POINTEWISE CHAIN RECURRENT MAPS OF THE SPACE $Y$

WENJING GUO, FANPING ZENG AND QIYING HU

Let $Y = \{z \in C : z^3 \in [0,1]\}$ (equipped with subspace topology of the complex space $C$) and let $f : Y \rightarrow Y$ be a continuous map. We show that if $f$ is pointwise chain recurrent (that is, every point of $Y$ is chain recurrent under $f$), then either $f^{12}$ is the identity map or $f^{12}$ is turbulent. This result is a generalisation to $Y$ of a result of Block and Coven for pointwise chain recurrent maps of the interval.

1. INTRODUCTION

In this paper we characterise the dynamics of maps of the space $Y = \{z \in C : z^3 \in [0,1]\}$ equipped with the subspace topology for with every point is chain recurrent. We prove the following.

**MAIN THEOREM.** Let $f$ be a continuous map of $Y$ to itself. If $f$ is pointwise chain recurrent, then either $f^{12}$ is the identity map or $f^{12}$ is turbulent.

Block and Coven (see [4]) proved that a pointwise chain recurrent map $h$ of the interval must satisfy that either $h^2$ is the identity map or $h^2$ is turbulent. So our theorem extends this result to maps of the space $Y$.

Firstly some notation and definitions are established. Let $(X, d)$ be a compact metric space and $g : X \rightarrow X$ be a continuous map. If $g^n(x) = x \neq g^k(x), k = 1, 2, \ldots, n - 1$, for some $x \in X$ and some positive integer $n$, then the point $x$ is called a periodic point of period $n$, where $g^0 = id, g^i = g \circ (g^{i-1})(i \geq 1)$. In particular, if $g(x) = x$, then $x$ is called a fixed point of $g$. Denoted by $P(g)$ and $F(g)$ the set of periodic points and fixed points set of $g$ respectively. For $x, y \in X$ and $\varepsilon > 0$, an $\varepsilon$-chain from $x$ to $y$ is a finite sequence $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ with $d(g(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i \leq n - 1$. We say $x$ is chain recurrent under $g$, if for each $\varepsilon > 0$, there is an $\varepsilon$-chain from $x$ to $y$. The map $g$ is said to be pointwise chain recurrent, if every point of $X$ is chain recurrent under $g$.

The following facts about chain recurrent are standard observations:

(a) If $g$ is pointwise chain recurrent, then $g$ maps $X$ onto $X$.

(b) $g$ is pointwise chain recurrent if and only if $g^n$ is pointwise chain recurrent for every $n > 0$. 

Received 8th May, 2002

Project supported by NNSF of China (19961001).

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/03 $A2.00+0.00.$
(c) [5, Theorem A] If $X$ is connected and $g : X \rightarrow X$ is pointwise chain recurrent, then there is no nonempty open set $U \neq X$ such that $g(U) \subseteq U$.

Being chain recurrent is an important dynamical property of a system and has been studied intensively in recent years. For more details see [1, 3, 5, 6, 9].

The space $Y$ is obviously a tree (see [7]) in which there are exactly three ends, denoted by $e_1$, $e_2$, and $e_3$, and exactly one vertex, denoted by $o$. For $a, b \in Y$, we shall use $[a, b]$, called a closed subinterval of $Y$, to denote the smallest closed connected subset containing $a$ and $b$. We define $(a, b) = [a, b] \setminus \{a, b\}$ and we can similarly define $(a, b)$ and $[a, b]$. For a subset $A$ of $Y$, we use $\text{int}(A)$, $\overline{A}$ and $\partial A$ to denote the interior, the closure and the boundary of $A$, respectively.

A map $g : Y \rightarrow Y$ is called turbulent if there are closed subintervals $J$ and $K$ with disjoint interiors such that $g(J) \cap g(K) \supseteq J \cup K$. Clearly, if $f$ is turbulent then $f^n$ is turbulent for any $n \geq 2$.

From the above definition of turbulence and the proof of [8, Theorem 1], the following result is clear.

**Theorem 1.1.** Let $f$ be a continuous map of space $Y$. If $f$ is turbulent, then $f$ has more than one fixed point.

Set $e \in \{e_1, e_2, e_3\}$. A partial order $<e$ on $Y$ defined as follows, which will be useful in dealing with continuous maps of the space $Y$. For $x, y \in Y$, $x < e y$ if $x \in [y, e]$ and $x \neq y$.

