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INTERACTION OF SOME MEROMORPHIC SOLUTIONS OF THE KdV EQUATION

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A necessary and sufficient condition for confluence of two poles of a class of meromorphic solutions of the KdV equation is introduced and proved.

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1. Introduction and statement of the main result

In this paper we study interaction of some meromorphic solutions of the Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad \lim_{x\to\pm\infty} u(t,x) = 0. \tag{1.1}$$

These solutions, sometimes referred to as positions [**11, 12, 14, 15**], and sometimes as harmonic breathers [**7**] are of the form

$$u = \frac{8\ell^2 \sin 2\ell(4\ell^2 t + x - \gamma)}{\sin 2\ell(4\ell^2 t+x-\gamma) - 2\ell(12\ell^2 t + x - p)} + 8\ell^2 \left[\frac{1-\cos 2\ell(4\ell^2 t+x-\gamma)}{\sin 2\ell(4\ell^2 t+x-\gamma)-2\ell(12\ell^2 t+x-p)}\right]^2. \tag{1.2}$$

Probably first studied in [**11, 12, 14, 15**], (1.2) naturally arises when one generates explicit solutions of (1.1) by means of the Darboux transform [**11, 12**] or attempts to define solutions of (1.1) with the "simplest continuous spectrum" [**7**].

Solutions (1.2) possess a pole whose location depends on time. This pole may play role of a "centre" of the corresponding harmonic breather in much the same way as the local minimum of the soliton solution $-\frac{2\ell^2}{\cosh^2 \ell(x-4\ell^2 t-\varphi)}$ plays that role for the corresponding soliton. The solution (1.2) can be written as

$$u(t,x) = -2\frac{\partial^2}{\partial x^2} \ln \tau(t,x) \tag{1.3a}$$

$$\tau(t,x) = \frac{\sin 2\ell(4\ell^2 t + x - \gamma)}{2\ell} - (12\ell^2 t + x - p) \tag{1.3b}$$

so the pole of (1.2) is exactly the zero of the τ-function in (1.3) and the study of motion of the pole of (1.2) can be reduced to the study of motion of the zero of $\tau(t, x)$. Since $\tau'(t, x) = \frac{\partial\tau(t,x)}{\partial x} = \cos 2\ell(4\ell^2 t + x - \gamma) - 1 \leq 0$, the τ-function itself is monotonically decreasing in x from $+\infty$ to $-\infty$ and thus always has exactly one zero. The zero is simple unless

$$2\ell(4\ell^2 t + x - \gamma) = 2\pi n, \quad n \in \mathbb{I} \tag{1.4a}$$

and

$$12\ell^2 t + x - p = 0 \tag{1.4b}$$

in which case the zero is of third order. The solution of (1.4) is of the form:

$$t = \frac{p-\gamma}{8\ell^2} + \frac{\pi n}{4\ell^3}, \quad x = \frac{3\gamma - p}{2} - \frac{3\pi n}{\ell}, \quad n \in \mathbb{N} \tag{1.5}$$

Such points are often referred to as resonances [**14**].

We can define superposition of two harmonic breathers [**7**, **13**] and study their interaction in a way similar to that for solitons. Due to the complicated form of the two-harmonic-breather solution, the interaction of the harmonic breathers when they are close to each other has, so far, been studied only numerically [**17**]. Here we obtain some analytical results similar to those of [**2**, **5**, **9**, **10**, **13**, **16**] for solitons.

To do this we use the following representation of the two-harmonic-breather solution obtained in [**7**]:

$$w(t, x) = -2\frac{\partial^2}{\partial x^2}\ln \tau(t, x), \quad \tau(t, x) = \tau_1(t, x)\tau_2(t, x) - q^2(t, x) \tag{1.6}$$

where

$$\tau_i(t, x) = \frac{\sin 2\ell_i\xi_i}{2\ell_i} - \eta_i, \quad \xi_i = x + 4\ell_i^2 t - \gamma, \quad \eta_i = x + 12\ell_i^2 t - p_i, \quad i = 1, 2;$$

$$q = \frac{\sin(\ell_1\xi_1 - \ell_2\xi_2)}{\ell_1 - \ell_2} - \frac{\sin(\ell_1\xi_1 + \ell_2\xi_2)}{\ell_1 + \ell_2} = \frac{2}{\ell_1^2 - \ell_2^2}(\ell_2 \sin\ell_1\xi_1 \cos\ell_2\xi_2 - \ell_1 \cos\ell_1\xi_1 \sin\ell_2\xi_2),$$

$$\ell_1 \neq \ell_2, \quad \ell_1 > 0, \quad \ell_2 > 0.$$

The analytical results are summarized in the following theorem.

