu} \xi_i(s, L) \Phi_i(f, s - \lambda),$$

$$Z^*(s, f^*, L^*) = \sum_{1 \leq i \leq \nu} \xi_i^*(s, L^*) \Phi_i^*(f^*, s - \lambda^*).$$

Let Ω (resp. Ω^*) be the convex hull of $(B^* U^{-1} + \lambda) \cup B$ (resp. $(B - \lambda) U \cup B^*$) in $\mathbf{C}^n$. Then $(\Omega - \lambda) U = \Omega^*$. Applying the Poisson summation formula and using (6.10), we obtain the following:

THEOREM 6.3. *Assume that $f^* \in \mathcal{S}(V_{\mathbf{R}}^*)$ and its Fourier transform $F^* f^*$ vanish on $S_{\mathbf{R}}^*$ and $S_{\mathbf{R}}$, respectively. Then $Z(s, F^* f^*, L)$ and $Z^*(s, f^*, L^*)$ are continued to Ω and Ω^* respectively as meromorphic functions of s. Furthermore the following functional equation holds:*

$$Z^*((s - \lambda) U, f^*, L^*) = \mathrm{vol}(V_{\mathbf{R}}^* / L^*)^{-1} Z(s, F^* f^*, L) \qquad (s \in \Omega).$$

For simplicity, we write $b_0(s)$ and $b_0^*(s)$ for $b_{\chi_1^* \cdots \chi_n^*}(s)$ and $b_{\chi_1 \cdots \chi_n}^*(s)$, where $b_{\chi^*}(s)$ (resp. $b_\chi^*(s)$) is the b-function of $\mathbf{D}$ (resp. $\mathbf{D}^*$). We can now state our main result, which follows from Theorem 5.2 and Theorem 6.3 in use of the standard argument in the theory of zeta functions of PV's (for example, see [8] and [3]).

THEOREM 6.4. *Assume that the conditions* (3.6), (6.1) *and* (6.5) *are satisfied.*
(i) *The Dirichlet series $\xi_1(s, L), \ldots, \xi_\nu(s, L)$ (resp. $\xi_1^*(s, L^*), \ldots, \xi_\nu^*(s, L^*)$) are continued to meromorphic functions of s on Ω (resp. Ω^*). Furthermore $b_0(s - \lambda) \cdot \xi_i(s, L)$ (resp. $b_0^*(s - \lambda^*) \cdot \xi_i^*(s, L^*)$) is holomorphic on Ω (resp. Ω^*) for $i = 1, \ldots, \nu$.*
(ii) *The following functional equations hold for $i = 1, \ldots, \nu$:*

$$\mathrm{vol}(V_{\mathbf{R}}^* / L^*) \cdot \xi_i^*((s - \lambda) U, L^*) = c(\lambda - s) \gamma(s - \lambda) \sum_{1 \leq j \leq \nu} A_{ji}(s - \lambda) \cdot \xi_j(s, L),$$

where $s \in \Omega$ and $A_{ji}(s)$ is a polynomial in $\exp(\pm \pi i s_1), \ldots, \exp(\pm \pi i s_n)$ given in Theorem 5.2.

§7. Examples

1. Let $\mathbf{D} = (G, V, \rho, \alpha)$ be an affine datum over $\mathbf{Q}$. For $a \in V_{\mathbf{Q}}$ and $b^* \in V_{\mathbf{Q}}^*$, define an algebraic homomorphism $\rho_{a,b^*} : G \to \mathrm{Aff}(V)$ and an affine 1-cocycle $\alpha_{a,b^*} : V \times G \to \mathbf{G}_a$ by

(7.1) $\qquad x \rho_{a,b^*}(g) = (x + a) \rho(g) - a,$

(7.2) $\qquad \alpha_{a,b^*}(x, g) = \alpha(x + a, g) + \langle (x + a)(\rho(g) - 1_V), b^* \rangle.$

Then $\mathbf{D}_{a,b^*} = (G, V, \rho_{a,b^*}, \alpha_{a,b^*})$ is also an affine datum, which we call the *shift* of $\mathbf{D}$ by (a, b^*). If $\mathbf{D}$ is a PAD with singular set S, then $\mathbf{D}_{a,b^*}$ is also prehomogeneous and its singular set is $S_a = \{x \in V \mid x + a \in S\}$. The dual of $\mathbf{D}_{a,b^*}$ is the shift $\mathbf{D}^*_{b^*,a}$ of the dual $\mathbf{D}^*$ of $\mathbf{D}$ by (b^*, a). It is easily verified that $\mathbf{D}_{a,b^*}$ is regular if and only if so is $\mathbf{D}$.

