Bull. Aust. Math. Soc. **106** (2022), 508–510
doi:10.1017/S0004972722000867

SYLOW CLASSES OF REFLECTION SUBGROUPS AND PSEUDO-LEVI SUBGROUPS

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(Received 18 July 2022; first published online 30 August 2022)

2020 *Mathematics subject classification*: primary 20F55; secondary 20D20, 20G40.

Keywords and phrases: reflection groups, finite groups of Lie type, Sylow subgroups, Levi subgroups.

Preliminaries and motivation. We study and classify classes of subgroups of finite reflection groups and finite groups of Lie type that minimally contain Sylow ℓ-subgroups, where ℓ is some prime different to the defining characteristic of the connected reductive group. In particular, for finite complex reflection groups we classify, up to conjugacy, the parabolic subgroups and reflection subgroups minimally containing a Sylow ℓ-subgroup. Analogously, for finite groups of Lie type, we classify, up to conjugacy, the Levi subgroups and pseudo-Levi subgroups minimally containing a Sylow ℓ-subgroup. We find connections between both these classifications while also using both to aid in describing Sylow ℓ-subgroups in their respective groups. Parabolic subgroups minimally containing a Sylow subgroup appear in the study of modular representation theory for finite groups of Lie type [1, Theorem 4.5], Furthermore, in [5, Theorem 4.2], the analogous idea of Levi subgroups minimally containing a Sylow ℓ-subgroup of a finite group of Lie type can help determine the semisimple vertex of the ℓ-modular Steinberg character.

Overview. We begin by classifying the parabolic subgroups and reflection subgroups minimally containing a Sylow ℓ-subgroup in a finite complex reflection group. This closely follows the previously published article [13] with some minor additions. It generalises the prior work [12] which achieves the same classification for finite real reflection groups. We classify these minimal subgroups via case-by-case calculations on the irreducible finite complex reflection groups described in [9], and their parabolic subgroups and reflection subgroups described in [11]. We first classify the parabolic subgroups minimally containing a Sylow ℓ-subgroup for each prime ℓ dividing the order of an irreducible reflection group. Since the class of parabolic subgroups

Thesis submitted to the University of Sydney in December 2021; degree approved on 31 March 2022; primary supervisor Anthony Henderson, auxiliary supervisor Zsuzsanna Dancso.

is closed under conjugation and intersection, the parabolic subgroup minimally containing a Sylow ℓ-subgroup is unique up to conjugacy. To classify the reflection subgroups minimally containing a Sylow ℓ-subgroup, we are able to focus on particular irreducible finite reflection groups due to the following result.

THEOREM 1. *Let G be a finite complex reflection group. The parabolic closure of a reflection subgroup minimally containing a Sylow ℓ-subgroup is a parabolic subgroup minimally containing a Sylow ℓ-subgroup.*

A property of the normaliser of the parabolic subgroups minimally containing a Sylow ℓ-subgroup motivates the next part of the thesis.

THEOREM 2. *Let G be a finite complex reflection group, S_ℓ a Sylow ℓ-subgroup and P_ℓ a parabolic subgroup minimally containing S_ℓ. Then $N_G(S_\ell) \leq N_G(P_\ell)$.*

The normaliser of a parabolic subgroup has been described for finite real reflection groups in [7] and for finite complex reflection groups in [8]. In both cases, the normaliser is described as a semidirect product of the parabolic subgroup and some complement, which we refer to as the H-complement in the real situation and the MT-complement in the complex situation. Case-by-case, we give a Sylow ℓ-subgroup contained in a minimal parabolic subgroup that is normalised by the complement. This gives us the following theorem for finite real reflection groups and an analogous version for finite complex reflection groups with the MT-complement.

THEOREM 3. *Let W be a finite real reflection group. Let P_ℓ be a parabolic subgroup minimally containing a Sylow ℓ-subgroup and U_ℓ be the the H-complement of P_ℓ. Then there exists a Sylow ℓ-subgroup $S_\ell \leq P_\ell$ of W such that $N_W(S_\ell) = N_{P_\ell}(S_\ell) \rtimes U_\ell$.*

Moving on to the analogous problem in finite groups of Lie type, let $\mathbf{G}$ be a connected reductive group and F a Steinberg endomorphism. Then we classify, up to $\mathbf{G}^F$-conjugacy, the Levi subgroups and pseudo-Levi subgroups of $\mathbf{G}^F$ that minimally contain a Sylow ℓ-subgroup. Using the split BN-pair structure of $\mathbf{G}^F$, the conjugacy class of Levi subgroups minimally containing a Sylow ℓ-subgroup is unique. Let $\mathbf{G}$ be defined over $\overline{\mathbb{F}}_q$, where $q = p^f$ is a prime power with $p \neq \ell$. The order of $\mathbf{G}^F$ can be factorised into a product of some power of q and cyclotomic polynomials over q. We do a preliminary calculation of the class of Levi subgroups minimally containing a Sylow ϕ-torus, where ϕ is a generalisation of a cyclotomic polynomial seen in [4]. This leads to a straightforward procedure to find the Levi subgroups minimally containing a Sylow ℓ-subgroup. Furthermore, we prove the following result.

THEOREM 4. *Let $\mathbf{G}^F$ be an $\mathbb{F}_q$-split finite group of Lie type and ℓ a prime such that* $\mathrm{ord}_\ell(q) = 1$. *Then the parabolic subsystem associated to the Levi subgroup minimally containing a Sylow ℓ-subgroup of $\mathbf{G}^F$ is the same type as the parabolic subgroup minimally containing a Sylow ℓ-subgroup of the Weyl group of $\mathbf{G}$.*

We then introduce pseudo-Levi subgroups of $\mathbf{G}^F$ as an analogue of reflection subgroups in finite reflection groups. We give a version of Theorem 4 generalised to

https://doi.org/10.1017/S0004972722000867 Published online by Cambridge University Press

reflection subsystems with a caveat of the additional requirement that the subsystem is p-closed. We then classify the pseudo-Levi subgroups minimally containing a Sylow ℓ-subgroup using a procedure based on the Borel–de-Siebenthal algorithm [2].

Finally, we investigate descriptions of Sylow subgroups of finite groups of Lie type focusing on the description provided in [4], which generalises results of [3]. These descriptions associate a complex reflection group to $\mathbf{G}^F$ via generalisations of Springer theory [10]. In particular, for specific cases we give explicit descriptions of Sylow ℓ-subgroups in terms of the action of a Sylow ℓ-subgroup of the associated complex reflection group on elements of the cocharacters of $\mathbf{G}$. Motivated by these descriptions, we introduce what we call twisted Levi subgroups and twisted pseudo-Levi subgroups. These subgroups allow us to restate Theorem 4 for general $\mathbf{G}^F$ and $\mathrm{ord}_\ell(q)$. They are similar to the split Levi subgroups appearing in generalisations of Harish-Chandra theory [6, 3.5].

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https://doi.org/10.1017/S0004972722000867 Published online by Cambridge University Press