Theorem. (a) *The τ-function of (1.6) always has at most two zeros, i.e. for each value of $t \in \mathbb{R}$, there are at most two values of x satisfying $\tau(t, x) = 0$.*

(b) *Two distinct roots of $\tau(t, x) = 0$, which we denote by $x_1(t)$ and $x_2(t)$, merge into one for some value of t if and only if the quantities*

$$
\begin{aligned}
n_1 &= \frac{2\ell_1}{3\pi(\ell_2^2-\ell_1^2)}[6\ell_2^2(p_1-\gamma_1)-6\ell_1^2(p_2-\gamma_2)-\ell_1^2(3\gamma_2-p_2-3\gamma_1+p_1)] \\
n_2 &= \frac{2\ell_2}{3\pi(\ell_1^2-\ell_2^2)}[6\ell_1^2(p_2-\gamma_2)-6\ell_2^2(p_1-\gamma_1)-\ell_2^2(3\gamma_1-p_1-3\gamma_2+p_2)]
\end{aligned} \tag{1.7}
$$

are integers and are either both even or both odd. If n_1 and n_2 are both even, then at time $t=\frac{p_2-\gamma_2}{8\ell_2^2}+\frac{\pi n_2}{16\ell_2^3}=\frac{p_1-\gamma_1}{8\ell_1^2}+\frac{\pi n_1}{16\ell_1^3}$, $\tau(t,x)$ has a single root $x=\frac{3\gamma_1-p_1}{2}-\frac{3\pi n_1}{4\ell_1}=\frac{3\gamma_2-p_2}{2}-\frac{3\pi n_2}{4\ell_2}$ of order 10. If n_1 and n_2 are both odd then at time $t=\frac{p_1-\gamma_1}{8\ell_1^2}+\frac{\pi n_1}{16\ell_1^3}=\frac{p_2-\gamma_2}{8\ell_2^2}+\frac{\pi n_2}{16\ell_2^2}$, $\tau(t,x)$ has a single root $x=\frac{3\gamma_1-p_1}{2}-\frac{3\pi n_1}{4\ell_1}=\frac{3\gamma_2-p_2}{2}-\frac{3\pi n_2}{4\ell_2}$ of order 6.

2. Proof of the Theorem

We break up the proof into a sequence of lemmas.

Lemma 1. *The τ-function defined in* (1.6) *and its components satisfy the following identities:*

$$\tau=\tau_1\tau_2-q^2 \tag{2.1}$$

$$\tau_i=\frac{\sin 2\ell_i\xi_i}{2\ell_i}-\eta_i,\quad \xi_i=x+4\ell_i^2t-\gamma_i,\quad \eta_i=x+12\ell_i^2t-p_i,\quad i=1,2 \tag{2.2}$$

$$q=\frac{2}{\ell_1^2-\ell_2^2}(\ell_2\sin\ell_1\xi_1\cos\ell_2\xi_2-\ell_1\cos\ell_1\xi_1\sin\ell_2\xi_2) \tag{2.3}$$

$$q'=\frac{\partial q}{\partial x}=2\sin\ell_1\xi_1\sin\ell_2\xi_2 \tag{2.4}$$

$$q''=\frac{\partial^2 q}{\partial x^2}=2\ell_1\cos\ell_1\xi_1\sin\ell_2\xi_2+2\ell_2\sin\ell_1\xi_1\cos\ell_2\xi_2 \tag{2.5}$$

$$\tau_i'=\frac{\partial\tau_i}{\partial x}=\cos 2\ell_i\xi_i-1=-2\sin^2\ell_i\xi_i,\quad i=1,2 \tag{2.6}$$

$$\tau_i''=\frac{\partial^2\tau_i}{\partial x^2}=-4\ell_i\sin\ell_i\xi_i\cos\ell_i\xi_i,\quad i=1,2 \tag{2.7}$$

$$\tau'=\frac{\partial\tau}{\partial x}=-2\tau_1\sin^2\ell_2\xi_2-2\tau_2\sin^2\ell_1\xi_1-2q\sin\ell_1\xi_1\sin\ell_2\xi_2 \tag{2.8}$$