Let $\mathbf{D}$ be a $\mathbf{Q}$-regular PAD satisfying the conditions (3.6) and (6.1). Then its shift $\mathbf{D}_{a,b^*}$ is also $\mathbf{Q}$-regular and satisfies (3.6) and (6.1). Let (β, β^*) be a normalized pair for $(\mathbf{D}, \mathbf{D}^*)$ defined over $\mathbf{Q}$ (for definition, see §3). Put

$$\beta_{a,b^*}(x) = \beta(x+a) + \left\langle x + \frac{1}{2}a,\, b^* \right\rangle \qquad (x \in V), \tag{7.3}$$

$$\beta^*_{b^*,a}(x^*) = \beta^*(x^* + b^*) + \left\langle a,\, x^* + \frac{1}{2}b^* \right\rangle \qquad (x^* \in V^*). \tag{7.4}$$

Then $(\beta_{a,b^*}, \beta^*_{b^*,a})$ is a normalized pair for $(\mathbf{D}_{a,b^*}, \mathbf{D}^*_{b^*,a})$ defined over $\mathbf{Q}$. We can easily verify that the a-function, the b-function and the data appearing in Theorem 5.2 (λ, U, $c(s)$, $\gamma(s)$ and $A(s)$) do not change for the shifting of PAD. Thus zeta functions attached to $\mathbf{D}_{a,b^*}$ have the same functional equations as those attached to $\mathbf{D}$.

2. We now consider the simplest PAD $\mathbf{D} = (G, V, \rho, \alpha)$ with $G_{\mathbf{Q}} = \mathbf{Q}^\times$, $V_{\mathbf{Q}} = \mathbf{Q}$, $x\rho(t) = xt$ and $\alpha(x, t) = 0$ $(x \in \mathbf{Q},\, t \in \mathbf{Q}^\times)$. The singular set is $S = \{0\}$ and $V_{\mathbf{R}} - S_{\mathbf{R}} = V_1 \cup V_2$, where $V_1 = \{x \in \mathbf{R} \mid x > 0\}$ and $V_2 = \{x \in \mathbf{R} \mid x < 0\}$. We identify V and V^* via the inner product $\langle x, y\rangle = xy$. In this case, the data appearing in Theorem 5.2 are given by

$$\lambda = 1,\ U = (-1),\ c(s) = (2\pi)^s \mathbf{e}\left[-\frac{s}{4}\right],\ \gamma(s) = \Gamma(s+1),$$

$$A(s) = \frac{-1}{2\pi i}\begin{pmatrix} 1 & -\mathbf{e}[-s/2] \\ -\mathbf{e}[-s/2] & 1 \end{pmatrix}.$$

Let $\mathbf{D}_{a,b}$ be the shift of $\mathbf{D}$ by a and b $(a, b \in \mathbf{Q})$. Since $\Gamma = G^+_{\mathbf{Z}} = \{1\}$, $L = \mathbf{Z}$ is a Γ-admissible lattice of $\mathbf{D}_{a,b}$ and associated zeta functions are given as follows:

$$\xi_1(s, L) = \mathbf{e}\left[-\frac{ab}{2}\right]\zeta_+(s, a, -b),\ \xi_2(s, L) = \mathbf{e}\left[-\frac{ab}{2}\right]\zeta_-(s, a, -b),$$

$$\xi_1^*(s, L^*) = \mathbf{e}\left[\frac{ab}{2}\right]\zeta_+(s, b, a),\ \xi_2^*(s, L^*) = \mathbf{e}\left[\frac{ab}{2}\right]\zeta_-(s, b, a),$$

Here $\zeta_\pm(s, a, b)$ are the usual Hurwitz-Lerch zeta functions given by

$$\zeta_{\pm}(s, a, b) = \sum_{n \in \mathbf{Z}, \pm(n+a)>0} \mathbf{e}[bn] \, |n+a|^{-s}.$$