Throughout this paper, $f$ denotes a pointwise chain recurrent map of $Y$ into itself. This paper is organised as follows. In Section 2 and Section 3, the pointwise chain recurrent maps of $Y$ with more than one fixed point are characterised, where the fixed points set is disconnected in Section 2 and connected in Section 3. In Section 4, the pointwise chain recurrent maps of $Y$ with exactly one fixed point are discussed.

**Examples.** Clearly, $Y = I \cup \{xe^{(2/3)\pi i} | x \in I\} \cup \{xe^{(4/3)\pi i} | x \in I\}$, where $I = [0, 1]$.

1. $f : Y \rightarrow Y$, $f(x) = xe^{(2/3)\pi i}$, $f(xe^{(2/3)\pi i}) = x$ and $f(xe^{(4/3)\pi i}) = xe^{(4/3)\pi i}$ for any $x \in [0, 1]$. Then $f$ is pointwise chain recurrent such that $f^2 = \text{id}_Y$, but $f \neq \text{id}_Y$.

2. $f : Y \rightarrow Y$ is a rotation of period 3. Then $f$ is pointwise chain recurrent such that $f$ has exactly one fixed point.

**2. Pointwise chain recurrent maps of $Y$ with disconnected fixed points set**

In this section, we assume that $f$ has a disconnected fixed points set. Then there exist two fixed points $a, b$ of $f$ with $(a, b) \cap F(f) = \phi$. 

**Theorem 2.1** If the closure of some component of $Y \setminus \{o\}$ contains $(a, b)$, then $f^2$ is turbulent.
PROOF: Without loss of generality, we assume that \( \{a, b\} \subseteq [a, e_1] \) and \( b < e_1 a \).

CASE 1. \( f(x) < e_1 x \) for all \( x \in (a, b) \). Then \( b \neq e_1 \), for otherwise \( U = [e_1, a') \) satisfies \( f(U) \subseteq U \) for any \( a' \in (a, b) \). Let \( c \) be the largest point in \((b, e_1] \) relative to \( < e_1 \) such that \( f(c) = a \). (If no such \( c \) exists, then there exists \( b' \in (a, b) \) such that \( f(x) < e_1 \) \( b' \) for all \( x \in (a, e_1] \). But then \( U = (b', e_1] \) satisfies \( f(U) \subseteq U \).) Let \( d \in (a, c) \) be the point with \( f(d) = c \). (Again if no such \( d \) exists, then there exists \( b' \in (b, c) \) such that \( c' < e_1 f(x) \) for all \( x \in (a, c) \). But then \( U = (d', c') \) satisfies \( f(U) \subseteq U \) for some \( d' \in (a, b) \).) Then \( J = [a, d] \) and \( K = [d, c] \) show that \( f \) is turbulent, and hence \( f^2 \) is turbulent.

CASE 2. \( x < e_1 f(x) \) for all \( x \in (a, b) \). There exists \( c \in Y \setminus [a, e_1] \) such that \( f(c) = b \), for otherwise, \( U = Y \setminus [b', e_1] \) for some \( b < e_1 b' < e_1 a \) satisfies \( f(U) \subseteq U \). The following three subcases are considered.

SUBCASE 2.1. There exists \( c_i \in [e_i, a] \) such that \( f(c_i) = b_i, i = 2, 3, \) and there exists \( d_2 \in [c_2, b] \) such that \( f(d_2) = c_2 \). (or there exists \( c_i \in [e_i, a] \) such that \( f(c_i) = b_i, i = 2, 3, \) and there exists \( d_3 \in [c_3, b] \) such that \( f(d_3) = c_3 \), the proof of this case is similar and omitted.) Taking \( J = [c_2, d_2] \) and \( K = [d_2, b] \), one gets that \( f(J) \cap f(K) \supseteq J \cup K \) and then \( f \) is turbulent. Thus \( f^2 \) is turbulent.