$$\tau'' = \frac{\partial^2 \tau}{\partial x^2} = -2\ell_1\tau_2 \sin 2\ell_1\xi_1 - 2\ell_2\tau_1 \sin 2\ell_2\xi_2 \\ - \frac{8}{\ell_1^2 - \ell_2^2}(\ell_2^2 \sin^2 \ell_1\xi_1 \cos^2 \ell_2\xi_2 - \ell_1^2 \sin^2 \ell_2\xi_2 \cos^2 \ell_1\xi_1) \tag{2.9}$$

$$\tau''' = -4\ell_1^2\tau_2 \cos 2\ell_1\xi_1 - 4\ell_2^2\tau_1 \cos 2\ell_2\xi_2 + \frac{4\ell_2}{\ell_1^2 - \ell_2^2} \sin 2\ell_2\xi_2(\ell_1^2 + \ell_1^2 \cos^2 \ell_1\xi_1 + \ell_2^2 \sin^2 \ell_1\xi_1) \\ - \frac{4\ell_1}{\ell_1^2 - \ell_2^2} \sin 2\ell_1\xi_1(\ell_2^2 + \ell_2^2 \cos^2 \ell_2\xi_2 + \ell_1^2 \sin^2 \ell_2\xi_3) \tag{2.10}$$

$$\tau = -\frac{\tau'\tau_1}{2\sin^2 \ell_1\xi_1} - \left(\frac{\sin \ell_2\xi_2}{\sin \ell_1\xi_1}\tau_1 + q\right)^2 \tag{2.11a}$$

$$\tau = -\frac{\tau'\tau_2}{2\sin^2 \ell_2\xi_2} - \left(\frac{\sin \ell_1\xi_1}{\sin \ell_2\xi_2}\tau_2 + q\right)^2. \tag{2.11b}$$

Proof of (2.1)-(2.10) is by direct computations. Equation (2.11a) is obtained by solving (2.8) for τ_2 and then substituting $\tau_2 = -\frac{\tau'}{2\sin^2 \ell_1\xi_1} - \frac{\sin^2 \ell_2\xi_2}{\sin^2 \ell_1\xi_1}\tau_1 - \frac{2q \sin \ell_2\xi_2}{\sin \ell_1\xi_1}$ into (2.1); (2.11b) is obtained in a similar manner.

Lemma 2. *If $\tau(t, x) = \tau'(t, x) = 0$, then $\tau''(t, x) = 0$.*

Proof. If neither $\sin \ell_1\xi_1 = 0$ nor $\sin \ell_2\xi_2 = 0$, (2.11) gives us $\tau_1 = -\frac{\sin \ell_1\xi_1}{\sin \ell_2\xi_2}q$, $\tau_2 = -\frac{\sin \ell_2\xi_2}{\sin \ell_1\xi_1}q$. Substituting these into (2.9) we obtain $\tau'' = 0$. If $\sin \ell_1\xi_1 = 0$ ($\sin \ell_2\xi_2 = 0$ is handled in exactly the same manner), then $\tau' = 0$ and (2.8) implies $\tau_1 \sin \ell_2\xi_2 = 0$ and therefore either $\sin \ell_2\xi_3 = 0$ or $\tau_1 = 0$. If $\sin \ell_2\xi_2 = 0$ then substituting $\sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$ into (2.9) we obtain $\tau'' = 0$. If $\tau_1 = 0$, then substituting this and $\tau = 0$ into (2.1) we obtain $q = 0$, which together with $\sin \ell_1\xi_1 = 0$ yields $\sin \ell_2\xi_2 = 0$, substituting $\sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$ into (2.9) again gives us $\tau'' = 0$.

Lemma 3. *Let $\tau(t, x)$, considered as a function of x for an arbitrary but fixed value of t, have an extremum at $x = x_0$. Then $\tau(t, x_0) \leq 0$.*

Proof. If $\sin \ell_1\xi_1 \neq 0$ at (t, x_0), then $\tau'(t, x_0) = 0$ and (2.11a) yields

$$\tau = -\left(\frac{\sin \ell_2\xi_2}{\sin \ell_1\xi_1}\tau_1 + q\right)^2 \leq 0.$$

If $\sin \ell_2\xi_2 \neq 0$ at (t, x_0), (2.11b) yields the result.

Consider now the case $\sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$ at (t, x_0). Substituting $\sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$ into (2.8) and (2.9) we obtain $\tau'(t, x_0) = \tau''(t, x_0) = 0$. The fact that $x = x_0$ is an extremum then implies $\tau'''(t, x_0) = 0$, which, using (2.10), gives us $\ell_1^2\eta_2 + \ell_2^2\eta_1 = 0$, with η_1, η_2 evaluated at (t, x_0) according to (2.2). Then $\ell_1^2\eta_2 + \ell_2^2\eta_1 = 0$ implies $\eta_1\eta_2 \leq 0$. Substituting now $\sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$ into (2.1) we obtain $\tau = \eta_1\eta_2 \leq 0$.