Then Theorem 6.4 implies the well-known functional equations of Hurwitz-Lerch zeta functions:

$$(\zeta_+(1-s, b, a), \zeta_-(1-s, b, a))$$
$$= \mathbf{e}[-ab](2\pi)^{-s}\Gamma(s)(\zeta_+(s, a, -b), \zeta_-(s, a, -b)) \times \begin{pmatrix} \mathbf{e}[s/4] & \mathbf{e}[-s/4] \\ \mathbf{e}[-s/4] & \mathbf{e}[s/4] \end{pmatrix}.$$

Remark. Zeta functions associated with a shift of a regular PV are considered as a generalization of the Hurwitz-Lerch zeta functions. Such zeta functions have been studied by F. Sato from the point of view different from ours (see [4, §4]). In fact, he has shown their functional equations by using his theory of *zeta distributions.*

3. Let n and m be positive integers and put $V = \mathrm{Sym}_n$ ($=$ the space of symmetric matrices of degree n) and $W = \mathrm{M}_{m,n}$. Let $\mathbf{G} = \{(\xi, g) = \begin{pmatrix} 1_m & \xi \\ 0 & 1_n \end{pmatrix} \cdot \begin{pmatrix} 1_m & 0 \\ 0 & g \end{pmatrix} \mid \xi \in W, g \in \mathrm{GL}_n\}$ and $\mathbf{V} = V \times W$. We define an algebraic homomorphism ρ of $\mathbf{G}$ into $\mathrm{GL}(\mathbf{V})$ by

$$[x, u]\,\rho((\xi, g)) = [g^{-1}x\,{}^tg^{-1}, (u - \xi x)\,{}^tg^{-1}]$$

for $[x, u] \in \mathbf{V}$ and $(\xi, g) \in \mathbf{G}$. The triplet $(\mathbf{G}, \mathbf{V}, \rho)$ is a *non-regular* PV with singular set $\mathbf{S} = \{[x, u] \in \mathbf{V} \mid P(x, u) := \det(x) = 0\}$. Fix a positive definite semi-integral symmetric matrix S of degree m and define an affine 1-cocycle α by $\alpha([x, u], (\xi, g)) = \mathrm{tr}(x\,{}^t\xi S\xi - 2\,{}^t\xi Su)$. Then it is easily verified that $\mathbf{D} = (\mathbf{G}, \mathbf{V}, \rho, \alpha)$ is a regular PAD. In this case, a function β on $\mathbf{V} - \mathbf{S}$ satisfying (2.1) is given by $\beta([x, u]) = \mathrm{tr}(x^{-1} \cdot {}^tuSu)$. Note that $\mathbf{D}$ is not a shift of any PV. Identifying $\mathbf{V}^*$ with $\mathbf{V}$ via the inner product $\langle [x, u], [y, v] \rangle = \mathrm{tr}(xy + 2\,{}^tuSv)$ ($[x, u]$, $[y, v] \in \mathbf{V}$), we see that the dual of $\mathbf{D}$ is given by $\mathbf{D}^* = (\mathbf{G}, \mathbf{V}, \rho^*, \alpha^*)$ where

$$[x, u]\,\rho^*((\xi, g)) = [{}^tg(x + {}^t\xi S\xi + {}^tuS\xi + {}^t\xi Su)g, (u + \xi)g], \ \alpha^* \equiv 0.$$

The singular set $\mathbf{S}^*$ of $\mathbf{D}^*$ is $\{[x, u] \in \mathbf{V} \mid P^*(x, u) := \det(x - {}^tuSu) = 0\}$. Note that the relative invariant $P^*(x, u)$ of $\mathbf{D}^*$ is *not* homogeneous. The associated zeta functions with $\mathbf{D}$ and $\mathbf{D}^*$ are the same ones as studied in [2] (for the precise form of their functional equations, see [2 Theorem 2.4]). Note that certain special values of the above zeta functions appear in the dimension formula for the space of Jacobi forms of degree n (see [2, Theorem 4.3]).