SUBCASE 2.2. \( b < e_1 f(x) \) for all \( x \in [e_3, a] \) and there exists \( c \in [e_2, o) \) such that \( f(c) = b \). (or \( b < e_1 f(x) \) for all \( x \in [e_2, a] \) and there exists \( c \in [e_3, o) \) such that \( f(c) = b \), the proof of this case is similar and omitted.) Assume that such point \( c \) is the largest one in \([e_3, o) \) relative to \( < e_2 \). Then there exists \( d \in [e_3, b] \cup [c, o) \) such that \( f(d) = c \). (If no such \( d \) exists, then \( U = [e_3, b') \cup (o, c') \) for some \( b' \in (a, b) \) and some \( c' \in (o, c) \) satisfies \( f(U) \subseteq U \).) If \( d \in (c, b) \), then, taking \( J = [c, d] \) and \( K = [d, b] \), one gets that \( f(J) \cap f(K) \supseteq J \cup K \) and thus \( f^2 \) is turbulent. Now, assume \( [c, b] \cap f^{-1}(c) = \phi \) and such \( d \in [e_3, o) \) is the largest one in \([e_3, o) \) relative to \( < e_3 \). Then there exists \( t \in [c, b] \cup [o, d) \) such that \( f(t) = d \). (If \( b < e_1 f(x) \) for all \( x \in [e_3, a] \) and there exists \( c \in [e_3, o) \) such that \( f(c) = b \), the proof of this case is similar and omitted.) Assume that such \( d \) is the largest one in \([e_i, o) \) relative to \( < e_i(i = 2, 3) \). Now a similar argument as that in Subcase 2.2 yields that \( f^2 \) is turbulent. The proof is complete.

THEOREM 2.2. If \( a, b \) lie in two distinct components of \( Y \setminus \{o\} \), \( f^2 \) is turbulent.

PROOF: Without loss of generality, assume that \( b \in (a, e_1], a \in (o, e_2] \).
CASE 1. \( x < e_1 f(x) \) for all \( x \in (o, b) \) (or \( x < e_2 f(x) \) for all \( x \in (o, a) \), the proof of this case is similar and omitted.) A similar proof as that of case 2 in Theorem 2.1 implies that \( f^2 \) is turbulent.

CASE 2. \( f(x) < e_1 x \) for all \( x \in (o, b) \) and \( f(x) < e_2 x \) for all \( x \in (o, a) \). Then \([a', b'] \cap F(f) \neq \emptyset\) for any \( a' \in (o, a) \) and any \( b' \in (o, b) \) (according to the proof of [8, Theorem 1], in fact, we have \( o \in F(f) \)). There is a contradiction. Therefore case 2 is impossible and proof is complete. 

3. POINTWISE CHAIN RECURRENT MAPS OF \( Y \) WITH CONNECTED FIXED POINTS SET

In this section, we assume that \( f \) has connected fixed points set. Then \( F(f) \) is a connected closed subset of \( Y \). If \( F(f) \) is degenerated, then \( f \) has exactly one fixed point. This case will be discussed in section 4. Now assume that \( F(f) \) is nondegenerated.

**THEOREM 3.1** If \( F(f) \) is contained in the closure of a component of \( Y \setminus \{o\} \), then \( f^2 = \text{id}_Y \) but \( f \neq \text{id}_Y \) or \( f^2 \) is turbulent.

**PROOF:** Without loss of generality, assume that \( F(f) = [p, q] \subseteq [o, e_1] \) and \( p < e_1 q \).

We first claim that \( q = o \). Suppose not. Then \( f(x) < e_1 x \) for all \( x \in [o, q] \). Note that \( p, q \) are fixed points of \( f \). There exists \( q' \in (o, q) \) such that \( f([q', p]) \subseteq (q', p) \) for some \( p' \in (p, e_1) \) (if \( p \neq e_1 \)) or \( f([q', p]) \subseteq (q', e_1) \) (if \( p = e_1 \)). There is a contradiction. By the claim, the following two cases will be considered.

CASE 1. \( p \neq e_1 \). Clearly, we have \( x < e_1 f(x) \) for all \( x \in (p, e_1) \); \( x < e_2 f(x) \) for all \( x \in (o, e_2) \) and \( x < e_3 f(x) \) for all \( x \in (o, e_3) \). Since \( f \) is onto, there exists \( x_0 \in [e_2, e_3] \setminus \{o\} \) such that \( f(x_0) = e_1 \). Without loss of generality, we assume that \( x_0 \in [e_2, o) \). Then, by the continuity of \( f \), there exists \( r \in (o, x_0) \) such that \( f(r) = p \). Furthermore, we may assume that such \( r \) is the largest one in \( [e_2, o) \) relative to \( < e_2 \).