Lemma 4. *Let x_0 be a local maximum of $\tau(t, x)$ considered as a function of x for an arbitrary but fixed value of t. Then $\tau(t, x_0) < 0$.*

Proof. In view of Lemma 3, it suffices to show that $\tau(t, x_0) \neq 0$. Assume $\tau(t, x_0) = 0$ and consider

$$f(\lambda_1, \lambda_2, t, x) = \left[\frac{\sin 2\ell_1\xi_1}{2\ell_1} - (x + 12\ell_1^2 t - \lambda_1)\right]\left[\frac{\sin 2\ell_2\xi_2}{2\ell_2} - (x + 12\ell_2^2 t - \lambda_2)\right] - q^2.$$

Due to continuity of f in all of its arguments, $f(\lambda_1, \lambda_2, t, x_0) \leq 0$ for λ_1 and λ_2 satisfying $|\lambda_1 - p_1| + |\lambda_2 - p_2| < \varepsilon$ for some sufficiently small ε. Thus $f(\lambda_1, \lambda_2, t, x_0)$ attains a local maximum as a function of λ_1 and λ_2 at $\lambda_1 = p_1$ and $\lambda_2 = p_2$ and therefore $\frac{\partial f}{\partial \lambda_1} = \tau_2 = 0$, $\frac{\partial f}{\partial \lambda_2} = \tau_1 = 0$ and the matrix $\|\frac{\partial f}{\partial \lambda_i \partial \lambda_j}\|$ is nonnegative definite. On the other hand direct computations give us $\|\frac{\partial f}{\partial \lambda_i \partial \lambda_j}\| = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ which is not nonnegative definite. The obtained contradiction proves that the assumption $\tau(t, x_0) = 0$ is false.

Lemma 5. *Let x_0 be a local minimum of $\tau(t, x)$ (considered as a function of x for an arbitrary but fixed value of t) and $\tau(t, x_0) = 0$. Then*

$$\tau_1 \sin \ell_2\xi_2 = \tau_2 \sin \ell_1\xi_1 = q \sin \ell_1\xi_1 \sin \ell_2\xi_2 = 0.$$

Proof. Substituting $\tau' = 0$ into (2.8) we obtain

$$\tau_1 \sin^2 \ell_2\xi_2 + \tau_2 \sin^2 \ell_1\xi_1 = -q \sin \ell_1\xi_1 \sin \ell_2\xi_2.$$

Squaring both sides and replacing $\tau_1\tau_2$ with q^2 gives us

$$\tau_1^2 \sin^4 \ell_2\xi_2 + \tau_2^2 \sin^4 \ell_1\xi_1 = -q^2 \sin^2 \ell_1\xi_1 \sin^2 \ell_2\xi_2.$$

Since the left-hand side is nonnegative and the right-hand side is nonpositive, they both must be zero, yielding the result.

Lemma 6. *Let x_0 be a local minimum of $\tau(t, x)$ (considered as a function of x for an arbitrary but fixed value of t) and $\tau(t, x_0) = 0$. Then either*

$$\tau_1(t, x_0) = \tau_2(t, x_0) = \sin \ell_1\xi_1 = \sin \ell_2\xi_2 = 0$$

or

$$\tau_1(t, x_0) = \tau_2(t, x_0) = \cos \ell_1\xi_1 = \cos \ell_2\xi_2 = 0.$$

Proof. By Lemma 5 one of (a) $\tau_1 = \sin \ell_1 \xi_1 = 0$, (b) $\tau_2 = \sin \ell_2 \xi_2 = 0$, (c) $\sin \ell_1 \xi_1 = \sin \ell_2 \xi_2 = 0$ or (d) $\tau_1 = \tau_2 = 0$ must hold. Consider each case separately.

(a) $\tau_1 = \sin \ell_1 \xi_1 = 0$. Since x_0 is a local extremum and $\tau(t, x_0) = 0$, Lemma 2 implies $\tau'' = 0$. Substituting $\tau_1 = \sin \ell_1 \xi_1 = \tau'' = 0$ into (2.9) we obtain $\sin \ell_2 \xi_2 = 0$. Again since x_0 is a local extremum and $\tau'' = 0$ we also have $\tau''' = 0$, which along with (2.10) implies $\tau_2 = 0$.