4. Let K be an imaginary quadratic field o_K be the integer ring of K. We denote by $\bar{a}$ the conjugate of $a \in K$ and put $x^* = {}^t\bar{x}$ for $x \in \mathrm{M}_r(K)$. Let τ be the trace of K to $\mathbf{Q}: \tau(a) = a + \bar{a}\,(a \in K)$. For positive integers n and m, let $V_{\mathbf{Q}} = \{x \in \mathrm{M}_n(K) \mid x^* = x\}$, $W_{\mathbf{Q}} = \mathrm{M}_{m,n}(K)$ and $\mathbf{V}_{\mathbf{Q}} = V_{\mathbf{Q}} \times W_{\mathbf{Q}}$. The group $\mathbf{G}_{\mathbf{Q}} = \left\{(\xi, g) = \begin{pmatrix} 1_m & \xi \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} 1_m & 0 \\ 0 & g \end{pmatrix} \mid \xi \in W_{\mathbf{Q}}, g \in \mathrm{GL}_n(K)\right\}$ acts on $\mathbf{V}_{\mathbf{Q}}$ by

$$[x, u]\,\rho((\xi, g)) = [g^{-1}xg^{*-1}, (u - \xi x)g^{*-1}) \quad ([x, u] \in \mathbf{V}_{\mathbf{Q}}, (\xi, g) \in \mathbf{G}_{\mathbf{Q}}).$$

We fix a positive definite Hermitian matrix $H = (h_{ij})$ of degree m with $h_{ii} = \mathbf{Z}$ and $h_{ij} \in \delta_K^{-1}$, where δ_K is the different of K over $\mathbf{Q}: \delta_K^{-1} = \{a \in K \mid \tau(ao_K) \subset \mathbf{Z}\}$. For $u, v \in W_{\mathbf{Q}}$, $H[u]$ and $H(u, v)$ stand for u^*Hu and u^*Hv, respectively. For $[x, u] \in \mathbf{V}_{\mathbf{Q}}$ and $(\xi, g) \in \mathbf{G}_{\mathbf{Q}}$, put $\alpha([x, u], (\xi, g)) = \mathrm{tr}(x \cdot H[\xi]) - \tau(\mathrm{tr}(H(\xi, u)))$. Then α is an affine 1-cocycle with respect to ρ. It is easily verified that $\mathbf{D} = (\mathbf{G}, \mathbf{V}, \rho, \alpha)$ is a PAD defined over $\mathbf{Q}$ with singular set $\mathbf{S} = \{[x, u] \in \mathbf{V} \mid P([x, u]) := \det x = 0\}$. Henceforth we identify the $\mathbf{Q}$-dual $\mathbf{V}_{\mathbf{Q}}^*$ of $\mathbf{V}_{\mathbf{Q}}$ with $\mathbf{V}_{\mathbf{Q}}$ itself via the inner product $\langle\,,\rangle$ of $\mathbf{V}_{\mathbf{Q}}$ defined by $\langle [x,u], [y, v]\rangle = \mathrm{tr}(xy) + \tau(\mathrm{tr}(H(u, v)))$. Then the dual $\mathbf{D}^* = (\mathbf{G}, \mathbf{V}, \rho^*, \alpha^*)$ is given by $[x, u]\,\rho^*((\xi, g)) = [g^*(x + H[u + \xi] - H[u])g, (u + \xi)g]$, $\alpha^* \equiv 0$. We easily see that $\mathbf{D}^*$ is also PAD and its singular set is $\mathbf{S}^* = \{[x, u] \in \mathbf{V} \mid P^*([x, u]) := \det(x - H[u]) = 0\}$. The corresponding character χ_1(resp. χ_1^*) to P (resp. P^*) is given by $\chi_1((\xi, g)) = \mathrm{N}_{K/\mathbf{Q}}(\det g)^{-1}$ (resp. $\chi_1^*((\xi, g)) = \mathrm{N}_{K/\mathbf{Q}}(\det g)$). Thus $\mathbf{G}_{1,\mathbf{Q}} = \{(\xi, g) \in \mathbf{G}_{\mathbf{Q}} \mid \mathrm{N}_{K/\mathbf{Q}}(\det g) = 1\}$. For simplicity we write (ξ, A) for each element $\begin{pmatrix} 0 & \xi \\ 0 & A \end{pmatrix}$ of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}$ of $\mathbf{G}_{\mathbf{Q}}$ ($\xi \in \mathrm{M}_{m,n}(K)$, $A \in \mathrm{M}_n(K)$). Then the differentials of ρ, α and χ_1 are given as follows:

$$\begin{aligned} &[x, u]\,d\rho((\xi, A)) = [-Ax - xA^*, -\xi x - uA^*], \\ &d\alpha([x, u]\,(\xi, A)) = -\tau(\mathrm{tr}\,H(\xi, u)), \; d\chi_1((\xi, A)) = -\tau(\mathrm{tr}\,A). \end{aligned}$$

