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A HIERARCHY ON THE CLASS OF PRIMITIVE RECURSIVE ORDINAL FUNCTIONS

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The class of primitive recursive ordinal functions (PR) has been studied recently by numerous recursion theorists and set theorists (see, for example, Platek [**3**] and Jensen-Karp [**2**]). These investigations have been part of an inquiry concerning a larger class of functions; in Platek's case, the class of ordinal recursive functions and in the case of Jensen and Karp, the class of primitive recursive set functions. In [**4**] I began to study PR in depth and this paper is a report on an attractive analogy between PR and its progenitor, the class of primitive recursive functions on the natural numbers (Prim. Rec.).

One of the pleasant properties of Prim. Rec. is that one can assign a number to each function in Prim. Rec. that, in a sense, measures how fast the function grows. In 1953 Grzegorczyk constructed a hierarchy on Prim. Rec. based on this property—each set of the hierarchy is limited to functions that do not grow too fast relative to other functions in the set. In this paper a hierarchy on PR will be constructed in a manner analogous to Grzegorczyk's construction on Prim. Rec.

No attempt will be made to make this paper self-contained. Grzegorczyk's construction can be found both in [**1**] and also in [**5**] (which is much more accessible). The foundations of the theory of PR are laid out quite adequately in the papers referred to above by Platek and Jensen-Karp.

Definition. PR is the smallest class of ordinal functions containing, for each $m \in \omega$, the function $\lambda\xi.m$; the successor function, sc; the function C defined for all α, β, γ, δ, and ξ by

$$C(\alpha, \beta, \gamma, \delta, \xi) = \begin{cases} \alpha & \text{if } \gamma < \delta \\ \beta & \text{if } \delta \leqq \gamma \end{cases}$$

and closed under composition and primitive recursion.

Definition. For all functions F, F is *almost normal* if and only if there is an m_F such that for all α and β, if $\alpha > \beta \geqq m_F$, then $F(\alpha) > F(\beta)$ and for all limit ordinals λ, $F(\lambda) = \operatorname{Sup}_{\alpha<\lambda} F(\alpha)$. If $m_F = 0$, then F is a normal function.

If F is an almost normal function then a classic theorem states that for all α, $F(\alpha) \geqq \alpha - m_F$ and that F possesses arbitrarily large fixed points. The function enumerating these fixed points will be denoted by D_F. If the following is

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taken as the formal definition of D_F then, whenever F is an almost normal PR function with no finite fixed points, D_F will be a normal PR function with no finite fixed points.

Definition. For any almost normal function F (or any function possessing arbitrarily large fixed points), D_F is the function defined for all α by

$$D_F(\alpha) = \begin{cases} \alpha + 1 & \text{if } \alpha \in \omega \\ \alpha - \omega\text{th fixed point of } F & \text{if } \alpha \notin \omega. \end{cases}$$

The *predecessor function*, pd, defined by $\mathrm{pd}(\alpha + 1) = \alpha$ and $\mathrm{pd}(\lambda) = \lambda$ when λ is a limit ordinal, is clearly almost normal (with $m_{\mathrm{pd}} = 1$).

Definition. For all α,

(i) $F_0(\alpha) = \mathrm{pd}(\alpha)$,

and for all n,

$$\text{(ii)} \quad F_{n+1}(\alpha) = \begin{cases} n + 1 + \alpha & \text{if } \alpha \in \omega; \\ D_{F_n}(\alpha) & \text{otherwise.} \end{cases}$$

It is easy to see that $F_1(\omega + \alpha) = \omega(1 + \alpha)$, $F_2(\omega + \alpha) = \omega^{(1+\alpha)}$, $F_3(\omega + \alpha) = \omega^{\omega^\alpha}$, and $F_4(\omega + \alpha) = \epsilon_\alpha$. F_5 then enumerates those ordinals which are fixed points of the sequence of ϵ-numbers and the process continues.

Recall the definition of the functions f_n from Grzegorczyk. We extend those functions to PR functions by defining them to be ω if either argument is not finite.

Definition. (i) For all F, G, and H, *F comes from G and H by limited recursion* if and only if for all α and ξ,

$$F(\alpha, \xi) = G(\mathrm{Sup}_{\gamma<\alpha} F(\gamma, \xi), \alpha, \xi) \quad \text{and}$$
$$F(\alpha, \xi) \leqq H(\alpha, \xi).$$

(ii) A class $\mathscr{C}$ of functions is *closed under limited recursion* if and only if for all F, G, and H, if F comes from G and H by limited recursion and G and H are in $\mathscr{C}$, then also $F \in \mathscr{C}$.

Definition. For each n, $\mathscr{F}_n$ is the smallest class of ordinal functions containing sc, pd, C, $\lambda\xi.m$ for each $m \in \omega$, f_n, F_n, and closed under composition and limited recursion.