(b) Similar to (a).

(c) $\sin \ell_1 \xi_1 = \sin \ell_2 \xi_2 = 0$. Substituting these into (2.9) we obtain $\tau'' = 0$ which along with the fact that x_0 is an extremum yields $\tau''' = 0$. Substituting $\tau''' = \sin \ell_1 \xi_1 = \sin \ell_2 \xi_2 = 0$ into (2.10) gives us $\ell_1^2 \tau_2 + \ell_2^2 \tau_1 = 0$ and therefore $\tau_1 \tau_2 \leq 0$. On the other hand substituting $\tau = 0$ into (2.1) results in $\tau_1 \tau_2 = q^2 \geq 0$ implying that either τ_1 or τ_2 is 0. But $\ell_1^2 \tau_2 + \ell_2^2 \tau_1 = 0$ and therefore once one of them vanishes so does the other one.

(d) $\tau_1 = \tau_2 = 0$. Substituting $\tau_1 = \tau_2 = \tau = 0$ into (2.1) we obtain $q = 0$ and thus if either one of $\sin \ell_1 \xi_1$ or $\sin \ell_2 \xi_2$ is zero then so must the other one, yielding the result.

Let us now assume that neither $\sin \ell_1 \xi_1$ nor $\sin \ell_2 \xi_2$ vanish at (t, x_0). Substituting $\tau = \tau_2 = q = 0$ into the expressions for τ''' and $\tau^{(4)}$ we obtain $\tau''' = \tau^{(4)} = 0$. Since $x = x_0$ is an extremum we must have $\tau^{(5)} = 0$ which combined with $\tau_1 = \tau_2 = q = 0$ yields $\cos \ell_1 \xi_1 = \cos \ell_2 \xi_2 = 0$.

Lemma 7. *Let x_0 be a local minimum of $\tau(t_0, x)$ (considered as a function of x) and let $r(t_0, x_0) = 0$. Then there exist two integers n_1 and n_2 either both even or both odd such that*

$$t_0 = \frac{p_1 - \gamma_1}{8\ell_1^2} - \frac{\pi n_1}{16\ell_1^3} = \frac{p_2 - \gamma_2}{8\ell_2^2} - \frac{\pi n_2}{16\ell_2^3} \tag{2.12a}$$

$$x_0 = \frac{3\gamma_1 - p_1}{2} + \frac{3\pi n_1}{4\ell_1} = \frac{3\gamma_2 - p_2}{2} + \frac{3\pi n_2}{4\ell_2}. \tag{2.12b}$$

Proof. Lemma 6 implies that there exist two integers n_1 and n_2 either both even or both odd such that

$$\begin{cases} \eta_1 = x_0 + 12\ell_1^2 t_0 - p_1 = 0 \\ \xi_1 = x_0 + 4\ell_1^2 t_0 - \gamma_1 = \frac{\pi n_1}{2\ell_1} \end{cases}$$

$$\begin{cases} \eta_2 = x_0 + 12\ell_2^2 t_0 - p_2 = 0 \\ \xi_2 = x_0 + 4\ell_2^2 t_0 - \gamma_2 = \frac{\pi n_2}{2\ell_2}. \end{cases}$$

Solving the first system we obtain

$$t_0 = \frac{p_1 - \gamma_1}{8\ell_1^2} - \frac{\pi n_1}{16\ell_1^3}, \quad x_0 = \frac{3\gamma_1 - p_1}{2} - \frac{3\pi n_1}{4\ell_1}$$

whereas solving the second system we get

$$t_0 = \frac{p_2 - \gamma_2}{8\ell_2^2} - \frac{\pi n_2}{16\ell_2^2}, \quad x_0 = \frac{3\gamma_2 - p_2}{2} + \frac{3\pi n_2}{4\ell_2}.$$

Proof of the Theorem. Part (a) If $\tau(t, x)$ had more than two zeros it would also have a nonnegative local maximum but that contradicts Lemma 4.

Part (b) Two zeros of the τ-function merge into one if and only if for some t_0 $\tau(t_0, x)$ has a single zero x_0 which is also a local (as well as global) minimum. But according to Lemma 7 this can happen only if (2.12) holds. Vice versa if (2.12) holds then one can verify by direct computations that $\tau(t_0, x)$ has a local minimum at $x = x_0$ as well as $\tau(t_0, x_0) = 0$. By solving (2.12) for n_1 and n_2 we obtain (1.7).

The order of zero at $x = x_0$ is easily verified by direct computations.

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