**SUBCASE 1.1.** \( p < e_1 f(x) \) for all \( x \in (o, e_3) \). Then there exists \( s \in (o, r) \cup (o, e_3) \) such that \( f(s) = r \). (If no such \( s \) exists, then \( U = (r', e_3) \cup (o, p') \) for some \( r' \in (o, r) \) and some \( p' \in (p, e_1) \) satisfies \( f(U) \subseteq U \).) Furthermore, we have \( s \in (o, e_3) \) for otherwise \((o, r) \cap F(f) \neq \emptyset \) and assume that such \( s \) is the largest one in \( (o, e_3) \) relative to \( < e_3 \). There exists \( t \in (o, r) \cup (o, s) \) such that \( f(t) = s \). (If no such \( t \) exists, then \( U = (r', s') \cup (o, p') \) for some \( r' \in (o, r) \), some \( s' \in (o, s) \) and some \( p' \in (p, e_1) \) satisfies \( f(U) \subseteq U \).) Furthermore, we have \( t \in (o, r) \) (for otherwise, \( (o, s) \cap F(f) \neq \emptyset \)). Taking \( J = [o, t], K = [t, r] \), one gets \( f^2(J) \cap f^2(K) \supseteq J \cup K \) and thus \( f^2 \) is turbulent.

**SUBCASE 1.2** There exists \( r_1 \in (o, e_3) \) such that \( f(r_1) = p \). Without loss of generality, assume that such \( r_1 \) is the largest one in \( [e_3, o) \) relative to \( < e_3 \). Then there exists \( s \in (o, r_1) \) such that \( f(s) = r \) or \( s_1 \in (o, r) \) such that \( f(s_1) = r_1 \). (If none of such \( s, s_1 \) exists, then \( U = (r', s') \cup (o, p') \) for some \( r' \in (o, r), s' \in (o, r_1) \) and some \( p' \in (p, e_1) \) satisfies \( f(U) \subseteq U \).) Without loss of generality, we assume that there exists \( s \in (o, r_1) \) such
Pointwise chain recurrent maps

If there exists $s_1 \in (o, r)$ such that $f(s_1) = r_1$, the proof of this case is similar and omitted. A similar argument as that in subcase 1.1 yields that $f^2$ is turbulent.

CASE 2. $p = e_1$. Clearly, we have $x < e_2 f(x)$ for all $x \in (o, e_2)$ and $x < e_3 f(x)$ for all $x \in (o, e_3)$.

If there exists $a \in [e_2, e_3] \setminus \{o\}$ such that $f(a) \in (o, e_1)$, then we can get $b \in (o, a) \cup (o, e_3)$ (without loss of generality, assume that $a \in (o, e_2)$. For $a \in (o, e_3)$, a similar argument will be done.) such that $f(b) = a$. (If no such $b$ exists, then there exists $a' < e_2 f(x)$ for all $x \in (o, a] \cup (o, e_3]$. But then $U = [e_1, e_2] \cup (o, a')$ satisfies $f(U) \subseteq U$.) In fact, we have $b \in (o, e_3)$. (For otherwise, $F(f) \cap (o, a) \neq \emptyset$.) Without loss of generality, assume that such $b$ is the largest one in $(o, e_3)$ relative to $< e_3$ such that $f(b) = a$. Furthermore, let $c$ be any point in $(a, b)$ such that $f(c) = b$. (Again if no such $c$ exists, then there exists $b' \in (o, b)$ such that $b' < e_3 f(x)$ for all $x \in [a, b] \cup (o, e_1]$. But then $U = (a', b') \cup (o, e_1]$ satisfies $f(U) \subseteq U$ for some $a' \in (o, a)$. In fact, we have $c \in (o, a)$ (for otherwise, $F(f) \cap (o, e_3) \neq \emptyset$). Taking $J = [o, c], K = [a, c]$, one gets $f^2(J) \cap f^2(K) \supseteq J \cup K$ and thus $f^2$ is turbulent.

If $f^{-1}((o, e_1)) \cap [e_2, e_3] = \emptyset$, then $f|_{[e_2, e_3]} : [e_2, e_3] \to [e_2, e_3]$ is pointwise chain recurrent and has exactly one fixed point. It follows from [4, Theorem] that $f^2|_{[e_2, e_3]} = id|_{[e_2, e_3]}$ or $f^2|_{[e_2, e_3]}$ is turbulent. If $f^2|_{[e_2, e_3]} = id|_{[e_2, e_3]}$ then $f^2 = id_Y$ but $f \neq id_Y$; if $f^2|_{[e_2, e_3]}$ is turbulent, then $f^2$ is certainly turbulent.

The proof is complete.