It is obvious that each $\mathscr{F}_n$ is a subset of PR. The main theorem of this paper is that the $\mathscr{F}_n$ form a hierarchy on PR. This is a consequence of the following three theorems:

THEOREM 1. *For all n, F_{n+1} enumerates the $\mathscr{F}_n$-closed ordinals.*

THEOREM 2. *For all n, $\mathscr{F}_n \subset \mathscr{F}_{n+1}$.*

THEOREM 3. *For all F in PR, there is an n such that $F \in \mathscr{F}_n$.*

Definition. For all n and α, ϵ_α^n is the α-th ordinal larger than ω which is closed under all functions in $\mathscr{F}_n$.

It is easy to see that, for all n and α, α is $\mathscr{F}_n$-closed if and only if α is a limit ordinal and for all $\beta < \alpha$, $F_n(\beta) < \alpha$. The proof does, however, make explicit use of the fact that all recursions are limited.

THEOREM 1. *For all n and α, $F_{n+1}(\omega + 1 + \alpha) = \epsilon_\alpha^n$.*

Proof. The proof is in two parts. We first show, by the above remark, that for all α, $F_{n+1}(\omega + 1 + \alpha)$ is $\mathscr{F}_n$-closed. It is then shown that for all α, ϵ_α^n is a fixed point of F_n and is therefore in the range of F_{n+1}. An induction on α then shows that the two sequences coincide.

Fix α and let $F_{n+1}(\omega + 1 + \alpha) = \rho$. It is clear that ρ is a limit ordinal. Since F_{n+1} enumerates the fixed points of F_n, it is also the case that $F_n(\rho) = \rho$. Therefore, if $\beta < \rho$, then $F_n(\beta) < F_n(\rho) = \rho$, so ρ is $\mathscr{F}_n$-closed.

Now fix ϵ_α^n. Then, since ϵ_α^n is $\mathscr{F}_n$-closed, $F_n(\beta) < \epsilon_\alpha^n$ for all $\beta < \epsilon_\alpha^n$. Since F_n is almost normal (and normal if $n > 0$) this implies

$$F_n(\epsilon_\alpha^n) = \sup_{\beta<\epsilon_\alpha^n} F_n(\beta) \leqq \epsilon_\alpha^n \leqq F_n(\epsilon_\alpha^n),$$

so that ϵ_α^n is a fixed point of F_n.

THEOREM 2. *For all n, $\mathscr{F}_n \subset \mathscr{F}_{n+1}$.*

Proof. It will be shown that for all n and all $m \leqq n$, $\mathscr{F}_m \subset \mathscr{F}_n$. Towards this end it is sufficient to show that $F_m \in \mathscr{F}_n$ whenever $m \leqq n$. This is trivial if $n = 0$ or $n = 1$, so suppose $n \geqq 2$. This guarantees that $\lambda\alpha, \beta.\alpha + \beta$ is in the class $\mathscr{F}_n$.

Suppose then that $n \geqq 2$ and that for all $t \leqq m < n$, $\mathscr{F}_t \subset \mathscr{F}_n$. It is shown that F_{m+1} is in $\mathscr{F}_n$. Define the following function H in $\mathscr{F}_n$:

$$H(\alpha, \beta) = \begin{cases} \alpha \cdot m + \beta & \text{if } \alpha + \beta \text{ is finite} \\ F_n(\alpha + \beta) & \text{otherwise.} \end{cases}$$

Now define, by recursion, the function It_{F_m} that iterates F_m:

For all α, β, and limit λ,

$$\mathrm{It}_{F_m}(0, \beta) = \beta;$$
$$\mathrm{It}_{F_m}(\alpha + 1, \beta) = F_m(\mathrm{It}_{F_m}(\alpha, \beta));$$
$$\mathrm{It}_{F_m}(\lambda, \beta) = \mathrm{Sup}_{\gamma<\lambda}(\mathrm{It}_{F_m}(\gamma, \beta)).$$

It is an easy induction on α to show that for all α and β, $\mathrm{It}_{F_m}(\alpha, \beta) \leqq H(\alpha, \beta)$, so that $\mathrm{It}_{F_m} \in \mathscr{F}_n$. Now let G be the one-place function in $\mathscr{F}_n$ that is the identity on ω and constantly ω above. F_{m+1} is definable from G and It_{F_m} by primitive recursion as follows:

$$F_{m+1}(0) = m + 1; F_{m+1}(\lambda) = \sup_{\gamma<\lambda} F_{m+1}(\gamma), \text{ if } \lim(\lambda);$$

$$F_{m+1}(\alpha + 1) = \begin{cases} m + 2 + \alpha & \text{if } \alpha \in \omega \\ \mathrm{It}_{F_m}(G(\alpha + 1), F_{m+1}(\alpha) + 1), & \text{otherwise.} \end{cases}$$

Since for all α, $F_{m+1}(\alpha) \leqq F_n(\alpha)$, the recursion is limited and F_{m+1} is in $\mathscr{F}_n$ and $\mathscr{F}_{m+1} \subset \mathscr{F}_n$.