Since $d\chi_1$ vanishes on the Lie algebra $\mathfrak{g}_1$ of $\mathbf{G}_1$, we can define a $\mathbf{G}$-equivariant mapping $\phi_s: \mathbf{V} - \mathbf{S} \to \mathbf{V} - \mathbf{S}^*$ for $s \in \mathbf{C}$ so that

$$\langle [x, u]\,d\rho((\xi, A)), \phi_s([x, u])\rangle + d\alpha([x, u], (\xi, A)) = -\frac{1}{2\pi i}\,s\,d\chi_1((\xi, A))$$

for every $[x, u] \in \mathbf{V}$ and $(\xi, A) \in \mathfrak{g}_{\mathbf{Q}}$ (see Proposition 2.3). In fact, we have $\phi_s([x, u]) = \left[-\frac{s}{2\pi i}x^{-1} + H[ux^{-1}], -ux^{-1}\right]$ and hence ϕ_s is dominant if $s \neq 0$. Thus $\mathbf{D}$ and $\mathbf{D}^*$ are $\mathbf{Q}$-regular. We normalize β and β^* satisfying (2.1) by

$\beta([x, u]) = \mathrm{tr}(x^{-1}H[u])$, $\beta^* \equiv 0$. Then $F(s)$ defined in §3 is equal to $-\frac{n}{2\pi i}s$ and hence the condition (3.6) is satisfied.

Let V_i be the set of $n \times n$ Hermitian matrices with i positive and $n-i$ negative eigenvalues $(0 \leq i \leq n)$. Put $\mathbf{V}_i = V_i \times W_{\mathbf{R}}$ and $\mathbf{V}_i^* = \{[x, u] \in \mathbf{V}_{\mathbf{R}} \mid x - H[u] \in V_i\}$. Then we have $\mathbf{V}_{\mathbf{R}} - \mathbf{S}_{\mathbf{R}} = \cup_{i=0}^n \mathbf{V}_i$ and $\mathbf{V}_{\mathbf{R}} - \mathbf{S}_{\mathbf{R}}^* = \cup_{i=0}^n \mathbf{V}_i^*$ (disjoint union). Let $dx = \Pi_{i=1}^n dx_{ii} \Pi_{i<j} d\mathrm{Re}(x_{ij})\, d\mathrm{Im}(x_{ij})$ $(x = (x_{ij}) \in V_{\mathbf{R}})$ and $du = \Pi_{i=1}^m \Pi_{j=1}^n d\mathrm{Re}(u_{ij})\, d\mathrm{Im}(u_{ij})$ be the usual Lebesgue measures on $V_{\mathbf{R}}$ and $W_{\mathbf{R}}$, respectively. For $f \in \mathcal{S}(\mathbf{V}_{\mathbf{R}})$, define

$$\Phi_i(f, s) = \int_{\mathbf{V}_i} f([x, u]) \mid \det x \mid^s \mathbf{e}[\mathrm{tr}(x^{-1}H[u])]\, dxdu,$$

$$\Phi_i^*(f, s) = \int_{\mathbf{V}_i^*} f([x, u]) \mid \det (x - H[u]) \mid^s dxdu.$$

Both functions $\Phi_i(f, s)$ and $\Phi_i^*(f, s)$ are continued to meromorphic functions on $\mathbf{C}$ (Proposition 5.1). Let F^* be the Fourier transform of $f \in \mathcal{S}(\mathbf{V}_{\mathbf{R}})$:

$$F^*f([x, u]) = \int_{\mathbf{V}_{\mathbf{R}}} f([y, v])\, \mathbf{e}[\langle [x, u], [y, v] \rangle]\, dy\, dv$$

The following functional equation is proved by a similar argument to that of [2, Proposition 1.4] and by using the results of [8, §4].