**THEOREM 3.2.** There does not exist $f$ such that $o \in \text{int} F(f)$ except the identity map $id_Y$.

**Proof:** Assume that such $f$ exists and $f$ is not the identity. Let $F(f) \cap [o, e_i] = [o, p_i], i \in \{1, 2, 3\}$. Note that each $p_i$ is the smallest fixed point in $[o, e_i]$ relative to $< e_i$. Then there exists $p'_i \in (p_i, e_i)$ (if $p_i \neq e_i$) such that $x < e_i f(x) < e_i p_j (i \in \{1, 2, 3\}, and $j \neq i$) for all $x \in (p_i, p'_i)$. Thus, taking

$$U = U_1 \cup U_2 \cup U_3,$$

where each $U_i = [o, p'_i] \cup [o, e_i] \cup [p_i, e_i]$ if $p_i \neq e_i$; and $[o, e_i]$ if $p_i = e_i$, one gets that $f(U) \subseteq U$. There is a contradiction. The proof is complete.

**4. Pointwise chain recurrent of $Y$ with exact one fixed point**

In this section, we assume that $f$ has exactly one fixed point, written by $p$.

**Lemma 4.1.**

1. If $p = o$, then $f^2$ has exactly one fixed point too, but then $f^3$ has more than one fixed point.

2. If $p \neq o$, then $f^2$ has more than one fixed point.
PROOF: (1) Assume that $f^2$ has a fixed point $p'$ different from $o$. Without loss of generality, we assume that $p' \in (o,e_1]$, then $f(p') \in (o,e_3] \cup (o,e_2]$ (for otherwise, there exists at least one fixed point of $f$ in $(o,e_1]$.) Without loss generality, we assume that $f(p') \in (o,e_2]$. Since $f$ is onto, there exist $a_1 \in (o,e_2] \cup (o,e_3]$ such that $f(a_1) = e_1$, $a_2 \in (o,e_1] \cup (o,e_3]$ such that $f(a_2) = e_2$ and $a_3 \in (o,e_1] \cup (o,e_2]$ such that $f(a_3) = e_3$. If $a_1 \in (o,e_2]$, then we claim that $a_2 \in (o,e_3]$ and $a_3 \in (o,e_1]$ (If $a_1 \in (o,e_3]$, we must have $a_2 \in (o,e_1]$ and $a_3 \in (o,e_2]$. A similar argument will be done.) In fact, if $a_2 \in (o,e_1]$, then $a_3 \in (o,e_2]$ or $a_3 \in (o,e_1]$. Without loss of generality, we assume that $a_3 \in (o,e_2]$ (If $a_3 \in (o,e_1]$, the proof of this case is similar and omitted.) Furthermore, we assume that $a_1 < e_2 a_3$ (If $a_3 < e_2 a_1$, the proof of this case is similar and omitted.), then by the continuity of $f$, $f(a_3) \in [o,e_1]$, which contradicts $f(a_3) = e_3$. Thus, we have $p',a_3 \in (o,e_1]$. By the continuity of $f$, if $p' < e_1 a_3$, then $f(a_3) \in [o,f(p')]$, which contradicts $f(a_3) = e_2$; If $a_3 < e_1 p'$, then $f(p') \in [o,e_3]$, which contradict $f(p') \in (o,e_2]$.

From the above discussion, we see that either there exist $a_1 \in (o,e_2]$, $a_2 \in (o,e_3]$, $a_3 \in (o,e_1]$, or $a_1 \in (o,e_3]$, $a_2 \in (o,e_1]$, $a_3 \in (o,e_2]$ such that $f(a_1) = e_1$, $f(a_2) = e_2$, $f(a_3) = e_3$. Since the proofs of the above two cases are similar. We only prove the former. Clearly, $[o,a_1] \subseteq f^3([o,a_1])$, hence there exists $a \in [o,a_1]$ such that $f^3(a) = a_1$. Then $f^3$ has a fixed point in $[a,e_2]$.

(2) In fact, if $p \neq o$, then we must have $p$ is in one component of $Y \backslash \{o\}$ and $p \notin \{e_1, e_2, e_3\}$. (For otherwise, there exist more than one fixed point of $f$.) The proof of this case is similar to that of [4, Lemma 3] and omitted. \[\square\]

THEOREM 4.1.

(1) If $p = o$, then $f^2$ can not be turbulent. But $f^6$ is turbulent or identity map.

(2) If $p \neq o$, then $f^4$ is turbulent or identity map.

PROOF: By the previous results, the theorem is clear. Now to prove the main theorem, by Theorems 2.1, 2.2, 3.1, 3.2 and Lemma 4.1, either $f^{12}$ is the identity map or $f^{12}$ is turbulent. \[\square\]

REFERENCES