THEOREM 3. *For all functions F in PR there is an n such that $F \in \mathscr{F}_n$.*

Proof. The proof is by induction on the class PR. All of the initial functions are in $\mathscr{F}_0$ and all of the $\mathscr{F}_n$ are closed under composition so we need only show that whenever $G \in \mathscr{F}_m$ and F comes from G by primitive recursion, then there is an n such that $F \in \mathscr{F}_n$. We may assume that $m \geqq 2$.

Since F is in PR, $F \upharpoonright \omega \in$ Prim. Rec., and by Grzegorczyk's work, there is an r for which $F \upharpoonright \omega \in \mathscr{E}_r$. Let F' extend $F \upharpoonright \omega$ by taking the value ω for infinite arguments. Then $F' \in \mathscr{F}_r$. Let $n = \max(r + 1, m + 2)$. We will show that F is in $\mathscr{F}_n$. Define $H \in \mathscr{F}_n$ by cases as follows: For all α and $\tilde{\xi}$,

$$H(\alpha, \tilde{\xi}) = \begin{cases} F'(\alpha, \tilde{\xi}) & \text{if } \max(\alpha, \tilde{\xi}) < \omega \\ F_n(\xi_0 + \ldots + \xi_{k-1} + \alpha + 1) & \text{otherwise.} \end{cases}$$

We claim that for all α and $\tilde{\xi}$, $F(\alpha, \tilde{\xi}) \leqq H(\alpha, \tilde{\xi})$ so that the recursion is bounded in $\mathscr{F}_n$. The proof is by induction on α and splits into four cases as α and $\max(\tilde{\xi})$ are finite or infinite. We illustrate the proof by doing the case when $\alpha > \omega$. Let G_0 be a function in $\mathscr{F}_m$ with the following two properties:

(i) For all $\tilde{\gamma}$, $G(\tilde{\gamma}) \leqq G_0(\tilde{\gamma})$, and

(ii) for all $\tilde{\gamma}$ and $\tilde{\delta}$, if $\gamma_1 \leqq \delta_1, \ldots, \gamma_{k-1} \leqq \delta_{k-1}$, then $G_0(\tilde{\gamma}) \leqq G_0(\tilde{\delta})$. (It is an easy matter to show that such a G_0 exists.)

Now, if $\alpha > \omega$, then

$$\begin{aligned} F(\alpha, \tilde{\xi}) &= G(\mathrm{Sup}_{\beta<\alpha} F(\beta, \tilde{\xi}), \alpha, \tilde{\xi}) \leqq G_0(\mathrm{Sup}_{\beta<\alpha} F(\beta, \tilde{\xi}), \alpha, \tilde{\xi}) \\ &\leqq G_0(\mathrm{Sup}_{\beta<\alpha} F_n(\xi_0 + \ldots + \xi_{k-1} + \beta + 1), \alpha, \tilde{\xi}) \\ &\leqq G_0(F_n(\xi_0 + \ldots + \xi_{k-1} + \alpha), \alpha, \tilde{\xi}) \leqq F_n(\xi_0 + \ldots + \xi_{k-1} + \alpha + 1) \\ &= H(\alpha, \tilde{\xi}). \end{aligned}$$

The last inequality follows from the fact that $G_0 \in \mathscr{F}_{n-1}$ and all of the arguments of G_0 are less than $F_n(\xi_0 + \ldots + \xi_{k-1} + \alpha + 1)$ so that G_0 applied to these arguments is also less than $F_n(\xi_0 + \ldots + \xi_{k-1} + \alpha + 1)$. The proofs in the other cases are merely perturbations of the proof in this case. This completes the proof.

The following theorem is a corollary of the theorem that we have a hierarchy on PR:

THEOREM 4. *PR is the smallest class of functions containing* sc, *C, all f_n, all*

λ, $\xi.m$, F_1 *and closed under composition, limited recursion and the schema:*

For all F, *if* $F \in PR$ *and* F *is normal, then* D_F *is in* PR.

Finally, some remarks about the class $\mathscr{F}_3$ are in order. This class is the natural generalization of the class of elementary functions to the transfinite. Just as the elementary functions have the property that "natural" functions on the natural numbers are elementary, $\mathscr{F}_3$ has the analogous property for ordinal functions. A non-trivial example of this is the fact that it can be shown that the truth-predicate for the constructible class L, which Platek had shown to be in PR, is, in fact, in $\mathscr{F}_3$. (See Chapter 7 of [**4**] for details.) This observation about the class $\mathscr{F}_3$ also gives greater understanding to the role played by the ϵ-numbers in mathematics, for they are just the $\mathscr{F}_3$-closed ordinals.

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