THEOREM 7.1. *For $f \in \mathcal{S}(\mathbf{V}_{\mathbf{R}})$, we have*

$$\Phi_i(F^*f, s) = c(-s)\gamma(s) \sum_{j=0}^n A_{ij}(s)\, \Phi_j^*(f, -s-m-n),$$

where

$$c(s) = \mathbf{e}\left[-\frac{ns}{4}\right] (2\pi)^{ns},\ \gamma(s) = \prod_{1 \leq i \leq n} \Gamma(s+m+i),$$

$$A_{ij}(s) = (\det 2H)^{-n}\, \mathbf{e}\left[\frac{n^2 - 2mi}{4}\right] (2\pi)^{-n(m+n)}\, u_{ij}(s+m+n),$$

with $u_{ij}(s) = (-1)^{(n-i)(n-1)} \pi^{n(n-1)/2} \times \sum_{k=\mathrm{Max}(0,i+j-n)}^{\mathrm{Min}(i,j)} \binom{j}{k} \binom{n-j}{i-k} \exp(\pi i\, s(2k-i-j))$

for $0 \leq i, j \leq n$.

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We normalize a right invariant measure dg on $\mathbf{G}_{\mathbf{R}}^{+}$ by $d((\xi, g)) = |\det g|^{-2m-2n}$ $\prod_{1\le i,j\le n} d\mathrm{Re}(g_{i,j})\, d\mathrm{Im}(g_{i,j}) \prod_{i=1}^{m}\prod_{j=1}^{n} d\mathrm{Re}(\xi_{i,j})\, d\mathrm{Im}(\xi_{i,j})$. Define the density $\mu(x)$ (resp. $\mu^*(x^*)$) for $\mathbf{x} \in \mathbf{V}_{\mathbf{Q}} - \mathbf{S}_{\mathbf{Q}}$ (resp. $\mathbf{x}^* \in \mathbf{V}_{\mathbf{Q}} - \mathbf{S}_{\mathbf{R}}^*$) as in §6 and set $\boldsymbol{\Gamma} = \{(\xi, \gamma) \in \mathbf{G}_{\mathbf{Q}} \mid \xi \in \mathrm{M}_{m,n}(o_K),\ \gamma \in GL_n(o_K)\}$. Let $\mathbf{L}$ be a $\boldsymbol{\Gamma}$-admissible lattice of $\mathbf{V}_{\mathbf{Q}}$ with respect to (ρ, α) and $\mathbf{L}^*$ its dual. Put $\mathbf{L}_i = \mathbf{L} \cap \mathbf{V}_i$ and $\mathbf{L}_i^* = \mathbf{L}^* \cap \mathbf{V}_i^*$ $(0 \le i \le n)$ and define zeta functions as follows:

$$\xi_i(s, \mathbf{L}) = \sum_{[x,u]\in \mathbf{L}_i/\rho(\boldsymbol{\Gamma})} \mu([x,u])\, \mathbf{e}[-\operatorname{tr}(x^{-1}H[u])]\, |\det x|^{-s},$$

$$\xi_i^*(s, \mathbf{L}^*) = \sum_{[x,u]\in \mathbf{L}_i^*/\rho^*(\boldsymbol{\Gamma})} \mu^*([x,u])\, |\det(x - H[u])|^{-s}.$$

Then the condition (6.5) is satisfied. The following result is a consequence of Theorem 6.4 and Theorem 7.1.

THEOREM 7.2. *Let the notation be the same as in Theorem* 7.1. *The Dirichlet series* $\xi_i(s, \mathbf{L})$ *and* $\xi_i^*(s, \mathbf{L}^*)$ *are continued to meromorphic functions on* $\mathbf{C}$ *with possible simple poles at* $s = 1, \dots, n$. *Furthermore they satisfy the functional equations*

$$\begin{aligned}&\mathrm{vol}(\mathbf{V}_{\mathbf{R}}/\mathbf{L}^*)\xi_i^*(m+n-s, \mathbf{L}^*)\\ &\quad = c(m+n-s)\gamma(s-m-n)\sum_{j=0}^{n} A_{ji}(s-m-n)\xi_j(s, \mathbf{L}) \quad (1 \le i \le n).\end{aligned}$$

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